Club-guessing, good points and diamond

Pierre Matet

Abstract. Shelah's club-guessing and good points are used to show that the two-cardinal diamond principle $\diamondsuit_{\kappa,\lambda}$ holds for various values of κ and λ.

Keywords: $P_\kappa(\lambda)$, diamond principle

Classification: 03E05

Let κ be a regular cardinal greater than or equal to ω_2, and λ be a cardinal greater than κ. In [4] Jech introduced the following notions. $P_\kappa(\lambda)$ denotes the collection of all subsets of λ of size less than κ. A subset C of $P_\kappa(\lambda)$ is *closed unbounded* if (a) it is cofinal in the partially ordered set $(P_\kappa(\lambda), \subset)$, and (b) for any infinite ordinal $\theta < \kappa$ and any sequence $\langle a_\alpha : \alpha < \theta \rangle$ of elements of C such that $a_\beta \subset a_\alpha$ whenever $\beta < \alpha < \theta$, $\bigcup_{\alpha<\theta} a_\alpha \in C$. A subset S of $P_\kappa(\lambda)$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded subset C of $P_\kappa(\lambda)$. The principle $\diamondsuit_{\kappa,\lambda}$ asserts the existence of a sequence $\langle s_a : a \in P_\kappa(\lambda) \rangle$ with $s_a \subseteq a$ such that for any $X \subseteq \lambda$, $\{a : s_a = X \cap a\}$ is a stationary subset of $P_\kappa(\lambda)$. Jech showed that $\diamondsuit_{\kappa,\lambda}$ could be introduced by forcing. Moreover, he proved that $\diamondsuit_{\kappa,\lambda}$ holds in the constructible universe L. It was shown in [2] that if $2^{<\kappa} < \lambda$, then $\diamondsuit_{\kappa,\lambda}$ holds. In this paper we show that if $2^{<\kappa} \leq \mu^+$ for some cardinal μ such that $\omega < \mathrm{cf}(\mu) < \kappa < \mu \leq \lambda$, then $\diamondsuit_{\kappa,\lambda}$ holds. So if either $2^{<\kappa} = \lambda$ and λ is the successor of a cardinal of uncountable cofinality less than κ, or $2^{<\kappa} = \lambda^+$ and $\omega < \mathrm{cf}(\lambda) < \kappa$, then $\diamondsuit_{\kappa,\lambda}$ holds (and hence the nonstationary ideal on $P_\kappa(\lambda)$ is not 2^λ-saturated). Our result is proved by modifying the argument used by Foreman and Magidor in [3] to show that if $\mathrm{cf}(\lambda) < \kappa$, then there is a family of λ^{++} stationary subsets of $P_\kappa(\lambda)$ such that any two of them have nonstationary intersection.

We need a few lemmas.

Lemma 1 (Solovay [9]). *Let ρ be a regular uncountable cardinal. Then every stationary subset of ρ is the union of ρ disjoint stationary sets.*

Given two regular infinite cardinals $\theta < \rho$, E^ρ_θ denotes the set of all infinite limit ordinals $\alpha < \rho$ such that $\mathrm{cf}(\alpha) = \theta$.

Lemma 2 (Shelah see [5])**.** *Let $\rho > \omega_1$ be a regular cardinal, and S be a stationary subset of E^ρ_ω. Then one can find a cofinal, order-type ω subset c_γ of γ for each $\gamma \in S$ so that $\{\gamma \in S : c_\gamma \subseteq C\}$ is stationary in ρ for any closed unbounded subset C of ρ.*

Our source for the following notions and facts is [1]. Let $\mu > \kappa$ be a singular cardinal of uncountable cofinality $\nu < \kappa$. Suppose $\langle \mu_i : i < \nu \rangle$ is an increasing sequence of regular cardinals such that $\kappa < \mu_0$ and $\sup\{\mu_i : i < \nu\} = \mu$. Given $f, g \in \prod_{i<\nu} \mu_i$, $f <^* g$ means that $|\{i : f(i) \geq (i)\}| < \nu$. Similarly, $f \leq^* g$ means that $|\{i : f(i) > g(i)\}| < \nu$. By a *scale of length* ξ, ξ an ordinal, we mean a sequence $\langle f_\alpha : \alpha < \xi \rangle$ of elements of $\prod_{i<\nu} \mu_i$ such that (a) $f_\beta <^* f_\alpha$ whenever $\beta < \alpha < \xi$, and (b) for every $g \in \prod_{i<\nu} \mu_i$, there is $\alpha < \xi$ with $g <^* f_\alpha$. Shelah proved that the μ_i can be chosen so that there exists a scale of length μ^+. Let $\langle f_\alpha : \alpha < \mu^+ \rangle$ be such a scale. Given an infinite limit ordinal $\alpha < \mu^+$, an *exact upper bound* for the sequence $\langle f_\beta : \beta < \alpha \rangle$ is an element g of $\prod_{i<\nu} \mu_i$ such that (i) $f_\beta <^* g$ for any $\beta < \alpha$, and (ii) for every $h \in \prod_{i<\nu} \mu_i$ with $h <^* g$, there is $\beta < \alpha$ with $h <^* f_\beta$. By a *good point*, an infinite limit ordinal $\alpha < \mu^+$ is meant such that $\mathrm{cf}(\alpha) > \nu$ and there exists an exact upper bound g_α for $\langle f_\beta : \beta < \alpha \rangle$ with the property that for any $i < \nu$, $g_\alpha(i)$ is an infinite limit ordinal of cofinality $\mathrm{cf}(\alpha)$. Letting S denote the set of good points α such that $\mathrm{cf}(\alpha) = \kappa$, S is stationary in μ^+. Now consider the sequence $\langle h_\alpha : \alpha < \mu^+ \rangle$ defined by: $h_\alpha = f_\alpha$ if $\alpha \notin S$, and $h_\alpha = g_\alpha$ otherwise. Then $\langle h_\alpha : \alpha < \mu^+ \rangle$ is a scale. Moreover, for each $\alpha \in S$, h_α is an exact upper bound for $\langle h_\beta : \beta < \alpha \rangle$. Let us sum it up all in the following.

Lemma 3 (Shelah see [1])**.** *Let $\mu > \kappa$ be a singular cardinal of uncountable cofinality $\nu < \kappa$. Then one can find sequences $\langle \mu_i : i < \nu \rangle$ and $\langle h_\alpha : \alpha < \mu^+ \rangle$ and a set S such that* (a) *$\langle \mu_i : i < \nu \rangle$ is an increasing sequence of regular cardinals such that $\kappa < \mu_0$ and $\sup\{\mu_i : i < \nu\} = \mu$,* (b) *$\langle h_\alpha : \alpha < \mu^+ \rangle$ is a scale of length μ^+ in $\prod_{i<\nu} \mu_i$,* (c) *S is a stationary subset of $E^{\mu^+}_\kappa$, and* (d) *for each $\alpha \in S$, $\mathrm{ran}(h_\alpha) \subseteq E^{\mu^+}_\kappa$ and h_α is an exact upper bound for $\langle h_\beta : \beta < \alpha \rangle$.*

Suppose μ is a cardinal greater than κ. For $n < \omega$, let R^μ_n be the set of all increasing functions from n to $E^{\mu^+}_\kappa$. Let $\mathcal{T}_\mu$ be the collection of all nonempty subsets T of $\bigcup_{n<\omega} R^\mu_n$ such that for any $n < \omega$ and any $t \in T \cap R^\mu_n$, $\{t \restriction \ell : \ell < n\} \subseteq T$ and $\{\alpha \in E^{\mu^+}_\kappa : t \cup \{n, \alpha\} \in T\}$ is stationary in μ^+.

Lemma 4 (Shioya [8])**.** *Suppose that $\mu > \kappa$ is a cardinal, $T \in \mathcal{T}_\mu, n < \omega$ and $\varphi : T \cap (\bigcup_{n<q<\omega} R^\mu_q) \to \mu^+$ is such that for every $t \in \mathrm{dom}(\varphi)$,* (a) *$\varphi(t) \in t(n)$, and* (b) *$\varphi(t \restriction q) \leq \varphi(t)$ for $n < q < \mathrm{dom}(t)$. Then one can find $T' \in \mathcal{T}_\mu \cap P(T)$ and $\psi : T \cap R^\mu_n \to \mu^+$ so that* (i) *$T' \cap R^\mu_n = T \cap R^\mu_n$ and* (ii) *$\varphi(t) \leq \psi(t \restriction n)$ for any $t \in T' \cap (\bigcup_{n<q<\omega} R^\mu_q)$.*

For $A \subseteq P_\kappa(\lambda), G_{\kappa,\lambda}(A)$ denotes the following two-person game lasting ω moves. Player I makes the first move. I and II alternately pick members of $P_\kappa(\lambda)$, thus building a sequence $\langle a_n : n < \omega\rangle$ with the condition that $a_0 \subseteq a_1 \subseteq \dots$ II wins the game just in case $\bigcup_{n<\omega} a_n \in A$. Let $NG_{\kappa,\lambda}$ be the set of all $B \subseteq P_\kappa(\lambda)$ such that II has a winning strategy in $G_{\kappa,\lambda}(P_\kappa(\lambda) \setminus B)$.

Lemma 5 (Matet [6]). *$NG_{\kappa,\lambda}$ is a normal ideal on $P_\kappa(\lambda)$.*

Proposition 6. *Suppose $2^{<\kappa} \le \mu^+$ for some cardinal μ such that $\kappa < \mu \le \lambda$ and $\omega < \mathrm{cf}(\mu) < \kappa$. Then there is a sequence $\langle s_a : a \in P_\kappa(\lambda)\rangle$ with $s_a \subseteq a$ such that for any $X \subseteq \lambda, \{a : s_a = X \cap a\} \in NG^+_{\kappa,\lambda}$.*

Proof: Let μ be a fixed cardinal such that $\omega < \mathrm{cf}(\mu) < \kappa < \mu \le \lambda$ and $2^{<\kappa} \le \mu^+$. Fix a stationary subset H of $E^{\mu^+}_\omega$. Using Lemma 2, select an increasing function $\overline{\gamma}$ from ω into γ for each $\gamma \in H$ so that $\mathrm{ran}(\overline{\gamma})$ is cofinal in γ for every $\gamma \in H$, and $\{\gamma \in H : \mathrm{ran}(\overline{\gamma}) \subseteq C\}$ is stationary in μ^+ for any closed unbounded subset C of μ^+.

Set $\nu = \mathrm{cf}(\mu)$ and let $\langle \mu_i : i < \nu\rangle$, $\langle h_\alpha : \alpha < \mu^+\rangle$ and S be as in the statement of Lemma 3. For $\alpha < \mu^+$, set $x_\alpha = \mathrm{ran}(h_\alpha)$. For $b \in P_\kappa(\mu)$, define $g_b \in \prod_{i<\nu} \mu_i$ by $g_b(i) = \sup(b \cap \mu_i)$. Define $\rho : P_\kappa(\mu) \to \mu^+$ by $\rho(b) =$ the least $\beta < \mu^+$ such that $g_b \le^* h_\beta$. For $\alpha \in S$ and $b \in P_\kappa(\mu)$, define $g^\alpha_b \in \prod_{i<\nu} h_\alpha(i)$ by $g^\alpha_b(i) = \sup(b \cap h_\alpha(i))$. For $\alpha \in S$, define $\rho_\alpha : P_\kappa(\mu) \to \alpha$ by $\rho_\alpha(b) =$ the least $\beta < \mu^+$ such that $g^\alpha_b \le^* h_\beta$. Note that given any sequence $\langle b_n : n < \omega\rangle$ of elements of $P_\kappa(\mu), \rho(\bigcup_{n<\omega} b_n) = \sup\{\rho(b_n) : n < \omega\}$. Moreover, for every $\alpha \in S$, $\rho_\alpha(\bigcup_{n<\omega} b_n) = \sup\{\rho_\alpha(b_n) : n < \omega\}$.

We will prove that there is a sequence $\langle s_a : a \in P_\kappa(\lambda)\rangle$ such that for any $X \subseteq \lambda$,

$$\{a : \rho(\mu \cap a) \in H \ \text{ and } \ s_a = X \cap a\} \in NG^+_{\kappa,\lambda}.$$

For $n < \omega$ and $0 < \zeta < \kappa$, let F^ζ_n be the set of all $(n+1)$-tuples $(f_0, \dots, f_n)$ of functions from ζ to 2. By Lemma 1, S can be partitioned into disjoint stationary subsets Z_n, $n < \omega$. Again by Lemma 1, for each n, Z_n can be decomposed into disjoint stationary subsets $Z(f_0, \dots, f_n)$, $(f_0, \dots, f_n) \in \bigcup_{0<\zeta<\kappa} F^\zeta_n$.

For $b \subseteq \lambda$, let $e(b) : \mathrm{o.t.}\,(b) \to b$ be the function that enumerates the elements of b in increasing order. For $a, b \in P_\kappa(\lambda)$ with $a \subseteq b$, let $\chi(a, b) : \mathrm{o.t.}\,(b) \to 2$ be defined by $(\chi(a, b))(\delta) = 1$ if and only if $(e(b))(\delta) \in a$.

The proof will go as follows. Given $A \in NG^*_{\kappa,\lambda}$ and $X \subseteq \lambda$, we will construct a_n and α_n for $n < \omega$, f^i_n for $i \le n < \omega$, and a and γ so that (a) $a_0, a_1, \dots \in P_\kappa(\lambda)$ and $a_0 \subseteq a_1 \subseteq \dots$, (b) $f^i_n = \chi(a_i, a_n)$ for $i < n$ and $f^n_n = \chi(X \cap a_n, a_n)$, (c) $a = \bigcup_{n<\omega} a_n, a \in A$ and $\gamma = \rho(a \cap \mu)$, (d) $\alpha_{n+1} =$ the least α such that $\overline{\gamma}(n) < \alpha < \mu^+$ and $\mathrm{ran}(h_\alpha) \subseteq a$, and (e) $\alpha_{n+1} \in Z(f^0_n, f^1_n, \dots, f^n_n)$.

The guessing sequence $\langle s_a : a \in P_\kappa(\lambda)\rangle$ is now defined in the obvious way. Given $a \in P_\kappa(\lambda)$, put $\xi = \text{o.t.}\,(\text{a})$ and $\gamma = \rho(a \cap \mu)$. Let $(*)$ assert that $\gamma \in H$ and there exist ζ_n and α_{n+1} for $n < \omega$ and f_n^i for $i \le n < \omega$ such that (0) $0 < \zeta_n < \kappa$, (1) f_n^i is a function from ζ_n to 2, (2) $\alpha_{n+1} =$ the least $\alpha < \mu^+$ such that $\alpha > \overline{\gamma}(n)$ and $\text{ran}(h_\alpha) \subseteq a$, and (3) $\alpha_{n+1} \in Z(f_n^0, f_n^1, \ldots, f_n^n)$. s_a can be any subset of a if $(*)$ does not hold. Now suppose that $(*)$ holds. By induction on θ, define a_n^θ for $\theta < \xi$ and $n < \omega$ as follows. Put $a_n^0 = \phi$ for every $n < \omega$. If θ is an infinite limit ordinal, set $a_n^\theta = \bigcup_{\eta<\theta} a_n^\eta$ for all $n < \omega$. Assuming a_n^θ has been defined for all n, look for a $j < \omega$ such that (α) for $j < n < \omega, \text{o.t.}\,(a_n^\theta) \in \text{dom}(f_n^j)$ and $f_n^j(\text{o.t.}\,(a_n^\theta)) = 1$, and (β) for $\ell < j \le n < \omega, \text{o.t.}\,(a_n^\theta) \in \text{dom}(f_n^\ell)$ and $f_n^\ell(\text{o.t.}\,(a_n^\theta)) = 0$. If there is no such j, set $a_n^{\theta+1} = a_n^\theta$ for every $n < \omega$. If there is one, it must be unique. Set $a_n^{\theta+1} = a_n^\theta$ for $n < j$, and $a_n^{\theta+1} = a_n^\theta \cup \{(e(a))(\theta)\}$ for $j \le n < \omega$. Finally, set $s_a = \bigcup_{n<\omega} s_n$, where $s_n = \{(e(a_n))(\eta) : \eta \in \text{dom}(f_n^n) \cap \text{o.t.}\,(a_n) \text{ and } f_n^n(\eta) = 1\}$.

Now fix $A \in NG^*_{\kappa,\lambda}$ and $X \subseteq \lambda$. We must find $a \in A$ such that $s_a = X \cap a$. Let τ be a winning strategy for player II in the game $G_{\kappa,\lambda}(A)$. Define $k : \bigcup_{n<\omega} R^\mu_{n+1} \to P_\kappa(\lambda)$ as follows. Set $k(t) = \tau(x_{t(0)})$ for any $t \in R^\mu_1$. Given $0 < n < \omega$ and $t \in R^\mu_{n+1}$, define a_m and b_m for $m \le n$ by: $a_0 = x_{t(0)}$, $a_m = b_{m-1} \cup x_{t(m)}$ for $m > 0$, and $b_m = \tau(a_0, \ldots, a_m)$, and set $k(t) = b_n$.

Define W_n for $n < \omega$ by induction as follows. Set $W_0 = R^\mu_0$, $W_1 = R^\mu_1$ and

$$W_2 = \left\{t \in R^\mu_2 : t(1) \in Z(\chi(X \cap k(t \restriction 1), k(t \restriction 1))\right\}.$$

For $n \ge 2$, let W_{n+1} be the set of all $t \in R^\mu_{n+1}$ such that $t \restriction n \in W_n$ and $t(n)$ belongs to $Z(f_0, \ldots, f_{n-1})$, where $f_i = \chi(k(t \restriction (i+1)), k(t \restriction n))$ for $i < n-1$, and $f_{n-1} = \chi(X \cap k(t \restriction n)), k(t \restriction n))$. Put $T_0 = \bigcup_{n<\omega} W_n$. For $0 < r < \omega$, define $\varphi_r : T_0 \cap (\bigcup_{r<q<\omega} R^\mu_q) \to \mu^+$ by $\varphi_r(t) = \rho_{t(r)}(\mu \cap k(t))$. Using Lemma 4, select $T_r \in \mathcal{T}_\mu$ and $\psi_r : T_r \cap R^\mu_r \to \mu^+$ for $0 < r < \omega$ so that $T_r \subseteq T_{r-1}, T_r \cap R^\mu_r = T_{r-1} \cap R^\mu_r$, and $\varphi_r(t) \le \psi_r(t \restriction r)$ for every $t \in T_r \cap (\bigcup_{r<q<\omega} R^\mu_q)$. Set $T = \bigcap_{r<\omega} T_r$.

Let C be the set of all γ with $\kappa < \gamma < \mu^+$ such that for any r with $0 < r < \omega$, and any $t \in T \cap R^\mu_r$ with $\text{ran}(t) \subseteq \gamma, \rho(\mu \cap k(t)) < \gamma, \psi_r(t) < \gamma$ and $\{\alpha < \gamma : t \cup \{(r, \alpha)\} \in T\}$ is cofinal in γ. Since C is a closed unbounded subset of μ^+, there is $\gamma \in H$ such that $\text{ran}(\overline{\gamma}) \subseteq C$. Pick $y : \omega \to \mu^+$ so that $\{y \restriction m : m < \omega\} \subseteq T$ and $y(0) < \overline{\gamma}(0) < y(1) < \overline{\gamma}(1) < \ldots$. Set $a_n = k(y \restriction (n+1))$ for $n < \omega$, and $a = \bigcup_{n<\omega} a_n$. Then for each $m < \omega$,

$$\overline{\gamma}(m) < y(m+1) \le \rho(\mu \cap a_{m+1}) < \overline{\gamma}(m+1)$$

since $x_{y(m+1)} \subseteq a_{m+1}$ and $\text{ran}(y \restriction (m+2)) \subseteq \overline{\gamma}(m+1)$. Hence

$$\rho(\mu \cap a) = \sup\{\rho(\mu \cap a_{m+1}) : m < \omega\} = \gamma.$$

For $0 < r < \omega$,

$$\rho_{y(r)}(\mu \cap a) = \sup\{\rho_{y(r)}(\mu \cap a_q) : r < q < \omega\} \leq \overline{\gamma}(r-1)$$

since $\rho_{y(r)}(\mu \cap a_q) = \varphi_r(y \restriction q) \leq \psi_r(y \restriction r) < \overline{\gamma}(r-1)$ whenever $r < q < \omega$. It follows that $y(r) =$ the least $\alpha < \mu^+$ such that $\alpha > \overline{\gamma}(r-1)$ and $x_\alpha \subseteq a$, since $\delta \leq \rho_{y(r)}(a \cap \mu)$ for any $\delta < y(r)$ such that $x_\delta \subseteq a$. Define f_n^j for $j \leq n < \omega$ by $f_n^j = \chi(a_j, a_n)$ if $j < n$, and $f_n^n = \chi(X \cap a_n, a_n)$. Then $y(n+1) \in Z(f_n^0, f_n^1, \dots, f_n^n)$ for all $n < \omega$. Finally, $s_a = \bigcup_{n<\omega}\{(e(a_n))(\eta) : \eta \in \text{o.t.}(a_n)$ and $f_n^n(\eta) = 1\} = \bigcup_{n<\omega}(X \cap a_n) = X \cap a$. □

In the case when κ is the successor of a cardinal of cofinality ω, the assumption of Proposition 6 can be weakened.

Let $\nu > 0$ be a cardinal. For $A \subseteq P_\kappa(\lambda)$, the game $G^\nu_{\kappa,\lambda}(A)$ is defined similarly to $G_{\kappa,\lambda}(A)$, where now the choices are made from $P_\nu(\lambda)$.

Lemma 7 (Matet [7]). *Suppose κ is the successor of a cardinal ν of cofinality ω. Then for any $A \subseteq P_\kappa(\lambda)$, $A \in NG^*_{\kappa,\lambda}$ if and only if II has a winning strategy in the game $G^\nu_{\kappa,\lambda}(A)$.*

It is now straightforward to modify the proof of Proposition 6 so as to get the following.

Proposition 8. *Suppose that κ is the successor of a cardinal ν of cofinality ω, and $2^{<\nu} \leq \mu^+$ for some cardinal μ such that $\omega < \text{cf}(\mu) < \kappa < \mu \leq \lambda$. Then there is a sequence $< s_a : a \in P_\kappa(\lambda) >$ with $s_a \subseteq a$ such that for any $X \subseteq \lambda$, $\{a : s_a = X \cap a\} \in NG^+_{\kappa,\lambda}$.*

References

[1] Cummings J., Foreman M., Magidor M., *Canonical structure in the universe of set theory, part one*, Ann. Pure Appl. Logic **129** (2004), 211–243.

[2] Donder H.-D., Matet P., *Two cardinal versions of diamond*, Israel J. Math. **83** (1993), 1–43.

[3] Foreman M., Magidor M., *Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $\mathcal{P}_\kappa(\lambda)$*, Acta Math. **186** (2001), 271–300.

[4] Jech T.J., *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic **5** (1973), 165–198.

[5] Kojman M., *The A, B, C of pcf: a companion to pcf theory, part I*, 1995, unpublished.

[6] Matet P., *Concerning stationary subsets of $[\lambda]^{<\kappa}$*, in: Set Theory and its Applications (J. Steprāns and S. Watson, eds.), Lecture Notes in Mathematics 1401, Springer, Berlin, 1989, pp. 119–127.

[7] Matet P., *Game ideals*, preprint.

[8] Shioya M., *Splitting $\mathcal{P}_\kappa\lambda$ into maximally many stationary sets*, Israel J. Math. **114** (1999), 347-357.

[9] Solovay R.M., *Real-valued measurable cardinals*, in: Axiomatic Set Theory (D.S. Scott, ed.), Proceedings of Symposia in Pure Mathematics, vol. 13, part 1, American Mathematical Society, Providence, 1971, pp. 397–428.

Université de Caen - CNRS, Laboratoire de Mathématiques, BP 5186,
14032 Caen Cedex, France

E-mail: matet@math.unicaen.fr

(*Received* May 9, 2006, *revised* February 1, 2007)