Locally soluble-by-finite groups with small deviation for non-subnormal subgroups

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Abstract. A group G has subnormal deviation at most 1 if, for every descending chain $H_0 > H_1 > \dots$ of non-subnormal subgroups of G, for all but finitely many i there is no infinite descending chain of non-subnormal subgroups of G that contain H_{i+1} and are contained in H_i. This property $\mathfrak{P}$, say, was investigated in a previous paper by the authors, where soluble groups with $\mathfrak{P}$ and locally nilpotent groups with $\mathfrak{P}$ were effectively classified. The present article affirms a conjecture from that article by showing that locally soluble-by-finite groups with $\mathfrak{P}$ are soluble-by-finite and are therefore classified.

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This article continues the study of groups with “small deviation for non-subnormal subgroups”; more precisely, of groups G for which this deviation is at most 1, where we write $\operatorname{dev}_{\text{non-sn}}(G) \leq 1$. The discussion of groups with this property was begun in [6], following some general results on groups G for which the non-subnormal deviation exists at all. We refer the reader to [6] for the more general case and for precisely what it means to say that the non-subnormal deviation of a group G *exists*, but we recall here that a group G has non-subnormal deviation at most 1 if, for every descending chain $H_0 > H_1 > H_2 > \dots$ of non-subnormal subgroups of G, almost all of the “intervals” $I[H_{i+1}, H_i]$ satisfy the minimal condition for non-subnormal subgroups of G, that is, for all but finitely many i there is no infinite descending chain of non-subnormal subgroups of G all of which contain H_{i+1} and are contained in H_i. It was observed in [6] that a group with the weak minimal condition for non-subnormal subgroups has non-subnormal deviation at most 1 but that the converse is false — groups with this weak minimal condition were the topic of discussion in [4] and [5]. The main results of [6] were concerned with soluble groups G satisfying $\operatorname{dev}_{\text{non-sn}}(G) \leq 1$ and with locally nilpotent groups G with this property. We summarize the results on soluble groups as follows.

Theorem 1. *Let G be a soluble group such that* $\operatorname{dev}_{\text{non-sn}}(G) \leq 1$, *and let T be the torsion subgroup of the Baer radical B of G.*

(i) *If B is minimax then G too is minimax.*

(ii) *If B/T is not minimax then $G = B$ and every subgroup of G is subnormal.*
(iii) *If B/T is minimax but T is not minimax then T contains all elements of G of finite order, G/T is minimax and T satisfies $\min -\langle g\rangle$ for each element $g \in G\backslash B$. Furthermore, every subgroup of B is subnormal.*

We were also able to show that a locally nilpotent group G with $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ is soluble and of a certain restricted structure. It is clear from the above theorem that the structure of a soluble-by-finite group with non-subnormal deviation at most 1 has also been determined, and we conjectured in [6] that a locally soluble-by-finite group with deviation at most 1 is soluble-by-finite. Verification of this conjecture is the main purpose of the present article.

Theorem 2. *Let G be a locally soluble-by-finite group such that $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. Then G is soluble-by-finite.*

We approach the proof of this theorem by means of a sequence of lemmas.

Lemma 1. *Let G be a locally finite group satisfying $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. Then G is soluble-by-finite.*

Proof: If G is Chernikov then the result is clear, while if G is not Chernikov then it is a Baer group, by Lemma 2.8 of [6], hence locally nilpotent and hence soluble, as pointed out above. $\square$

Recall next that a group G is *radical* if it is the union of an ascending series of normal subgroups with successive factor groups locally nilpotent.

Lemma 2. *Let G be a radical group satisfying $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. Then G is soluble.*

Proof: By Theorem 5.7 of [6] every locally nilpotent section of G is soluble. Let H_1 be the locally nilpotent radical of $G, H_2/H_1$ the locally nilpotent radical of G/H_1, and let T denote the maximal normal torsion subgroup of G. By Lemma 1 T is soluble and, factoring, we may assume that $T = 1$. Let $B = B(G)$ denote the Baer radical of G; then $B = B(H_2)$. If B is not minimax then $H_2 = B$, by Theorem 1, and so $G = H_1$, which is soluble. But if B is minimax then every abelian subgroup of G is minimax, by Corollary 2.3 of [6], and then G is minimax, by Theorem 10.35 of [8], and hence soluble. This establishes the result. $\square$

Lemma 3. *Let G be a finitely generated soluble-by-finite group satisfying $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. Then G is minimax.*

Proof: We may suppose that G is soluble. If G is not minimax then it has infinite (Prüfer) rank, by Theorem 10.38 of [8], and hence a section K isomorphic to the wreath product of a cyclic group of prime order by an infinite cyclic group, by [3]. But the Baer radical of K is abelian of prime exponent and clearly does not satisfy $\min -K$, and Theorem 1(iii) gives a contradiction. $\square$

Our next result reduces the proof of Theorem 2 to that of the locally soluble case. This is followed by two straightforward lemmas, the first of which is well-known. These will assist in establishing a Proposition that, in turn, provides a key step in the final stage of the proof of the theorem. In the statement of Lemma 6 (and elsewhere) we employ for convenience the term "periodic index" to indicate that a subgroup H of a group G has the property that every element of G has a non-zero power in H.

Lemma 4. *Suppose that every locally soluble group with non-subnormal deviation at most* 1 *is soluble. Then every locally soluble-by-finite group with non-subnormal deviation at most* 1 *is soluble-by-finite.*

PROOF: Let G be a locally soluble-by-finite group with $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ and suppose that G is not soluble-by-finite. If there are integers d and k such that every finitely generated subgroup of G has its soluble radical of derived length at most d and index at most k then G too has this property, by Proposition 1.K.2 of [2]. Thus we may assume that G is countable and, by Lemma 2, we may further assume (by factoring if necessary) that G has trivial Hirsch-Plotkin radical. G is an ascending union of finitely generated subgroups $F_1 < F_2 < \dots$; for each i the soluble radical S_i of F_i has finite index, and it is easy to see that the subgroup S generated by all the S_i is locally soluble and hence, by hypothesis, soluble. Since G has trivial Baer radical we deduce from Corollary 2.3 of [6] that every abelian subgroup of G is minimax and hence that S is minimax, by Theorem 10.35 of [8]. It follows that there is a finite upper bound r for the torsion-free ranks of the subgroups F_i. By Lemma 3 each F_i is minimax and hence nilpotent-by-abelian-by-finite (by Theorem 3.25 of [8]), and we may apply Proposition 1 of [1] to deduce that G has a finite ascending normal series whose factors are soluble or locally finite. But each locally finite factor of this series is soluble-by-finite, by Lemma 1, and since each finite G-invariant factor of G has its centralizer of finite index in G we easily obtain the contradiction that G is soluble-by-finite. $\square$

Lemma 5. *Let X be a finitely generated nilpotent-by-finite group, Y a subgroup of X, and let U be a normal subgroup of finite index in Y. Then there is a normal subgroup N of finite index in X such that $N \cap Y \leq U$.*

PROOF: There is a normal nilpotent subgroup V of finite index in X, and if $M \cap (Y \cap V) \leq U \cap V$ for some normal subgroup M of finite index in V then, with $N = \mathrm{Core}_X(M)$, we have X/N finite and $N \cap Y = N \cap (Y \cap V) \leq (U \cap V) \leq U$, and the result follows. Thus we may suppose that $X = V$ and in particular that Y is subnormal in X. If Y is normal in X then $Y/\mathrm{Core}_X(U)$ is finite, and since $X/\mathrm{Core}_X(U)$ is residually finite the result follows in this case. By induction on the subnormal defect of Y in X we may assume that there is a normal subgroup W of X containing Y, and a normal subgroup K of finite index in W, such that $K \cap Y \leq U$; replacing K by its core if necessary we may assume that K is normal in X. Since X/K is residually finite there is a normal subgroup N of finite index in X such that $N \cap W = K$, and then $N \cap Y = N \cap W \cap Y = K \cap Y \leq U$, and the lemma is proved. □

Lemma 6. *Let G be a locally soluble group with* $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ *and suppose that H is a subnormal subgroup of G that has periodic index in G. Then there is a positive integer d such that $G^{(d)} \leq H$ (where $G^{(d)}$ denotes the dth term of the derived series of G).*

Proof: There is a finite subnormal series from H to G, and each factor of this series is locally finite and hence soluble, by Lemma 1. □

The following Proposition, with its rather technical hypotheses, will allow us to deal, in particular, with the case where G is locally nilpotent-by-finite, and its generality will also allow us to reduce to that case. Much of the proof of the Proposition is similar to that of Theorem 4 of [1], though there are significant differences.

Proposition. *Let G be a locally soluble group that is the ascending union of a chain of finitely generated subgroups $F_1 < F_2 < \dots$ and, for each i, let K_i be a normal subgroup of F_i such that F_i/K_i is nilpotent-by-finite, F_1/K_1 and each index $[F_{i+1} : F_iK_{i+1}]$ is infinite, and $K_{i+1} \cap F_i \leq K_i$. Suppose also that F_i has derived length exactly d_i and that F_{i+1}/K_{i+1} has derived length at least $2d_i$ (for each i). Then there are subgroups M and L of G such that $L \leq M, L$ has periodic index in G and $G^{(d)} \not\leq M$ for any positive integer d, and there is an infinite descending chain of subgroups of M that contain L.*

Proof: There is a nilpotent normal subgroup M_1/K_1 of finite index in F_1/K_1 and a proper F_1-invariant subgroup L_1/K_1 of finite index in M_1/K_1. By Lemma 5, applied to the group F_2/K_2, there is a normal subgroup M_2/K_2 of finite index in F_2/K_2 such that $M_2 \cap F_1 \leq K_2L_1$ and so $M_2 \cap F_1 \leq K_2L_1 \cap F_1 = L_1(K_2 \cap F_1) = L_1$. Clearly we may choose M_2 such that M_2/K_2 is nilpotent and F_2/M_2 has derived length equal to that of F_2/K_2 (using the residual finiteness of F_2/K_2). Since F_2/K_2 is polycyclic, the intersection of all subgroups U of finite index in F_2 that contain K_2F_1 is K_2F_1, which has infinite index in F_2. So we may choose such a U that does not contain M_2, and then $L_2 := \mathrm{Core}_{F_2}(U \cap M_2)$ contains K_2, has finite index in M_2, and is such that M_2 is not contained in F_1L_2. We continue in this manner: for each i there is a normal nilpotent subgroup M_{i+1}/K_{i+1} of finite index in F_{i+1}/K_{i+1} such that $M_{i+1} \cap F_i \leq L_i$ and F_{i+1}/M_{i+1} has derived length equal to that of F_{i+1}/K_{i+1}, and an F_{i+1}-invariant subgroup L_{i+1}/K_{i+1} of finite index in M_{i+1}/K_{i+1} such that $M_{i+1} \not\leq F_iL_{i+1}$. Let $L = \langle L_i : i \geq 1\rangle, M = \langle M_i : i \geq 1\rangle$; clearly L has periodic index in G and $L \leq M$.

Suppose next that $G^{(d)} \leq M$ for some integer d, and choose $i > 1$ such that $d_{i-1} > d$. We have $F_i^{(d)} \leq M \cap F_i$, and since $M_{j+1} \cap F_j \leq M_j$ for all j it follows that $M \cap F_i = M_1 \dots M_i$ (see Lemma 2 of [1]) and hence that $F_i^{(d)} \leq F_{i-1}M_i$.

But this gives $F_i^{(d+d_{i-1})} \leq M_i$, so $d_i \leq (d + d_{i-1}) < 2d_{i-1}$, a contradiction.

Now let $H_0 = M$ and, for $i \geq 1$, let $H_i = (L_1 \dots L_i)(M_{i+1}M_{i+2} \dots)$; thus $L \leq H_i \leq M$ for all i. Assume for a contradiction that the chain $H_0 \geq H_1 \geq H_2 \geq \dots$ terminates after finitely many steps, so that $M_t \leq (L_1 \dots L_t)(M_{t+1}M_{t+2} \dots)$ for some t. Now if $x \in (M_{t+1} \dots M_{t+k}) \cap F_t$ for some $k > 1$ then

$$x \in (M_{t+1} \dots M_{t+k-1})(M_{t+k} \cap F_{t+k-1}) \cap F_t = (M_{t+1} \dots M_{t+k-1}) \cap F_t,$$

and an easy induction on k shows that $x \in M_{t+1} \cap F_t \leq L_t$. Thus

$$\begin{aligned} M_t \leq (L_1 \dots L_t)(M_{t+1}M_{t+2} \dots) \cap F_t &= (L_1 \dots L_t)((M_{t+1}M_{t+2} \dots) \cap F_t) \\ &= (L_1 \dots L_t) \leq F_{t-1}L_t, \end{aligned}$$

contradicting the choice of L_t. Thus the H_i form an infinite descending chain, and the Proposition is proved. □

Proof of Theorem 2: By Lemma 4 we may assume that G is locally soluble, and our aim is to show that G is soluble, so we may also assume that G is countable. Suppose the result false. The iterated Hirsch-Plotkin radical of G is soluble, by Lemma 2, so we may factor and assume that G has no nontrivial subnormal soluble subgroups; as in the proof of Lemma 4, every soluble subgroup of G is therefore minimax. If every insoluble subgroup of G is subnormal then G satisfies the weak minimal condition for non-subnormal subgroups (since soluble minimax groups satisfy the weak minimal condition for *all* subgroups), and Theorem B of [5] implies that either G is soluble or G has all subgroups subnormal, in which case G is again soluble [7], a contradiction. Thus we see that every insoluble subgroup of G contains a non-subnormal insoluble subgroup, and since $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ there exists an insoluble subgroup G_0 of G such that, if $X < Y$ are any non-subnormal insoluble subgroups of G_0 then the interval $I[X, Y]$ satisfies the minimal condition for non-subnormal subgroups of G. We may assume that $G = G_0$ and, again factoring if necessary, we may retain the condition that the Hirsch-Plotkin radical of G is trivial and hence that every soluble subgroup is minimax.

By Lemma 3, G is the ascending union of finitely generated subgroups $F_1 < F_2 < \dots$, each of which is minimax. For each i, let K_i denote the locally nilpotent radical of F_i; then K_i is nilpotent and F_i/K_i is abelian-by-finite, by Theorem 3.25 of [8]. Certainly $F_i \cap K_{i+1} \leq K_i$ for each i. If there is an upper bound b for the derived lengths of the factors F_i/K_i then $G^{(b)}$ is locally nilpotent and therefore trivial, and we have the contradiction that G is soluble. Passing to an appropriate subsequence of the F_i if necessary, we may therefore assume that the derived length of F_{i+1}/K_{i+1} is at least twice that of F_i, for each i.

If there is no upper bound for the torsion-free ranks (that is, Hirsch lengths) of the F_i/K_i then we may again re-label the F_i and assume that the torsion-free rank of F_{i+1}/K_{i+1} is greater than that of F_i for each i and that F_1/K_1 is infinite. But now G satisfies the hypotheses and hence the conclusion of the Proposition and, with the notation of the Proposition, we may apply Lemma 6 to deduce that neither L nor any of the subgroups H_i is subnormal in G, so the interval $I[L, M]$ does not satisfy min for non-subnormal subgroups of G, so that L must be soluble and hence minimax, which gives, since L has periodic index in G, the contradiction that the torsion-free ranks of the F_i are bounded. It follows that the torsion-free rank of each F_{i+1}/K_{i+1} is at most r, for some fixed positive integer r.

For each i there is a normal torsion-free abelian subgroup U_i/K_i of finite index in F_i/K_i, and the rank of each U_i/K_i is at most r. By Theorem 3.23 of [8], there is an integer $d = d(r)$ such that (for all i), if C_i denotes the centralizer in F_i of U_i/K_i, then F_i/C_i has derived length at most d. As C_i/K_i is centre-by-finite it has finite derived group, and so $F_i^{(d+1)}$ is nilpotent-by-finite, which gives $G^{(d+1)}$ locally nilpotent-by-finite, so there is no loss in assuming that G is locally nilpotent-by-finite, that is, F_i/K_i is finite for each i. Observe that, by factoring by the iterated Hirsch-Plotkin radical if necessary, we may again retain the condition that all soluble subgroups of G are minimax.

If the torsion-free ranks of the F_i are bounded above then, as in the proof of Lemma 4, we may apply Proposition 1 of [1] to deduce that G has a finite ascending normal series whose factors are soluble or locally finite and hence to obtain the contradiction that G is soluble. Thus we may assume that the torsion-free ranks of the F_i increase with i. But now G satisfies the hypotheses of the Proposition, with each "K_i" of those hypotheses trivial, and as in the earlier part of the present proof we have subgroups L and M and an infinite descending chain of subgroups H_i of M, each containing L, such that no H_i is subnormal in G. Thus L is soluble and hence minimax and, since L has periodic index in G, we obtain the contradiction that the torsion-free ranks of the F_i are bounded. This completes the proof of Theorem 2. □

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