# Covering $\mathbf{\Sigma}^0_\xi$-generated ideals by $\mathbf{\Pi}^0_\xi$ sets

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*Abstract.* We develop the theory of topological Hurewicz test pairs: a concept which allows us to distinguish the classes of the Borel hierarchy by Baire category in a suitable topology. As an application we show that for every $\mathbf{\Pi}^0_\xi$ and not $\mathbf{\Sigma}^0_\xi$ subset $P$ of a Polish space $X$ there is a $\sigma$-ideal $\mathcal{I} \subseteq 2^X$ such that $P \notin \mathcal{I}$ but for every $\mathbf{\Sigma}^0_\xi$ set $B \subseteq P$ there is a $\mathbf{\Pi}^0_\xi$ set $B' \subseteq P$ satisfying $B \subseteq B' \in \mathcal{I}$. We also discuss several other results and problems related to ideal generation and Hurewicz test pairs.

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## 1. Introduction

Let $(X,\tau)$ be an uncountable Polish space. For every $0 < \xi < \omega_1$ and $P \subseteq X$ let $\mathcal{S}^0_\xi(P)$ and $\mathcal{P}^0_\xi(P)$ denote the collections of $\mathbf{\Sigma}^0_\xi(\tau)$ and $\mathbf{\Pi}^0_\xi(\tau)$ subsets of $P$. In this paper we prove in particular the following result.

**Theorem 1.** *Let $\xi$ be a successor ordinal such that $1 < \xi < \omega_1$. Let $(X,\tau)$ be an uncountable Polish space and $P \subseteq X$ be a $\mathbf{\Pi}^0_\xi(\tau)$ and not $\mathbf{\Sigma}^0_\xi(\tau)$ set. Then there is a mapping $\Phi\colon \mathcal{S}^0_\xi(P) \to \mathcal{P}^0_\xi(P)$ such that for every $B, B^i \in \mathcal{S}^0_\xi(P)$ $(i < \omega)$,*

$$B \subseteq \Phi(B) \quad \text{and} \quad P \setminus \bigcup_{i<\omega} \Phi(B^i) \neq \emptyset.$$

At first sight this result may seem not to be informative so we discuss why the effort is made for it. Our first motivation is the effort itself: the proof is based on the concept of *topological Hurewicz test pairs*. The main innovation in this method is that for every (nice) $\mathbf{\Pi}^0_\xi(\tau)$ set $P \subseteq X$ we are able to construct a Polish topology $\tau_P$ on $X$ such that it is true in particular that every $\mathbf{\Sigma}^0_\xi(\tau)$ subset of $P$ is meager in the relative topology $\tau_P|_P$ (compare to a result of S. Solecki [8, Theorem 2.2, p. 526]). Thus Theorem 1 is plausible in the sense that every $\mathbf{\Sigma}^0_\xi(\tau)$

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subset of $P$, i.e. the kind of subsets $B$ of $P$ that we intend to cover with $\Phi(B)$, are small in Baire category relatively to $P$. In this paper we develop the theory of topological Hurewicz test pairs; Theorem 1 will be a nontrivial application of this technique.

The second motivation for Theorem 1 is that we think that it is the best result one can obtain in ZFC: we conjecture that it is independent whether Theorem 1 fails for limit ordinals or not and that it is also independent whether $\Phi(B)$ can be chosen to be $\mathbf{\Pi}^0_\vartheta(\tau)$ for $\vartheta < \xi$ or not. We will discuss the problem of limit ordinals at the end of the paper after the proof of Theorem 1. Here we only argue for the independence of the problem arising from the quest for the optimal Borel class for $\Phi(B)$. First we give a definition and two results.

**Definition 2.** Let $\mathcal{I}$ be a $\sigma$-ideal and $\mathcal{F} \subseteq \mathcal{I}$. We say that $\mathcal{I}$ *is generated by* $\mathcal{F}$ if for every $G \in \mathcal{I}$ there exist $F_i \in \mathcal{F}$ $(i < \omega)$ such that $G \subseteq \bigcup_{i<\omega} F_i$. We say that $\mathcal{F}$ is *cofinal in* $\mathcal{I}$ if for every $G \in \mathcal{I}$ there is an $F \in \mathcal{F}$ such that $G \subseteq F$.

S. Solecki proved the following result (see [7, Theorem 1, p. 1023]).

**Theorem 3** (S. Solecki). *In the Polish space $(X,\tau)$ let $\mathcal{I} \subseteq 2^X$ be a $\sigma$-ideal generated by its $\mathbf{\Pi}^0_1(\tau)$ members. Let $A \subseteq X$ be a $\mathbf{\Sigma}^1_1(\tau)$ set. Then either $A \in \mathcal{I}$ or there is a $\mathbf{\Pi}^0_2(\tau)$ set $G \subseteq A$ such that $G \notin \mathcal{I}$, moreover $F \cap G$ is relatively $\tau|_G$-meager in $G$ for every $F \in \mathcal{I}$.*

It is natural to ask whether the same result holds for higher Borel classes (see [6, Question 1.9]).

**Question 4** (A. Miller). *Let $(X,\tau)$ be a Polish space and fix an ordinal $\xi$ satisfying $2 \leq \xi < \omega_1$. Let $\mathcal{I} \subseteq 2^X$ be a $\sigma$-ideal which is generated by its $\mathbf{\Pi}^0_\xi$ members. Is it true that for every analytic set $A \subseteq X$, either $A \in \mathcal{I}$ or there is a $\mathbf{\Pi}^0_{\xi+1}$ set $B \subseteq A$ such that $B \notin \mathcal{I}$?*

Before all further comments it is important to point out that this question is already refuted by the following unpublished result of A. Kechris and M. Zelený.

**Theorem 5** (A. Kechris-M. Zelený). *Assume $V = L$. Then there is an analytic set $A \subseteq 2^\omega$ and a $\sigma$-ideal $\mathcal{I} \subseteq 2^{2^\omega}$ such that $A \notin \mathcal{I}$ but $\mathcal{I}$ contains every Borel subset of $A$ and the $\mathbf{\Pi}^0_2(\tau_{2^\omega})$ members of $\mathcal{I}$ are cofinal in $\mathcal{I}$.*

That is the answer to Question 4 is consistently negative. Moreover, in Theorem 5 instead of the condition $V = L$ it is enough to assume the Continuum Hypothesis, which is a consequence of $V = L$ (see e.g. [1]), and for a consistent counterexample we do not need one special analytic set (see [5, Theorem 6]).

**Theorem 6.** *Let $(X,\tau)$ be a Polish space and $P \subseteq X$ be a Borel and not $\mathbf{\Sigma}^0_3(\tau)$ set. By assuming the Continuum Hypothesis there is a $\sigma$-ideal $\mathcal{I}$ such that*

1. *the $\mathbf{\Pi}^0_2(\tau)$ members of $\mathcal{I}$ are cofinal in $\mathcal{I}$;*

2. $\mathcal{S}^0_3(P) \subseteq \mathcal{I}$;
3. $P \notin \mathcal{I}$.

Observe that by Theorem 1 for $\xi = 3$ we have in particular that for every $\mathbf{\Pi}^0_3(\tau)$ and not $\mathbf{\Sigma}^0_3(\tau)$ set $P \subseteq X$ the $\sigma$-ideal

$$\mathcal{I} = \left\{ A \subseteq X : \exists\, B_i \in \mathcal{S}^0_3(P)\ (i < \omega) \left( A \subseteq \bigcup_{i<\omega} \Phi(B_i) \right) \right\}$$

satisfies $P \notin \mathcal{I}$ but for every $B \in \mathcal{S}^0_3(P)$ there is a $B' \in \mathcal{P}^0_3(P) \cap \mathcal{I}$ satisfying $B \subseteq B'$. So Theorem 6 strengthens this by providing a $\sigma$-ideal with cofinal $\mathbf{\Pi}^0_2$ members instead of a $\mathbf{\Pi}^0_3$-generated one. We think that it is consistently true for every $1 < \xi < \omega_1$ that in Theorem 1, $\Phi(B)$ can be taken in $\mathbf{\Pi}^0_2(\tau)$ $(B \in \mathcal{S}^0_\xi(P))$. Moreover, for this only Theorem 6 should be established for $3 < \xi < \omega_1$; that is for every Borel and not $\mathbf{\Sigma}^0_\xi(\tau)$ set $P \subseteq X$ we need to construct a $\sigma$-ideal $\mathcal{I}$ with cofinal $\mathbf{\Pi}^0_2$ members such that $\mathcal{S}^0_\xi(P) \subseteq \mathcal{I}$ but $P \notin \mathcal{I}$. On the other hand, a consistently positive answer to Question 4 could give that $\Phi(B)$ cannot be taken in $\mathbf{\Pi}^0_\vartheta(\tau)$ for $\vartheta < \xi$, that is $\Phi(B) \in \mathcal{P}^0_\xi(P)$ is optimal $(B \in \mathcal{S}^0_\xi(P))$. However, up to our knowledge these problems are open.

The paper is structured as follows. In Section 2 we recall and introduce some key notions related to the refinement of topologies in Polish spaces. Next, in Section 3, we prove Theorem 1 for $\xi = 2$; this case is treated separately because no Hurewicz tests appear in the proof, moreover the argument which covers the $3 \le \xi < \omega_1$ case is not applicable for $\xi = 2$. We define our topological Hurewicz test pairs and discuss their basic properties in Section 3. In Section 4 we obtain sufficient criteria for sets to be in a topological Hurewicz test pair and we prove Proposition 34, which is the main lemma toward Theorem 1 but might be interesting on its own right. In Section 5 we construct Hurewicz test pairs, we prove Theorem 1 for $3 \le \xi < \omega_1$ and we close the paper with a short analysis of the proof.

## 2. Preliminaries

Our terminology and notation follow [2]. As usual, $\mathbf{\Pi}^0_\xi(\tau)$ and $\mathbf{\Sigma}^0_\xi(\tau)$ $(0 < \xi < \omega_1)$ stand for the $\xi^{\text{th}}$ multiplicative and additive Borel class in the Polish space $(X, \tau)$, starting with $\mathbf{\Pi}^0_1(\tau) =$ closed sets, $\mathbf{\Sigma}^0_1(\tau) =$ open sets. A set is called *proper* $\mathbf{\Pi}^0_\xi(\tau)$ if it is $\mathbf{\Pi}^0_\xi(\tau)$ and not $\mathbf{\Sigma}^0_\xi(\tau)$ $(0 < \xi < \omega_1)$.

Let $(C, \tau_C)$ denote the Polish space $2^\omega$ with its usual product topology. For two finite sequences $s, t \in \omega^{<\omega}$, we write $s \subseteq t$ and $s \subset t$ if $t$ is an extension of $s$ and if $t$ is a proper extension of $s$. The length of $s$ is denoted by $|s|$. If $s = (s_0 s_1 \dots s_{n-1})$ and $i < \omega$, then $s^\frown i$ stands for the sequence $(s_0 s_1 \dots s_{n-1} i)$.

If $T \subseteq \omega^{<\omega}$ is a subtree and $s \in \omega^{<\omega}$ we set $T_s = \{t \in \omega^{<\omega}: s^\frown t \in T\}$. The terminal nodes of $T$ are denoted by $\mathfrak{T}(T)$.

Let $\xi, \vartheta_i$ $(i < \omega)$ be ordinals. We write $\vartheta_i \twoheadrightarrow \xi$ if $\xi$ is successor and $\vartheta_i + 1 = \xi$ $(i < \omega)$ or if $\xi$ is limit, $\vartheta_i \leq \vartheta_j$ $(i \leq j < \omega)$ and $\sup_{i<\omega} \vartheta_i = \xi$.

For every ordinal $\xi < \omega_1$ we fix once and for all a sequence $(\vartheta_i)_{i<\omega}$ such that $\vartheta_i \twoheadrightarrow \xi$. To avoid complicated notations, we do not indicate the dependence of the sequence $(\vartheta_i)_{i<\omega}$ on $\xi$, it will be always clear which pair of ordinal and sequence is considered.

In this note we will notoriously refine Polish topologies by turning countably many closed sets into open sets. We do this as described in [2], that is the open sets of the ancient topology together with their portions on the members of our collection of closed sets serve as a subbase of the new, finer topology. We will use that the topology obtained in this way is also Polish.

**Definition 7.** Let $(X, \tau)$ be a Polish space, $\mathcal{P} = \{P_i : i < \omega\}$ be a countable collection of $\mathbf{\Pi}^0_1(\tau)$ sets. Then $\tau[\mathcal{P}]$ denotes that Polish topology refining $\tau$ where each $P_i$ $(i < \omega)$ is turned successively into an open set. For the precise procedure we refer to [2, (13.2) Lemma and (13.3) Lemma, p. 82].

It is easy to see that the resulting finer topology $\tau[\mathcal{P}]$ is independent from the enumeration of $\mathcal{P}$. This will be clear shortly when we fix a base of $\tau[\mathcal{P}]$. We also use the notation $\tau[\mathcal{P}]$ when the countable collection of not necessarily $\mathbf{\Pi}^0_1(\tau)$ sets $\mathcal{P}$ can be enumerated on such a way that $P_n$ is $\mathbf{\Pi}^0_1(\tau[\{P_i : i < n\}])$.

**Definition 8.** If $\tau_n$ $(n < \omega)$ is a Polish topology on some base set $X$ then $\bigvee_{n<\omega} \tau_n$ denotes the coarsest topology on $X$ which refines each $\tau_n$ $(n < \omega)$.

The resulting topology is also Polish and we will shortly fix a countable base for it. Before doing this we need a precise notion of basic open sets in our spaces.

**Definition 9.** Let $(X_i, \tau_i)$ $(i \in I)$ be Polish spaces; if a basis $\mathcal{G}_i$ is fixed in the spaces $(X_i, \tau_i)$ $(i \in I)$, which are meant to be the basic open sets in $(X_i, \tau_i)$, then the *basic open sets* of $(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i)$ are the open sets of the form

$$\prod_{i\in J} G_i \times \prod_{i\in I\setminus J} X_i,$$

where $J \subseteq I$ is finite and $G_i \in \mathcal{G}_i$ for every $i \in J$.

If the family $\mathcal{G}$ basic open sets is fixed in the Polish space $(X, \tau)$ and $\tau[\mathcal{P}]$ makes sense for a countable collection $\mathcal{P}$ of subsets of $X$, then *the basic open sets* of $\tau[\mathcal{P}]$ are of the form $G \cap F_0 \cap \cdots \cap F_{n-1}$ or $G$ with $G \in \mathcal{G}$, $F_i \in \mathcal{P}$ $(i < n)$; a basic $\tau[\mathcal{P}]$-open set is said to be *proper* if it is not $\tau$-open.

If the basic open sets $\mathcal{G}_n$ are fixed for the Polish topologies $\tau_n$ $(n < \omega)$ then the *basic open sets* for $\bigvee_{n<\omega} \tau_n$ are the sets of the form $\bigcap_{i<m} G_{n_i}$ where $G_{n_i} \in \mathcal{G}_{n_i}$ $(m < \omega,\ n_i < \omega\ (i < m))$.

Observe that the basic open sets defined on this way form a basis of $\prod_{i\in I}\tau_i$, $\tau[\mathcal{P}]$ and $\bigvee_{n<\omega}\tau_n$, respectively. From now on whenever a Polish space $(X,\tau)$ appears we assume that a countable basis comprised of basic $\tau$-open sets is fixed; and this is done with respect to the convention of Definition 9 if it is applicable. We take $X$ to be basic $\tau$-open. Basic open sets are assumed to be regular open. In zero dimensional spaces we assume that our basic $\tau$-open sets are $\mathbf{\Pi}^0_1(\tau)$; note that our procedure of refinement results a zero dimensional space from a zero dimensional one with $\mathbf{\Pi}^0_1(\tau[\mathcal{P}])$ basic $\tau[\mathcal{P}]$-open sets. If two Polish topologies $\tau$ and $\tau'$ are given on the same base set and $\tau'$ is finer than $\tau$ we assume that the basic $\tau$-open and the basic $\tau'$-open sets are fixed such that the every basic $\tau$-open set is also basic $\tau'$-open.

The closure of a set $A\subseteq(X,\tau)$ is denoted by $\mathrm{cl}_\tau(A)$. We will never have to fix a special compatible metric on our Polish spaces but we will condition on the diameter of sets. In this case $\mathrm{diam}_\tau$ denotes the diameter in an arbitrary fixed metric generating $\tau$. We assume that $\mathrm{diam}_\tau(X)\le 1$.

We recall that a $\mathbf{\Pi}^0_2(\tau)$ subset $G$ of the Polish space $(X,\tau)$ is itself a Polish space with the restricted topology $\tau|_G$ (see e.g. [2, (3.11) Theorem]). In particular, the notions related to category in the topology $\tau$ make sense relative to $G$.

We will have to return to the topologies on the coordinates in product spaces. If $(X,\sigma)$, $(Y,\tau)$ are arbitrary topological spaces and $(\mathcal{X},\mathcal{S})=(X\times Y,\sigma\times\tau)$, then we define $\mathrm{Pr}_X(\mathcal{S})=\sigma$. The projection of product sets in product spaces is defined analogously. If $G_X\subseteq X$ and $G_Y\subseteq Y$, we say that the set of product form $G=G_X\times G_Y\subseteq\mathcal{X}$ is *nontrivial on the $X$ coordinate* if $G_X\neq X$.

Finally we recall a result which was the strongest motivation for the topologization of Hurewicz tests and will be used in the proof of Theorem 1 (see [3]).

**Theorem 10** (A. Louveau, J. Saint Raymond). *Let $3\le\xi<\omega_1$ and $(X,\tau)$ be a Polish space. If $P_\xi\subseteq C$ is proper $\mathbf{\Pi}^0_\xi(\tau_C)$ and $A_0,A_1\subseteq X$ is any pair of disjoint analytic sets, then either $A_0$ can be separated from $A_1$ by a $\mathbf{\Sigma}^0_\xi(\tau)$ set or there is a continuous one-to-one map $\varphi\colon(C,\tau_C)\to X$ with $\varphi(P_\xi)\subseteq A_0$ and $\varphi(2^\omega\setminus P_\xi)\subseteq A_1$.*

*The same conclusion holds for $\xi=2$ if $P_2\subseteq C$ is the complement of a dense countable set.*

## 3. The $\xi=2$ case

From now on $(X,\tau)$ denotes a Polish space. In this section we prove Theorem 1 for $\xi=2$. The proof does not use topological Hurewicz test pairs; instead, it is based on a compactness argument which was pointed out to the author by Zoltán Szentmiklóssy. The fact that this argument, which brakes down for $\xi>2$, can be substituted by a reasoning based on topological Hurewicz test pairs may motivate other applications of the topological Hurewicz test pairs.

In this section $(\mathcal{N},\tau_{\mathcal{N}})$ stands for $\omega^\omega$ with its usual product topology. For $f,g\in\mathcal{N}$ we write $f\prec g$ if the set $\{n<\omega\colon f(n)\ge g(n)\}$ is finite. We will use the following well-known results (see e.g. [2, (7.7) Theorem, p. 37], [2, (4.11) Excercise, p. 20] and [1, Lemma 29.6, p. 575]).

**Proposition 11.** *Let $N \subseteq C$ be the complement of a dense countable set. Then $(N, \tau_C|_N)$ is homeomorphic to $(\mathcal{N}, \tau_{\mathcal{N}})$.*

**Proposition 12.** *Let $K \subseteq \mathcal{N}$ be a $\mathbf{\Pi}^0_1(\tau_{\mathcal{N}})$ set. Then $K$ is compact if and only if $K$ is dominated, i.e. there exists $f\colon \omega \to \omega$ such that for every $g \in K$ we have $g(n) \leq f(n)$ $(n < \omega)$.*

**Proposition 13.** *Let $(f_i)_{i<\omega} \subseteq \mathcal{N}$ be arbitrary. Then there exists an $f \in \mathcal{N}$ satisfying $f_i \prec f$ $(i < \omega)$.*

Proof of Theorem 1 for $\xi = 2$: Take $N \subseteq C$ to be the complement of a dense countable set. Then by Proposition 11, $(N, \tau_C|_N)$ is homeomorphic to $(\mathcal{N}, \tau_{\mathcal{N}})$ so in the sequel, with a little abuse of notation we identify $(N, \tau_C|_N)$ with $(\mathcal{N}, \tau_{\mathcal{N}})$.

We apply Theorem 10 for $\xi = 2$ with $P_2 = \mathcal{N} \subseteq C$ in $(X, \tau)$ for the $\mathbf{\Pi}^0_2(\tau)$ and not $\mathbf{\Sigma}^0_2(\tau)$ set $P$. We get a continuous one-to-one map $\varphi\colon (C, \tau_C) \to (X, \tau)$ such that $\varphi^{-1}(P) = \mathcal{N}$.

We define $\Phi\colon \mathcal{S}^0_2(P) \to \mathcal{P}^0_2(P)$ as follows. For every $B \in \mathcal{S}^0_2(P)$ fix a decomposition $B = \bigcup_{j<\omega} B_j$ where $B_j$ is $\mathbf{\Pi}^0_1(\tau)$ $(j < \omega)$. Then $\varphi^{-1}(B_j) \subseteq C$ is $\mathbf{\Pi}^0_1(\tau_C)$ hence compact $(j < \omega)$. Since $B_j \subseteq P$ and $\varphi^{-1}(P) = \mathcal{N}$, we obtained that $\varphi^{-1}(B_j) \subseteq \mathcal{N}$ is a compact set $(j < \omega)$. By Proposition 13, for every $j < \omega$ there is a function $f_j\colon \omega \to \omega$ such that for every $g \in \varphi^{-1}(B_j)$ we have $g(n) \leq f_j(n)$ $(n < \omega)$. Let $f_B\colon \omega \to \omega$ be such that $f_j \prec f_B$ $(j < \omega)$ and set

$$F(B) = \{g \in \mathcal{N}\colon \forall\, N < \omega\ \exists\, n > N\ (g(n) \leq f_B(n))\}.$$

We define $\Phi(B) = (P \setminus \varphi(\mathcal{N})) \cup \varphi(F(B))$.

Clearly, $P \setminus \varphi(\mathcal{N}) = P \setminus \varphi(C)$ is a $\mathbf{\Pi}^0_2(\tau)$ set. Since $F(B)$ is $\mathbf{\Pi}^0_2(\tau_{\mathcal{N}})$ and homeomorphisms keep the Borel class of sets we have that $\Phi(B) \in \mathcal{P}^0_2(P)$. We show that $B \subseteq \Phi(B)$; take $g \in B$. If $x \in B \setminus \varphi(\mathcal{N})$ then $x \in P \setminus \varphi(\mathcal{N}) \subseteq \Phi(B)$. If $x \in B \cap \varphi(\mathcal{N})$, say $x \in B_j \cap \varphi(\mathcal{N})$ then $[\varphi^{-1}(x)](n) \leq f_j(n)$ $(n < \omega)$ hence $\varphi^{-1}(x) \in F(B)$ so again $x \in \Phi(B)$.

Finally, for $i < \omega$ let $B^i \in \mathcal{S}^0_2(P)$ with its decomposition $B^i = \bigcup_{j<\omega} B^i_j$, eventual dominator $f_{B^i}$ and covering set $F(B^i)$. Take $g\colon \omega \to \omega$ such that $f_{B^i} \prec g$ $(i < \omega)$. Then $g \notin F(B^i)$ $(i < \omega)$ hence $\varphi(g) \in \varphi(\mathcal{N}) \setminus \bigcup_{i<\omega} \varphi(F(B^i))$, that is $\varphi(g) \in P \setminus \bigcup_{i<\omega} \Phi(B^i)$. This completes the proof. $\square$

## 4. Topological Hurewicz test pairs

In this section we construct our Hurewicz test pairs. In order to produce a sufficiently big family of test pairs we need a machinery which allows us to condition on the construction of a given Borel set from simpler sets. For this, we handle a $\mathbf{\Pi}^0_\xi(\tau)$ set by coding its construction from closed sets in a tree. The following inductive definition makes this concrete.

**Definition 14.** Let $0 < \xi < \omega_1$ and $\vartheta_i \twoheadrightarrow \xi$. For $\xi = 1$, $[P, (P_\emptyset)]$ is called a $\mathbf{\Pi}^0_1(\tau)$ *set with presentation* if $P = P_\emptyset$ is a $\mathbf{\Pi}^0_1(\tau)$ set.

Suppose that the $\mathbf{\Pi}^0_\vartheta(\tau)$ sets with presentation are defined for $\vartheta < \xi$. Then $[P, (P_t)_{t\in T}]$ is a $\mathbf{\Pi}^0_\xi(\tau)$ *set with presentation* if $T \subseteq \omega^{<\omega}$ is a subtree such that $\{(i) : i < \omega\} \subseteq T$, $P = X \setminus \bigcup_{i<\omega} P_{(i)}$ and $[P_{(i)}, (P_{i\frown t})_{t\in T_{(i)}}]$ is a $\mathbf{\Pi}^0_{\vartheta_i}(\tau)$ set with presentation ($i < \omega$).

It is important to note that a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation is not necessarily a proper $\mathbf{\Pi}^0_\xi(\tau)$ set. For example such a set can easily be empty.

Next we define the test sets and the corresponding topologies.

**Definition 15.** Let $0 < \xi < \omega_1$ and $P \subseteq X$. We call the pair $\{P, \tau_P\}$ a $\mathbf{\Pi}^0_\xi(\tau)$ *topological Hurewicz test pair in* $(X, \tau)$ if

1. $P$ is a $\mathbf{\Pi}^0_\xi(\tau)$ set;
2. $\tau_P$ is a Polish topology on $X$ refining $\tau$;
3. $P$ is a $\tau_P$-nowhere dense $\mathbf{\Pi}^0_1(\tau_P)$ set;
4. (a) $\xi = 1$: if for a $\tau$-open set $A \subseteq X$ and a basic $\tau_P$-open set $G$ with $G \cap P \neq \emptyset$ we have $A \cap P$ is $\tau_P|_P$-residual in $G \cap P$ then $A$ is $\tau_P$-residual in a $\tau_P$-open set $G' \subseteq G$ such that $G \cap P \subseteq \mathrm{cl}_{\tau_P}(G' \cap P)$.
   (b) $1 < \xi$ is a successor ordinal: if for a $\vartheta < \xi$, a $\mathbf{\Pi}^0_\vartheta(\tau)$ set $A \subseteq X$ and a basic $\tau_P$-open set $G$ with $G \cap P \neq \emptyset$ we have $A \cap P$ is $\tau_P|_P$-residual in $G \cap P$ then $A$ is $\tau_P$-residual in $G$.
   (c) $1 < \xi$ is a limit ordinal: for every $\vartheta < \xi$ there is a $\tau_P$-open set $\mathcal{H}_{X,P}(\vartheta)$ such that $P \subseteq \mathcal{H}_{X,P}(\vartheta') \subseteq \mathcal{H}_{X,P}(\vartheta)$ ($\vartheta \leq \vartheta' < \xi$) and if for a $\vartheta < \xi$, a $\mathbf{\Pi}^0_\vartheta(\tau)$ set $A \subseteq X$ and a basic $\tau_P$-open set $G$ with $G \cap P \neq \emptyset$ we have $A \cap P$ is $\tau_P|_P$-residual in $G \cap P$ then $A$ is $\tau_P$-residual in $G \cap \mathcal{H}_{X,P}(\vartheta)$.

Notice the paradox behavior of a test pair $\{P, \tau_P\}$. Even if $P$ is a $\tau_P$-nowhere dense $\mathbf{\Pi}^0_1(\tau_P)$ set, that is it is negligible in the sense of Baire category, from the information that a set $A$ has a Borel class lower than the Borel class of $P$ and $A \cap P$ is big in category relative to $P$ we conclude that $A$ is big in the whole space $X$, that is $A$ is of $\tau_P$-second category.

Observe also the following fact.

**Lemma 16.** *Let $(X, \tau')$ be a Polish space and $\tau$ be a Polish refinement of $\tau'$. Let $\{P, \tau_P\}$ be a $\mathbf{\Pi}^0_\xi(\tau)$ topological Hurewicz test pair in $(X, \tau)$. If $P$ is also a $\mathbf{\Pi}^0_\xi(\tau')$ set then $\{P, \tau_P\}$ is a $\mathbf{\Pi}^0_\xi(\tau')$ topological Hurewicz test pair in $(X, \tau')$.*

Proof: Since every $\mathbf{\Sigma}^0_1(\tau')$ set is $\mathbf{\Sigma}^0_1(\tau)$ and every $\mathbf{\Pi}^0_\vartheta(\tau')$ set is $\mathbf{\Pi}^0_\vartheta(\tau)$ ($\vartheta < \xi$) the statement follows. $\square$

We associate topologies $\tau_P^<$ and $\tau_P$ to $\mathbf{\Pi}^0_\xi(\tau)$ sets with presentation $[P, (P_t)_{t\in T}]$.

**Definition 17.** Consider a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation $[P,(P_t)_{t\in T}]$. For $\xi = 1$ we define $\tau_P^< = \tau_P = \tau$. If $1 < \xi < \omega_1$ and $\tau_Q$ is defined for $\mathbf{\Pi}^0_\vartheta(\tau)$ sets $Q$ with presentation for $\vartheta < \xi$, set

$$\mathcal{P} = \left\{P_{(n)} \cap \bigcap_{i<n}(X \setminus P_{(i)}) \colon n < \omega\right\}.$$

We define $\tau_P^< = \bigvee_{i<\omega} \tau_{P_{(i)}}$ and $\tau_P = \tau_P^<[\mathcal{P}]$.

Note that the topologies $\tau_P^<$ and $\tau_P$ depend on the presentation of $P$. However, we do not indicate this in the notation, the presentation will always be fixed in advance. Observe also that $P$ is disjoint to the members of $\mathcal{P}$ and that the sets in $\mathcal{P}$ are pairwise disjoint.

Next we prove an auxiliary claim on how $P$ is related to the topologies $\tau_P^<$, $\tau_P$. For its proof we will need the Kuratowski-Ulam Theorem in the following form (see [2, (8.41) Theorem]).

**Theorem 18** (Kuratowski-Ulam)**.** *Let $(X,\tau)$ and $(Y,\sigma)$ be Polish spaces, let $G = G_X \times G_Y$ be a basic $\tau\times\sigma$-open set in $X \times Y$ and consider a Borel set $A \subseteq X \times Y$. Set $A^y = \{x \in X \colon (x,y) \in A\}$. Then $A$ is $\tau\times\sigma$-residual in $G$ if and only if*

$$\{y \in G_Y \colon A^y \text{ is } \tau\text{-residual in } G_X\}$$

*is $\sigma$-residual in $G_Y$.*

**Proposition 19.** *With the notation of Definition* 17 *we have the following.*

1. *$P$ is $\mathbf{\Pi}^0_2(\tau_P^<)$ and $\mathbf{\Pi}^0_1(\tau_P)$.*
2. *If $G$ is basic $\tau_P$-open and $G \cap P \neq \emptyset$ then $G$ is in fact basic $\tau_P^<$-open.*
3. *The topologies $\tau_P|_P$ and $\tau_P^<|_P$ coincide.*
4. *The topologies*
$$\tau_P|_{P_{(n)}\setminus\bigcup_{i<n}P_{(i)}} \text{ and } \tau_P^<|_{P_{(n)}\setminus\bigcup_{i<n}P_{(i)}} \ (n<\omega)$$
*coincide.*
5. *If $(Y,\sigma)$ is any nonempty Polish space and $\{P,\tau_P\}$ is a $\mathbf{\Pi}^0_\xi(\tau)$ topological Hurewicz test pair in $(X,\tau)$ then $\{P\times Y, \tau_P\times\sigma\}$ is a $\mathbf{\Pi}^0_\xi(\tau\times\sigma)$ topological Hurewicz test pair in $(X\times Y, \tau\times\sigma)$; and if $\xi$ is a limit ordinal then $\mathcal{H}_{X\times Y,P\times Y}(\vartheta) = \mathcal{H}_{X,P}(\vartheta)\times Y$ fulfills the requirements.*

Proof: We prove the first statement by induction on $\xi$. For $\xi = 1$ the statement is obvious. Let now $1 < \xi < \omega_1$ and suppose that the statement holds for $\vartheta < \xi$. We have

$$P = X \setminus \bigcup_{n<\omega} P_{(n)} = X \setminus \bigcup_{n<\omega}\left(P_{(n)} \cap \bigcap_{i<n}(X\setminus P_{(i)})\right), \tag{1}$$

where $P_{(n)}$ is $\mathbf{\Pi}^0_1(\tau_P^<)$ $(n<\omega)$ by the inductive hypothesis and $P_{(n)} \cap \bigcap_{i<n}(X \setminus P_{(i)})$ is $\tau_P$-open $(n<\omega)$ by definition, so 1 follows.

By Definition 17 proper basic $\tau_P$-open sets do not intersect $P$, which shows 2. This immediately implies 3.

Since the sets in $\mathcal{P}$ are pairwise disjoint, if $G$ is a proper basic $\tau_P$-open set which intersects $P_{(n)} \setminus (\bigcup_{i<n} P_{(i)})$ then $G = G' \cap P_{(n)} \setminus (\bigcup_{i<n} P_{(i)})$ where $G'$ is basic $\tau_P^<$-open, so 4 holds.

For 5, let $G$ be a basic $\tau_P \times \sigma$-open set, say $G = G_X \times G_Y$ where $G_X$ is basic $\tau_P$-open in $X$ and $G_Y$ is basic $\sigma$-open in $Y$. If $\xi = 1$, let $A$ be a $\tau \times \sigma$-open set such that $A \cap (P \times Y)$ is $(\tau_P \times \sigma)|_{P\times Y}$-residual in $G \cap (P \times Y) = (G_X \cap P) \times G_Y$. Let

$$G' = \bigcup\{H \subseteq G\colon H \text{ is basic } \tau_P \times \sigma\text{-open, } A \text{ is } \tau_P \times \sigma\text{-residual in } H\}.$$

Then $G' \subseteq G$ and $A$ is $\tau_P \times \sigma$-residual in $G'$ so it remains to show that

$$G \cap (P \times Y) \subseteq \mathrm{cl}_{\tau_P\times\sigma}(G' \cap (P \times Y)).$$

Suppose that $K = K_X \times K_Y \subseteq G$ is a nonempty basic $\tau_P \times \sigma$-open set such that

$$K \cap (P \times Y) \subseteq G \cap (P \times Y) \setminus \mathrm{cl}_{\tau_P\times\sigma}(G' \cap (P \times Y)).$$

Then $A \cap (P \times Y)$ is $(\tau_P \times \sigma)|_{P\times Y}$-residual in $(K_X \cap P) \times K_Y$, so by Theorem 18,

$$W = \{y \in K_Y\colon A^y \cap P \text{ is } \tau_P|_P\text{-residual in } K_X \cap P\}$$

is $\sigma$-residual in $K_Y$. Since $\{P, \tau_P\}$ is a $\mathbf{\Pi}^0_1(\tau)$-topological Hurewicz test pair, by Definition 15.4(a), for every $y \in W$ there is a $\tau_P$-open set $K'_X(y) \subseteq K_X$ such that $A^y$ is $\tau_P$-residual in $K'_X(y)$ and $K_X \cap P \subseteq \mathrm{cl}_\tau(K'_X(y) \cap P)$. Since $(X, \tau_P)$ has countable base there is a basic $\tau_P$-open set $K'_X \subseteq K_X$ such that $\{y \in K_Y\colon K'_X \subseteq K'_X(y)\}$ is $\sigma$-nonmeager, hence residual in a basic $\sigma$-open set $K'_Y \subseteq K_Y$. Thus we obtained that

$$\left\{y \in K'_Y\colon A^y \text{ is } \tau_P\text{-residual in } K'_X\right\}$$

is $\sigma$-residual in $K'_Y$. Then by Theorem 18, $A$ is $\tau_P \times \sigma$-residual in $K' = K'_X \times K'_Y$, that is $K' \subseteq G'$, a contradiction.

Let now $1 < \xi < \omega_1$ be a successor ordinal, $\vartheta < \xi$ and $A$ be a $\mathbf{\Pi}^0_\vartheta(\tau \times \sigma)$ set such that $A \cap (P \times Y)$ is $(\tau_P \times \sigma)|_{P\times Y}$-residual in $(G_X \cap P) \times G_Y$. We show that $A$ is $\tau_P \times \sigma$-residual in $G$. By Theorem 18,

$$W = \{y \in G_Y\colon A^y \cap P \text{ is } \tau_P|_P\text{-residual in } G_X \cap P\}$$

is $\sigma$-residual in $G_Y$. Since $\{P,\tau_P\}$ is a $\mathbf{\Pi}^0_\xi(\tau)$-topological Hurewicz test pair, by Definition 15.4(b), $A^y$ $(y\in W)$ is $\tau_P$-residual in $G_X$. Then again by Theorem 18, $A$ is $\tau_P\times\sigma$-residual in $G$, as stated.

Let now $1<\xi<\omega_1$ be a limit ordinal. We show that

$$\mathcal{H}_{X\times Y,P\times Y}(\vartheta)=\mathcal{H}_{X,P}(\vartheta)\times Y\ \ (\vartheta<\xi)$$

fulfills the requirements. Let $\vartheta<\xi$ and $A$ be a $\mathbf{\Pi}^0_\vartheta(\tau\times\sigma)$ set such that $A\cap(P\times Y)$ is $(\tau_P\times\sigma)|_{P\times Y}$-residual in $(G_X\cap P)\times G_Y$. By Theorem 18,

$$W=\{y\in G_Y\colon A^y\cap P\ \text{ is } \tau_P|_P\text{-residual in } G_X\cap P\}$$

is $\sigma$-residual in $G_Y$. Since $\{P,\tau_P\}$ is a $\mathbf{\Pi}^0_\xi(\tau)$-topological Hurewicz test pair, by Definition 15.4(c), $A^y$ $(y\in W)$ is $\tau_P$-residual in $G_X\cap\mathcal{H}_{X,P}(\vartheta)$. Then again by Theorem 18, $A$ is $\tau_P\times\sigma$-residual in

$$G\cap\mathcal{H}_{X\times Y,P\times Y}(\vartheta)=\big(G_X\cap\mathcal{H}_{X,P}(\vartheta)\big)\times G_Y,$$

as stated. This completes the proof. $\square$

The following claim describes the behavior of a topological test pair with respect to $\mathbf{\Sigma}^0_\xi(\tau)$ sets. Compare it to Definition 15.4(a).

**Proposition 20.** *Let $0<\xi<\omega_1$ and let $\{P,\tau_P\}$ be a $\mathbf{\Pi}^0_\xi(\tau)$ topological Hurewicz test pair. If for a $\mathbf{\Sigma}^0_\xi(\tau)$ set $W$ and $\tau_P$-open set $G$ with $G\cap P\neq\emptyset$ we have $W\cap P$ is $\tau_P|_P$-residual in $G\cap P$, then $W$ is $\tau_P$-residual in a $\tau_P$-open set $H$ satisfying that $G\cap P\subseteq\mathrm{cl}_{\tau_P}(H\cap P)$.*

Proof: For $\xi=1$ the statement follows from the definition. Let now $1<\xi<\omega_1$ and write $W=\bigcup_{i<\omega}Q_i$, where $Q_i$ is $\mathbf{\Pi}^0_{\vartheta_i}(\tau)$ and $\vartheta_i\twoheadrightarrow\xi$. If $W\cap G\cap P$ is $\tau_P|_P$-residual in $G\cap P$ then let $H_i$ denote the maximal $\tau_P$-open set in which $Q_i$ is $\tau_P$-residual $(i<\omega)$. By Definition 15.4, the $\tau_P$-open set $H=\bigcup_{i<\omega}H_i$ meets every $\tau_P|_P$-open set intersecting $G\cap P$, which proves the statement. $\square$

In the following theorem we give a method allowing to build up inductively a topological Hurewicz test pair from simpler test sets.

**Theorem 21.** *Let $0<\xi<\omega_1$, $\vartheta_i\twoheadrightarrow\xi$ and let $[P,(P_t)_{t\in T}]$ be a nonempty $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation. If $\xi=1$ and $P$ is $\tau$-nowhere dense then $\{P,\tau_P\}$ is a topological Hurewicz test pair.*

*For $1<\xi<\omega_1$ suppose that $\bigcup_{i<\omega}P_{(i)}$ is $\tau_P^{<}$-dense in $X$ and $\{P_{(i)},\tau_P^{<}\}$ is a $\mathbf{\Pi}^0_{\vartheta_i}(\tau)$ topological Hurewicz test pair in $(X,\tau)$ $(i<\omega)$. Then*

1. *$P$ is $\tau_P^{<}$-residual;*
2. *$\{P,\tau_P\}$ is a $\mathbf{\Pi}^0_\xi(\tau)$ topological Hurewicz test pair.*

Proof: If $\xi = 1$, $G' = A \cap G$ does the job. Let now $1 < \xi < \omega_1$. Since $\{P_{(n)}, \tau_P^<\}$ $(n < \omega)$ is a $\mathbf{\Pi}^0_{\vartheta_n}(\tau)$ topological Hurewicz test pair, $P_{(n)}$ $(n < \omega)$ is $\tau_P^<$-nowhere dense so statement 1 follows from equation (1).

For 2 we have to check the conditions of Definition 15; 1 holds by the choice of $P$, 2 follows from Definition 17.

For 3, by Proposition 19.1 it remains to show that $P$ does not contain any nonempty basic $\tau_P$-open set. Suppose that $G \subseteq P$ and $G$ is nonempty basic $\tau_P$-open. Then by Proposition 19.2, $G$ is basic $\tau_P^<$-open, we have $\bigcup_{i<\omega} P_{(i)}$ is $\tau_P^<$-dense hence $P_{(n)} \cap G \neq \emptyset$ for some $n < \omega$, a contradiction.

Let now $\vartheta < \xi$, $A \subseteq X$ be $\mathbf{\Pi}^0_\vartheta(\tau)$, $G$ be a basic $\tau_P$-open set with $G \cap P \neq \emptyset$ and suppose that $A \cap P$ is $\tau_P|_P$-residual in $G \cap P$. By Proposition 19.2, $G$ is actually $\tau_P^<$-open while by 1 and Proposition 19.3, $A$ is $\tau_P^<$-residual in $G$.

Set $G' = G$ if $\xi$ is a successor. If $\xi$ is limit let $I < \omega$ be minimal such that $\vartheta \leq \vartheta_I$, set $\mathcal{H}_{X,P}(\vartheta) = \bigcap_{i<I} X \setminus P_{(i)}$ and $G' = G \cap \mathcal{H}_{X,P}(\vartheta) = G \setminus \bigcup_{i<I} P_{(i)}$. We have

$$P \subseteq \mathcal{H}_{X,P}(\vartheta') \subseteq \mathcal{H}_{X,P}(\vartheta) \ (\vartheta \leq \vartheta' < \xi)$$

by Definition 14. It remains to show that $A$ is $\tau_P$-residual in $G'$. Note that $G'$ is $\tau_P^<$-open and that $A$ is $\tau_P^<$-residual in $G'$.

Suppose that $A$ is not $\tau_P$-residual in $G'$; that is we have a nonempty basic $\tau_P$-open set $\tilde{G} \subseteq G'$ such that $A \cap \tilde{G}$ is $\tau_P$-meager in $\tilde{G}$. By passing to a nonempty basic $\tau_P$-open subset we can assume that

$$\tilde{G} = G_0 \cap P_{(n)} \cap \bigcap_{i<n} \left(X \setminus P_{(i)}\right) = G_0 \cap P_{(n)} \setminus \bigcup_{i<n} P_{(i)}$$

where $G_0$ is basic $\tau_P^<$-open and $n < \omega$. Note that if $\xi$ is limit then $I \leq n$ by the choice of $G'$. So we can assume $G_0 \cap \bigcap_{i<n}(X \setminus P_{(i)}) \subseteq G'$.

We obtained that the $\mathbf{\Sigma}^0_\vartheta(\tau)$ set $X \setminus A$ is $\tau_P|_{P_{(n)} \setminus \bigcup_{i<n} P_{(i)}}$-residual in the $\tau_P$-open set $P_{(n)} \cap G_0 \cap \bigcap_{i<n} X \setminus P_{(i)}$. Thus by Proposition 19.4, $X \setminus A$ is $\tau_P^<|_{P_{(n)} \setminus \bigcup_{i<n} P_{(i)}}$-residual in the $\tau_P^<|_{P_{(n)}}$-open set $P_{(n)} \cap G_0 \cap \bigcap_{i<n} X \setminus P_{(i)}$. Since $\vartheta \leq \vartheta_n$, we can apply Proposition 20 for the $\mathbf{\Sigma}^0_{\vartheta_n}(\tau)$ set $\underline{W} = X \setminus A$, the $\mathbf{\Pi}^0_{\vartheta_n}(\tau)$ topological Hurewicz test pair $\{P_{(n)}, \tau_P^<\}$ and the $\tau_P^<$-open set $\underline{G} = G_0 \cap \bigcap_{i<n} X \setminus P_{(i)}$ satisfying $\underline{G} \cap P_{(n)} \neq \emptyset$. We get $X \setminus A$ is $\tau_P^<$-residual in a $\tau_P^<$-open set $H$ such that $\underline{G} \cap P_{(n)} \subseteq \mathrm{cl}_{\tau_P^<}(H \cap P_{(n)})$, in particular $H' = H \cap \underline{G} \neq \emptyset$ and $H' \subseteq G'$. Thus both $A$ and $X \setminus A$ are $\tau_P^<$-residual in the nonempty $\tau_P^<$-open set $H$, a contradiction. This completes the proof. □

**Remark 22.** It is very important to note that for the topological Hurewicz test sets $P$ satisfying the conditions of Theorem 21, $\mathcal{H}_{X,P}(\vartheta)$ $(\vartheta < \xi)$ *does not* depend on the topology $\tau$, it is a function of the presentation of $P$ and $\vartheta$. Moreover, $\mathcal{H}_{X,P}(\vartheta)$ $(\vartheta < \xi)$ is a $\tau_P^<$-dense $\tau_P^<$-open set.

The conditions of Theorem 21 concern the presentation of the $\mathbf{\Pi}^0_\xi(\tau)$ set $P$ instead of $P$ itself. This handicap seems to be inevitable. First, because up to our knowledge there are no results providing some method to build up $\mathbf{\Sigma}^0_\xi(\tau)$ sets from simpler sets on a canonical way, there is not even a canonical decomposition of $\mathbf{\Sigma}^0_2(\tau)$ sets into $\mathbf{\Pi}^0_1(\tau)$ sets. It is easy to see that by taking a wrong presentation the topology $\tau_P$ becomes wrong either, that is we cannot just condition our test set to be proper $\mathbf{\Pi}^0_\xi(\tau)$, a suitably chosen, not necessarily natural presentation must be involved. Second, because the only way to build up a proper $\mathbf{\Pi}^0_\xi(\tau)$ set for $\xi > 4$ is to use induction, so in view of our first reason one could hardly imagine Theorem 21 without some inductive condition on the presentation. Even if well explained, this handicap remains painful and this is responsible for most of the complication we have to face later. We close this section with some corollaries of Theorem 21, and in fact the first four statements can be considered as its reformulation. The fifth statement points out an obvious fact for every set $P$ of a topological Hurewicz test pair $\{P, \tau_P\}$.

**Corollary 23.** *For a $0 < \xi < \omega_1$, let $\{P, \tau_P\}$ be a $\mathbf{\Pi}^0_\xi(\tau)$ topological Hurewicz test pair as in Theorem 21. Let $G$ be a nonempty $\tau_P^<$-open set. Then the following hold.*

1. *If $\vartheta < \xi$, $A \subseteq X$ is $\mathbf{\Pi}^0_\vartheta(\tau)$ and $\tau_P^<$-residual in $G$ then $A$ is $\tau_P$-residual in $G$ if $\xi$ is a successor, while $A$ is $\tau_P$-residual in $G \cap \mathcal{H}_{X,P}(\vartheta)$ if $\xi$ is a limit ordinal.*
2. *If $A \subseteq X$ is $\mathbf{\Sigma}^0_\xi(\tau)$ and of $\tau_P^<$-second category in $G$ then $A$ is of $\tau_P$-second category in $G$.*
3. *If $\vartheta < \xi$, $A \subseteq X$ is $\mathbf{\Sigma}^0_\vartheta(\tau)$ and of $\tau_P$-second category in $G$ then $A$ is of $\tau_P^<$-second category in $G$ if $\xi$ is a successor, while $A$ is of $\tau_P^<$-second category in $G \cap \mathcal{H}_{X,P}(\vartheta)$ if $\xi$ is a limit ordinal.*
4. *If $A \subseteq X$ is $\mathbf{\Pi}^0_\xi(\tau)$ and $\tau_P$-residual in $G$ then $A$ is $\tau_P^<$-residual in $G$.*
5. *$P$ is a proper $\mathbf{\Pi}^0_\xi(\tau)$ set.*

Proof: Let $A$ be $\mathbf{\Pi}^0_\vartheta(\tau)$ and $\tau_P^<$-residual in $G$. By Theorem 21.1 and Proposition 19.1, $P$ is a $\tau_P^<$-residual $\mathbf{\Pi}^0_2(\tau_P^<)$ set so $A \cap P$ is $\tau_P^<|_P$-residual in $G \cap P$. By Proposition 19.3 the topologies $\tau_P^<|_P$ and $\tau_P|_P$ coincide so $A \cap P$ is $\tau_P|_P$-residual in $G \cap P$. Thus Definition 15.4(b) or Definition 15.4(c) applies and we conclude that $A$ is $\tau_P$-residual in $G$ if $\xi$ is a successor while $A$ is $\tau_P$-residual in $G \cap \mathcal{H}_{X,P}(\vartheta)$ if $\xi$ is a limit ordinal, which proves 1.

For 2, if $\xi = 1$ the statement follows from $\tau_P = \tau_P^< = \tau$. If $1 < \xi < \omega_1$ let $A = \bigcup_{i<\omega} A_i$ with $\mathbf{\Pi}^0_{\vartheta_i}(\tau)$ set $A_i$ $(i < \omega)$ where $\vartheta_i \twoheadrightarrow \xi$. By Theorem 21.1 and

Proposition 19.1, $P$ is a $\tau_P^<$-residual $\mathbf{\Pi}^0_2(\tau_P^<)$ set. If $A$ is of $\tau_P^<$-second category in $G$ then for an $i < \omega$, $A_i \cap P$ is $\tau_P^<|_P$-residual in $G' \cap P$ for some basic $\tau_P^<$-open set $G' \subseteq G$. Since by Proposition 19.3 the topologies $\tau_P^<|_P$ and $\tau_P|_P$ coincide, we have $A_i \cap P$ is $\tau_P|_P$-residual in the basic $\tau_P$-open set $G'$ with $G' \cap P \neq \emptyset$ so by Definition 15.4(b) or Definition 15.4(c), $A_i$ is $\tau_P$-residual in some nonempty $\tau_P$-open set $G'' \subseteq G'$ thus $A$ is of $\tau_P$-second category in $G$, as required.

Statements 3 and 4 follow from 1 and 2 by taking complements and using that $\mathcal{H}_{X,P}(\vartheta)$ $(\vartheta < \xi)$ is a $\tau_P^<$-dense $\tau_P^<$-open set, as pointed out in Remark 22.

For 5, suppose that $P$ is $\mathbf{\Sigma}^0_\xi(\tau)$. From Proposition 20 for $W = P$ we get $P$ is of $\tau_P$-second category in $X$. But by Definition 15.3, $P$ is $\tau_P$-nowhere dense, a contradiction. This completes the proof. $\square$

There is an asymmetry in our approach to topological Hurewicz test sets: the test set is of some multiplicative class and the sets tested are of the dual additive class. The reason for this is that $\mathbf{\Sigma}^0_\xi$ is closed under taking countable union while $\mathbf{\Pi}^0_\xi$ is not. However, there is a testing theorem like Theorem 21 for special $\mathbf{\Sigma}^0_\xi$ sets but the statement of this theorem cannot go beyond Corollary 23. So we do not work for that.

## 5. Intersection criteria

Toward the proof of Theorem 1 we need to find many topological Hurewicz test pairs. For this we analyze the conditions of Theorem 21. It turns out that the conditions of Theorem 21 are combinatorial, they are the same as requiring that countably many intersections are nonempty. We write up these intersections in Definition 24. Our purpose is to show in particular that if Theorem 21 proves that a set $P$ is a topological Hurewicz test set then $P$ remains a test set and satisfies the conditions of Theorem 21 if the initial topology $\tau$ of the Polish space is changed.

**Definition 24.** Let $0 < \xi < \omega_1$ and $[P, (P_t)_{t \in T}]$ be a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation. If $\xi = 1$, set $\mathcal{C}_1(X, \tau, P) = \{(X \setminus P, G)\colon\ G \in \tau \setminus \{\emptyset\}\}$. If $\mathcal{C}_\vartheta(X', \tau', P')$ is defined for every $\vartheta < \xi$, Polish space $(X', \tau')$ and $\mathbf{\Pi}^0_\vartheta(\tau)$ set with presentation $P'$ then let $\vartheta_i \twoheadrightarrow \xi$ and set

$$\mathcal{C}_\xi(X, \tau, P) = \big\{(X \setminus P, G)\colon G \in \tau_P^< \setminus \{\emptyset\}\big\} \cup \bigcup_{i<\omega} \mathcal{C}_{\vartheta_i}\Big(X, \bigvee_{j<\omega,\ j \neq i} \tau_{P_{(j)}}, P_{(i)}\Big).$$

We say that $[P, (P_t)_{t \in T}]$ *satisfies* $\mathsf{C}_\xi$ *in* $(X, \tau)$ if

$$\forall\, (C, G) \in \mathcal{C}_\xi(X, \tau, P)\ (C \cap G \neq \emptyset).$$

**Proposition 25.** *Fix an ordinal $\xi$ satisfying $0<\xi<\omega_1$. If a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation $[P,(P_t)_{t\in T}]$ satisfies $\mathsf{C}_\xi$ in $(X,\tau)$ then $\{P,\tau_P\}$ satisfies the conditions of Theorem 21, so in particular $\{P,\tau_P\}$ is a $\mathbf{\Pi}^0_\xi(\tau)$ topological Hurewicz test pair.*

Proof: We prove the statement by induction on $\xi$. For $\xi=1$, $\mathsf{C}_1$ means that $P$ is $\tau$-nowhere dense in $X$, that is $\{P,\tau_P\}$ is a $\mathbf{\Pi}^0_1(\tau)$ topological Hurewicz test pair by Theorem 21. Suppose now that the statement holds for $\vartheta<\xi$ and let $\vartheta_i \twoheadrightarrow \xi$. By $\mathsf{C}_\xi$, $X\setminus P=\bigcup_{i<\omega}P_{(i)}$ is $\tau_P^<$-dense in $X$ and $[P_{(i)},(P_{i^\frown t})_{t\in T_{(i)}}]$ satisfies $\mathsf{C}_{\vartheta_i}$ in the Polish space $(X,\bigvee_{j<\omega,\ j\neq i}\tau_{P_{(j)}})$. By Definition 17 we have

$$\tau_P^< = \bigvee_{j<\omega}\tau_{P_{(j)}} = \Big(\bigvee_{j<\omega,\ j\neq i}\tau_{P_{(j)}}\Big)_{P_{(i)}},$$

so by the induction hypothesis $\{P_{(i)},\tau_P^<\}$ is a $\mathbf{\Pi}^0_{\vartheta_i}(\tau)$ topological Hurewicz test pair $(i<\omega)$. Thus the conditions of Theorem 21 are satisfied, Theorem 21.2 can be applied and we conclude that $\{P,\tau_P\}$ is a $\mathbf{\Pi}^0_\xi(\tau)$ topological Hurewicz test pair. □

We need that if $P$ lives in a product space but it is nontrivial only on one coordinate then $\mathsf{C}_\xi$ puts conditions also only on one coordinate.

**Proposition 26.** *Let $0<\xi<\omega_1$ and let $[P,(P_t)_{t\in T}]$ be a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation in the Polish space $(X,\tau)$. Let $(Y,\sigma)$ be a Polish space and set $Q=P\times Y$, $Q_t=P_t\times Y$ $(t\in T)$. Then $[Q,(Q_t)_{t\in T}]$ is a $\mathbf{\Pi}^0_\xi(\tau\times\sigma)$ set with presentation and for every $(C,G)\in\mathcal{C}_\xi(X\times Y,\tau\times\sigma,Q)$, $C$ is of product form and it is nontrivial only on the $X$ coordinate, i.e. $C=\mathrm{Pr}_X(C)\times Y$.*

Proof: The statement easily follows by induction on $\xi$. □

The next claim gives that $\mathsf{C}_\xi$ remains true if the initial topology gets coarser. Compare this with Lemma 16.

**Proposition 27.** *Let $\tau'$ be a Polish topology on $X$ refining $\tau$. If for some $0<\xi<\omega_1$ a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation $[P,(P_t)_{t\in T}]$ satisfies $\mathsf{C}_\xi$ in $(X,\tau')$ then it satisfies $\mathsf{C}_\xi$ in $(X,\tau)$ as well.*

Proof: We prove by induction on $\xi$ that $\mathcal{C}_\xi(X,\tau,P)\subseteq\mathcal{C}_\xi(X,\tau',P)$ $(0<\xi<\omega_1)$. From this the statement follows.

For $\xi=1$ we have $\tau\subseteq\tau'$ and so $\mathcal{C}_1(X,\tau,P)\subseteq\mathcal{C}_1(X,\tau',P)$. Suppose now that the statement holds for $\vartheta<\xi$ and let $\vartheta_i\twoheadrightarrow\xi$. Since $\tau\subseteq\tau'$ we also have $\tau_P^<\subseteq{\tau'}_P^<$, $\tau_{P_{(i)}}\subseteq\tau'_{P_{(i)}}$ $(i<\omega)$ so

$$\{(X\setminus P,G)\colon G\in\tau_P^<\}\subseteq\{(X\setminus P,G)\colon G\in{\tau'}_P^<\}$$

and by the induction hypothesis,

$$\mathcal{C}_{\vartheta_i}\Big(X, \bigvee_{j<\omega,\ j\neq i} \tau_{P_{(j)}}, P_{(i)}\Big) \subseteq \mathcal{C}_{\vartheta_i}\Big(X, \bigvee_{j<\omega,\ j\neq i} \tau'_{P_{(j)}}, P_{(i)}\Big).$$

This proves $\mathcal{C}_\xi(X,\tau,P) \subseteq \mathcal{C}_\xi(X,\tau',P)$ and completes the proof. $\square$

From now on in this section we work to prove the main lemma of Theorem 1. The technique of the proof is to exploit the low Borel class Hurewicz test sets appearing in the construction of a $\mathbf{\Pi}^0_\xi(\tau)$ test set. For this we need some more topologies.

**Definition 28.** Let $0<\xi<\omega_1$ and let $[P,(P_t)_{t\in T}]$ be a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation which satisfies $\mathsf{C}_\xi$ in $(X,\tau)$. We define the topologies

$$\tau(n) = \bigvee_{i<n} \tau_{P_{(i)}} \vee \bigvee_{n<i<\omega} \tau^{<}_{P_{(i)}} \ (n<\omega) \tag{2}$$

and

$$\tau_P(n) = \bigvee_{i\leq n} \tau_{P_{(i)}} \vee \bigvee_{n<i<\omega} \tau^{<}_{P_{(i)}} \ (n<\omega). \tag{3}$$

**Lemma 29.** *Let $0<\xi<\omega_1$, $\vartheta_i \twoheadrightarrow \xi$ and let $[P,(P_t)_{t\in T}]$ be a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation satisfying $\mathsf{C}_\xi$ in $(X,\tau)$. Then for every $n<\omega$, with the notation of Definition 28, $\{P_{(n)},\tau_P(n)\}$ satisfies the conditions of Theorem 21 in the Polish space $(X,\tau(n))$. Also for every $n<\omega$, $\{P_{(n)},\tau_P(n)\}$ is a $\mathbf{\Pi}^0_{\vartheta_n}(\tau)$ topological Hurewicz test pair in $(X,\tau)$. Moreover we have*

$$\tau_P(0)^{<} = \bigvee_{i<\omega} \tau^{<}_{P_i}$$

*and $\tau_P(n+1)^{<} = \tau_P(n)$ $(n<\omega)$.*

Proof: Since $[P,(P_t)_{t\in T}]$ satisfies $\mathsf{C}_\xi$, $[P_{(n)},(P_{n^\frown t})_{t\in T_{(n)}}]$ satisfies $\mathsf{C}_{\vartheta_n}$ in the Polish space

$$\Big(X, \bigvee_{i<\omega,\ i\neq n} \tau_{P_{(i)}}\Big).$$

Then by Proposition 27, $[P_{(n)},(P_{n^\frown t})_{t\in T_{(n)}}]$ satisfies $\mathsf{C}_{\vartheta_n}$ in the Polish space $(X,\tau(n))$. By (2) and (3) we have

$$\tau_P(n) = \tau_{P_{(n)}} \vee \bigvee_{i<n} \tau_{P_{(i)}} \vee \bigvee_{n<i<\omega} \tau^{<}_{P_{(i)}} = \tau(n)_{P_{(n)}}$$

so by Proposition 27, $\{P_{(n)}, \tau_P(n)\}$ satisfies the conditions of Theorem 21 in the Polish space $(X, \tau(n))$. So it is a $\mathbf{\Pi}^0_{\vartheta_n}(\tau)$ topological Hurewicz test pair in $(X, \tau(n))$ hence by Lemma 16 in $(X, \tau)$, as well. Similarly,

$$\tau_P(n+1)^< = \tau^<_{P_{(n+1)}} \vee \bigvee_{i<n+1} \tau_{P_{(i)}} \vee \bigvee_{n+1<i<\omega} \tau^<_{P_{(i)}} = \tau_P(n) \ (n<\omega)$$

and

$$\tau_P(0)^< = \tau^<_{P_{(0)}} \vee \bigvee_{0<i<\omega} \tau^<_{P_{(i)}} = \bigvee_{i<\omega} \tau^<_{P_{(i)}},$$

which completes the proof. $\square$

The next lemma is an application of our newly found Hurewicz test sets.

**Lemma 30.** *Let $2 < \xi < \omega_1$ and $\vartheta_i \twoheadrightarrow \xi$. Let $[P, (P_t)_{t\in T}]$ be a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation which satisfies $\complement_\xi$. Fix an $n < \omega$. If $A$ is a $\mathbf{\Sigma}^0_\vartheta(\tau)$ set where $\vartheta < \vartheta_{n+1}$ and $A$ is $\tau_P(n)$-meager in a $\tau_P(n)$-open set $G$ then*

1. *$A$ is $\tau_P(n+1)$-meager in $G \cap \mathcal{H}_{X,P_{(n+1)}}(\vartheta)$ if $\xi = \xi'+1$ for a limit ordinal $\xi'$;*
2. *$A$ is $\tau_P(n+1)$-meager in $G$ if $\xi = \xi'+1$ for no limit ordinal $\xi'$.*

Proof: By Lemma 29 we have $\{P_{(n+1)}, \tau_P(n+1)\}$ is a $\mathbf{\Pi}^0_{\vartheta_{n+1}}(\tau)$ topological Hurewicz test pair in $(X, \tau)$ and $\tau_P(n+1)^< = \tau_P(n)$. So by Corollary 23.3 our $A$ cannot be of $\tau_P(n+1)$-second category in $G \cap \mathcal{H}_{X,P_{(n+1)}}(\vartheta)$ in case 1 or in $G$ in case 2. This completes the proof. $\square$

In order to avoid $\mathbf{\Pi}^0_\vartheta(\tau)$ sets when $3 \le \vartheta < \omega_1$ we have to reduce complicated sets to $\mathbf{\Pi}^0_2(\tau)$ sets. This is the motivation of the following concept.

**Definition 31.** Let $T \subseteq \omega^{<\omega}$ be a tree. We say that a subtree $T' \subseteq T$ is *even* if

$$(4) \quad t \in T',\ |t| \text{ odd} \implies \big\{t|_{|t|-1}{}^\frown i : i<\omega\big\} \cap T' = \{t\} \ \& \\ \big\{t^\frown i : i<\omega\big\} \cap T \neq \emptyset \ \& \ \big\{t^\frown i : i<\omega\big\} \cap T \subseteq T'.$$

We say that $T' \subseteq T$ is *even-complete* if it is a maximal even subtree of $T$.

If $T' \subseteq T$ is an even subtree, $t \in T' \setminus \mathfrak{T}(T')$ with $|t|$ even then $t^{+T} \in T'$ denotes the unique extension of $t$ with $|t^{+T}| = |t|+1$.

Our first observation immediately follows from the definition.

**Lemma 32.** *With the notation of Definition 31, if $T' \subseteq T$ is even-complete, $t \in T'$ and $|t|$ is even $T'_t$ is an even-complete subtree of $T_t$.*

**Lemma 33.** *For some $0 < \xi < \omega_1$ let $[P,(P_t)_{t\in T}]$ be a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation. If $T' \subseteq T$ is an even-complete subtree and $x \notin P_t$ $(t \in \mathfrak{T}(T'))$ then $x \notin P$.*

Proof: We prove the statement by induction on $\xi$. For $\xi = 1$ and $\xi = 2$ the only even-complete subtree of $T$ is $T' = \{\emptyset\}$ that is $x \notin P_\emptyset = P$, as stated. Suppose now that $3 \le \xi$ and the statement holds for $\vartheta < \xi$. By maximality we have $T' \neq \emptyset$, hence there is a unique $i < \omega$ such that $(i) \in T'$. By Lemma 32, $T'_{i\frown j}$ is an even-complete subtree of $T_{i\frown j}$ $(j < \omega,\ i\frown j \in T)$ so by the induction hypothesis $x \notin P_{i\frown j}$ $(j < \omega)$. That is $x \in P_{(i)}$ and so $x \notin P$, which completes the proof. □

Now we can prove the main result of the section.

**Proposition 34.** *Let $\xi$ be a successor ordinal such that $2 < \xi < \omega_1$, say $\xi = \xi'+1$. Let $[P,(P_t)_{t\in T}]$ be a $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation which satisfies $\mathsf{C}_\xi$ in $(X,\tau)$. For every $n < \omega$ let $A^n$ be a $\mathbf{\Pi}^0_{\xi'}(\tau)$ set such that $A^n \cap P_{(n)} = \emptyset$ $(n < \omega)$. Then $P \setminus \bigcup_{n<\omega} A^n \neq \emptyset$.*

Proof: Let $\xi_n \le \xi'$ be such that we have a presentation $[A^n,(A^n_t)_{t\in T^n}]$ of $A^n$ as a $\mathbf{\Pi}^0_{\xi_n}(\tau)$ set $(n < \omega)$. Let $\vartheta_i \twoheadrightarrow \xi'$. Take maps $\eta_1\colon \omega \to \omega$ and $\eta_2\colon \omega \to \omega^{<\omega}$ such that

$$\eta = (\eta_1,\eta_2)\colon \omega \to \bigcup_{n<\omega} \{n\} \times T^n$$

is a bijection satisfying

$$t, t' \in T^n,\ t \subseteq t' \implies \eta^{-1}(n,t) \le \eta^{-1}(n,t')\ (n < \omega) \tag{5}$$

and

$$\eta_1(n) \le n\ (n < \omega). \tag{6}$$

We construct inductively a basic $\tau_P(n)$-open set $G_n$ $(n < \omega)$ and an even-complete subtree $F^n \subseteq T^n$ $(n < \omega)$ such that

(7) $\quad \mathrm{cl}_{\tau_P(n)}(G_{n+1}) \subseteq G_n\ (n < \omega)$;

(8) $\quad G_n \cap P_{(n)} = \emptyset\ (n < \omega)$;

(9) $\quad G_n \cap A^n$ is $\tau_P(n)$-meager $(n < \omega)$;

(10) $\quad$ if $\eta_2(n) \in F^{\eta_1(n)} \setminus \mathfrak{T}(F^{\eta_1(n)})$ and $|\eta_2(n)|$ is even, then

$$G_n \cap A^{\eta_1(n)}_{\eta_2(n)^{+F^{\eta_1(n)}}\frown i} \text{ is } \tau_P(n)\text{-meager } (n, i < \omega);$$

(11) $\quad$ if $\eta_2(n) \in \mathfrak{T}(F^{\eta_1(n)})$ then $G_n \subseteq X \setminus A^{\eta_1(n)}_{\eta_2(n)}$ $(n < \omega)$.

Since the topology $\tau_P^{<}$ is finer than (or equals) $\tau_P(n)$ $(n<\omega)$, $\mathrm{cl}_{\tau_P(n)}(G_{n+1}) \subseteq G_n$ implies that $\mathrm{cl}_{\tau_P^{<}}(G_{n+1}) \subseteq G_n$ $(n<\omega)$, so we have $\bigcap_{n<\omega} G_n \neq \emptyset$ by (7) and $\bigcap_{n<\omega} G_n \subseteq P$ by (8). By Lemma 33, (11) gives

$$\bigcap_{n<\omega} G_n \subseteq P \setminus \bigcup_{n<\omega} A^n.$$

To start with, set $F^n = \{\emptyset\}$ $(n<\omega)$. In the following construction we will successively grow the trees $F^n$, so a node can be terminal after some intermediate step but not in the final tree. To be concrete, the $m^{\text{th}}$ step of the construction will have two parts: in the first part we grow $F^m$ at $\emptyset$ if and only if $2<\xi_m$; while in the second part, with $\eta(m)=(n,s)$, we grow $F^n$ at $s$ if and only if $s\neq\emptyset$ and $[A^n_s,(A^n_{s^\frown t})_{t\in T^n_s}]$ is a $\mathbf{\Pi}^0_\vartheta(\tau)$ set with presentation where $2<\vartheta$. We will declare when a tree does not grow any more from a node, so that this node remains terminal.

For every $0<n<\omega$ after completing the $n-1^{\text{th}}$ step of the construction we define the ordinal

$$\begin{aligned}\rho_n = \sup\big\{\vartheta<\omega_1\colon \exists\, m<n\ \exists\, s\in F^m\setminus\{\emptyset\}\\ \big([A^m_s,(A^m_{s^\frown t})_{t\in T^m_s}]\ \text{ is a } \mathbf{\Pi}^0_\vartheta(\tau) \text{ set with presentation}\big)\big\}\end{aligned}$$

if the sup is taken on a nonempty set, else we set $\rho_n=0$. Observe that if $[A^m_s,(A^m_{s^\frown t})_{t\in T^m_s}]$ is a $\mathbf{\Pi}^0_\vartheta(\tau)$ set with presentation for some $m<n$ and $s\in F^m\setminus\{\emptyset\}$ then $\vartheta\le\vartheta_{s|_{\{0\}}}=\vartheta_{\emptyset+F^m}<\xi_m$. Since after the $n-1^{th}$ step of the construction $F^m\neq\{\emptyset\}$ holds only for finitely many $m<\omega$ and $[A^m,(A^m_t)_{t\in T^m}]$ is a $\mathbf{\Pi}^0_{\xi_m}(\tau)$ set with presentation where $\xi_m\le\xi'$ $(m<\omega)$ we get $\rho_n<\xi'$ $(0<n<\omega)$.

If $\xi'$ is a limit ordinal we will choose $G_n$ $(0<n<\omega)$ such that in addition to (7) it will satisfy

$$G_n \subseteq G_{n-1}\cap\mathcal{H}_{X,P_{(n)}}(\rho_n)\ (0<n<\omega). \tag{12}$$

We prove first that for every $0<N<\omega$,

(13) if $k,m<N$, $s\in F^m\setminus\{\emptyset\}$ after the $k^{\text{th}}$ step of the construction and $A^m_s$ is $\tau_P(k)$-meager in $G_k$ then $A^m_s$ is $\tau_P(N)$-meager in $G_N$.

First we show that $A^m_s$ is $\tau_P(k+1)$-meager in $G_{k+1}$. We apply Lemma 30 for $n=k$, $A=A^m_s$, $G=G_k$. If $[A^m_s,(A^m_{s^\frown t})_{t\in T^m_s}]$ is a $\mathbf{\Pi}^0_\vartheta(\tau)$ set with presentation then $\vartheta\le\rho_{k+1}$. Thus if $\xi'$ is limit we have $\mathcal{H}_{X,P_{(k+1)}}(\rho_{k+1})\subseteq\mathcal{H}_{X,P_{(k+1)}}(\vartheta)$ so by (7) if $\xi'$ is a successor and by (12) if $\xi'$ is a limit we get $A^m_s$ is $\tau_P(k+1)$-meager in $G_{k+1}$. By repeating the argument $N-k-1$ times the statement follows.

We turn to the construction. For $n=0$, by (5) and (6) we have $\eta(0)=(0,\emptyset)$. To find our $G_0$ observe that $X\setminus A^0$ is a $\mathbf{\Sigma}^0_{\xi'}(\tau)$ set containing $P_{(0)}$. By Lemma 29, $\{P_{(0)},\tau_P(0)\}$ satisfies the conditions of Theorem 21 in $(X,\tau(0))$, in particular it is a $\mathbf{\Pi}^0_{\xi'}(\tau)$ topological Hurewicz test pair in $(X,\tau)$. By Theorem 21.1, $P_{(0)}$ and hence $X\setminus A^0$ is $\tau_P(0)^<$-residual, by Corollary 23.2, $X\setminus A^0$ is of $\tau_P(0)$-second category, that is $A^0$ is $\tau_P(0)$-meager in some nonempty basic $\tau_P(0)$-open set $G$.

If $A^0$ is $\mathbf{\Pi}^0_\vartheta(\tau)$ with $3\le\vartheta<\omega_1$ we have $A^0=\bigcap_{i<\omega}X\setminus A^0_{(i)}$, so for some $k_0<\omega$ and nonempty basic $\tau_P(0)$-open set $G'\subseteq G$ we have $X\setminus A^0_{(k_0)}$ is $\tau_P(0)$-meager in $G'$. Put $k_0^\frown i\in F^0$ $(i<\omega)$. So if we take $G_0\subseteq G'$ then (10) holds for $n=0$ and (11) does not apply. Since $P_{(0)}$ is $\tau_P(0)$-nowhere dense we can pass to some basic $\tau_P(0)$-open subset $G_0\subseteq G'$ such that $G_0\cap P_{(0)}=\emptyset$; so (8)–(11) are satisfied for $n=0$.

Else we have $A^0$ is $\mathbf{\Pi}^0_1(\tau)$ or $\mathbf{\Pi}^0_2(\tau)$. Since $\tau_P(0)$ is finer than $\tau$, the set $A^0$, which is $\tau_P(0)$-meager in $G$, is actually $\tau_P(0)$-nowhere dense in $G$. In this case choose the basic $\tau_P(0)$-open set $G_0$ so that $G_0\subseteq G\setminus(P_{(0)}\cup A^0)$ and $F^0$ does not grow at all, that is $\eta_2(0)=\emptyset\in\mathfrak{T}(F^0)$. Now (10) does not apply and (11) holds; so we again have (8)–(11) for $n=0$. This finishes the first part of the construction for $n=0$ and there is no second part.

Suppose that $G_n$ $(n<N)$ is already defined such that (8)–(13) hold for $n<N$ and (7) holds for $n<N-1$; we find our $G_N$. If $\xi'$ is a limit ordinal by Proposition 19.2 and Theorem 21.1, $P_{(N)}\subseteq\mathcal{H}_{X,P_{(N)}}(\rho_N)$ implies that $\mathcal{H}_{X,P_{(N)}}(\rho_N)$ is a $\tau_P(N)^<$-residual $\tau_P(N)^<$-open set. By Lemma 29, $\tau_P(N)^<=\tau_P(N-1)$ so $G_{N-1}$ is $\tau_P(N)^<$-open hence $G_{N-1}\cap\mathcal{H}_{X,P_{(N)}}(\rho_N)\neq\emptyset$. Thus by passing to a basic $\tau_P(N)^<$-open subset we can assume that $G_{N-1}\subseteq\mathcal{H}_{X,P_{(N)}}(\rho_N)$. Then (12) will hold for $n=N$ if $G_N\subseteq G_{N-1}$.

Again, $X\setminus A^N$ is a $\mathbf{\Sigma}^0_{\xi'}(\tau)$ set containing $P_{(N)}$. By Lemma 29, $\{P_{(N)},\tau_P(N)\}$ satisfies the conditions of Theorem 21 in $(X,\tau(N))$, it is a $\mathbf{\Pi}^0_{\xi'}(\tau)$ topological Hurewicz test pair, $\tau_P(N)^<=\tau_P(N-1)$ so $G_{N-1}$ is $\tau_P(N)^<$-open. By Theorem 21.1, $P_{(N)}$ and hence $X\setminus A^N$ is $\tau_P(N)^<$-residual in $G_{N-1}$, so by Corollary 23.2, $X\setminus A^N$ is of $\tau_P(N)$-second category in $G_{N-1}$, that is $A^N$ is $\tau_P(N)$-meager in some nonempty basic $\tau_P(N)$-open set $G\subseteq G_{N-1}$.

If $[A^N,(A^N_t)_{t\in T^N}]$ is not a $\mathbf{\Pi}^0_1(\tau)$ or a $\mathbf{\Pi}^0_2(\tau)$ set with presentation then we have $A^N=\bigcap_{i<\omega}X\setminus A^N_{(i)}$. So for some $k_N<\omega$ and nonempty basic $\tau_P(N)$-open set $G'\subseteq G$ we have $X\setminus A^N_{(k_N)}$ is $\tau_P(N)$-meager in $G'$. We put $k_N^\frown i\in F^N$, $(i<\omega)$. If $[A^N,(A^N_t)_{t\in T^N}]$ is a $\mathbf{\Pi}^0_1(\tau)$ or a $\mathbf{\Pi}^0_2(\tau)$ set with presentation then we set $G'=G$, and $F^N$ does not grow at all so $\emptyset\in\mathfrak{T}(F^N)$.

Since $P_{(N)}$ is $\tau_P(N)$-nowhere dense, we can pass to some basic $\tau_P(N)$-open subset $G'' \subseteq G'$ such that $G'' \cap P_{(N)} = \emptyset$ and $\mathrm{cl}_{\tau_P(N)}(G'') \subseteq G_{N-1}$. So (7)–(9) hold for every basic $\tau_P(N)$-open set $G_N \subseteq G''$. The first part on the $N^{\text{th}}$ step of the construction is complete. We turn to the second part.

If $\eta_2(N) \notin \mathfrak{T}(F^{\eta_1(N)})$ then set $G_N = G''$. If $|\eta_2(N)|$ is odd neither (10) nor (11) apply so the inductive step is complete. If $|\eta_2(N)|$ is even then (11) does not apply so it remains to show (10). If $\eta_2(N) = \emptyset$ then after the first part of the $\eta_1(N)^{\text{th}}$ step of the construction we had that $[A^{\eta_1(N)}, (A_t^{\eta_1(N)})_{t\in T^{\eta_1(N)}}]$ is neither a $\mathbf{\Pi}_1^0(\tau)$ nor a $\mathbf{\Pi}_2^0(\tau)$ set with presentation and

$$X \setminus A^{\eta_1(N)}_{(k_{\eta_1(N)})} = X \setminus A^{\eta_1(N)}_{\emptyset + F^{\eta_1(N)}}$$

is $\tau_P(\eta_1(N))$-meager in $G_{\eta_1(N)}$; hence $G_{\eta_1(N)} \cap A^{\eta_1(N)}_{\emptyset + F^{\eta_1(N)} \frown i}$ is $\tau_P(\eta_1(N))$-meager $(i < \omega)$. By (6) we have $\eta_1(N) < N$ so (13) for $k = m = \eta_1(N)$ and $s = \emptyset + F^{\eta_1(N)} \frown i$ gives that $A^{\eta_1(N)}_{\emptyset + F^{\eta_1(N)} \frown i}$ is $\tau_P(N)$-meager in $G_{\eta_1(N)}$ hence also in $G_N$ $(i < \omega)$, as required.

If $\eta_2(N) \notin \mathfrak{T}(F^{\eta_1(N)})$, $|\eta_2(N)|$ is even but $\eta_2(N) \neq \emptyset$ we show that $\eta_2(N)$ will never be a node of $F^{\eta_1(N)}$. Let $u$ be the terminal node of $F^{\eta_1(N)}$ on the branch of $\eta_2(N)$ in $T^{\eta_1(N)}$. By (5) there is an $m < N$ such that $\eta_1(m) = \eta_1(N)$ and $\eta_2(m) = u$. After the $m^{\text{th}}$ step of the construction $u$ remained a terminal node of $F^{\eta_1(N)}$, that is according to our growing convention $F^{\eta_1(N)}$ never grows from $u$ so $\eta_2(N)$ will never be a node of $F^{\eta_1(N)}$. So again neither (10) nor (11) apply and the inductive step is complete.

If $\eta_2(N) \in \mathfrak{T}(F^{\eta_1(N)})$ and $\eta_2(N) = \emptyset$ then by (6) and by our growing convention $[A^{\eta_1(N)}, (A_t^{\eta_1(N)})_{t\in T^{\eta_1(N)}}]$ is a $\mathbf{\Pi}_1^0(\tau)$ or a $\mathbf{\Pi}_2^0(\tau)$ set with presentation, in the $\eta_1(N)^{\text{th}}$ step of the construction by (11) we obtained $G_{\eta_1(N)} \subseteq X \setminus A^{\eta_1(N)}$ so $G'' \subseteq X \setminus A^{\eta_1(N)}$, as well. So (10) does not apply and (11) holds, thus $G_N = G''$ completes the inductive step.

If $\eta_2(N) \in \mathfrak{T}(F^{\eta_1(N)})$, $\eta_2(N) \neq \emptyset$ then we do the following. Let $k < \omega$ be such that $\eta_1(k) = \eta_1(N)$ and $\eta_2(k) = \eta_2(N)|_{|\eta_2(N)|-2}$. By (5) we have $k < N$. Since $\eta_2(N) \neq \emptyset$, $[A^{\eta_1(N)}, (A_t^{\eta_1(N)})_{t\in T^{\eta_1(N)}}]$ is neither a $\mathbf{\Pi}_1^0(\tau)$ nor a $\mathbf{\Pi}_2^0(\tau)$ set with presentation. We had (10) in the $k^{\text{th}}$ step of the construction so by (13) for $k$, $m = \eta_1(N)$ and $s = \eta_2(N)$, $G'' \subseteq G_{N-1}$ implies that $A^{\eta_1(N)}_{\eta_2(N)}$ is $\tau_P(N)$-meager in $G''$. If $[A^{\eta_1(N)}_{\eta_2(N)}, (A^{\eta_1(N)}_{\eta_2(N)\frown t})_{t\in T^{\eta_1(N)}_{\eta_2(N)}}]$ is a $\mathbf{\Pi}_1^0(\tau)$ or a $\mathbf{\Pi}_2^0(\tau)$ set with presentation then since $\tau_P(N)$ is finer than $\tau$, $A^{\eta_1(N)}_{\eta_2(N)}$ is actually $\tau_P(N)$-nowhere dense in $G''$. So we can find a nonempty basic $\tau_P(N)$-open set $G_N \subseteq G'' \setminus A^{\eta_1(N)}_{\eta_2(N)}$. We do not grow $F^{\eta_1(N)}$ from the node $\eta_2(N)$, so (10) does not apply and (11) holds.

If $[A^{\eta_1(N)}_{\eta_2(N)}, (A^{\eta_1(N)}_{\eta_2(N)\frown t})_{t\in T^{\eta_1(N)}_{\eta_2(N)}}]$ is a $\mathbf{\Pi}^0_\vartheta(\tau)$ set with presentation for some $3 \le \vartheta < \xi'$ since $A^{\eta_1(N)}_{\eta_2(N)} = \bigcap_{i<\omega} X \setminus A^{\eta_1(N)}_{\eta_2(N)\frown i}$, for some $l_N < \omega$ and nonempty basic $\tau_P(N)$-open set $G_N \subseteq G''$ we have $X \setminus A^{\eta_1(N)}_{\eta_2(N)\frown l_N}$ is $\tau_P(N)$-meager in $G_N$. We put $\eta_2(N)\frown l_N \frown i \in F^{\eta_1(N)}$, $(i < \omega)$, then (10) holds and (11) does not apply. This completes the second part of the inductive step and finishes the proof. □

## 6. Constructing coverings

In order to proceed we need to construct at least one concrete $\mathbf{\Pi}^0_\xi(\tau)$ topological Hurewicz test pair for every $\xi < \omega_1$. We do this in the Polish space $(C, \tau_C)$.

**Definition 35.** We set $(C_1, \tau_{C_1}) = (C, \tau_C)$,

$$P_1 = \{x \in C_1 \colon \forall\, m \in \omega\ (x(m) = 1)\},$$

$T_1 = \{\emptyset\}$ and $P^1_\emptyset = P_1$. Suppose that the spaces $(C_\vartheta, \tau_{C_\vartheta})$ and the $\mathbf{\Pi}^0_\vartheta(\tau_{C_\vartheta})$ sets with presentation $[P_\vartheta, (P^\vartheta_t)_{t\in T_\vartheta}]$ are defined for every $\vartheta < \xi$. Then with $\vartheta_i \twoheadrightarrow \xi$ let

$$C_\xi = \prod_{i<\omega} C_{\vartheta_i},\ \tau_{C_\xi} = \prod_{i<\omega} \tau_{C_{\vartheta_i}},$$

$$P_\xi = \{x \in C_\xi \colon \forall\, i < \omega\ (x(i,.) \in C_{\vartheta_i} \setminus P_{\vartheta_i})\}, \tag{14}$$

$$T_\xi = \{n\frown t \colon t \in T_{\vartheta_n},\ n < \omega\}, \tag{15}$$

$$P^\xi_{n\frown t} = \prod_{i<n} C_{\vartheta_i} \times P^{\vartheta_n}_t \times \prod_{n<i<\omega} C_{\vartheta_i}\ (t \in T_{\vartheta_n},\ n < \omega). \tag{16}$$

**Proposition 36.** *Let $0 < \xi < \omega_1$ and $\vartheta_i \twoheadrightarrow \xi$. The Polish space $(C_\xi, \tau_{C_\xi})$ is homeomorphic to $(C, \tau_C)$. The $\mathbf{\Pi}^0_\xi(\tau)$ set with presentation $[P_\xi, (P^\xi_t)_{t\in T_\xi}]$ satisfies $\mathsf{C}_\xi$ in $(C_\xi, \tau_{C_\xi})$, so it is a $\mathbf{\Pi}^0_\xi(\tau_{C_\xi})$ topological Hurewicz test pair in $(C_\xi, \tau_{C_\xi})$. We have $\tau^<_{P_\xi} = \prod_{i<\omega} \tau_{P_{\vartheta_i}}$ and*

$$\tau_{P_\xi} = \tau^<_{P_\xi}[\{U_{\xi,n} \colon n < \omega\}]\ (1 < \xi < \omega_1)$$

*where*

$$\begin{aligned} U_{\xi,n} &= \prod_{i<n} \big(C_{\vartheta_i} \setminus P_{\vartheta_i}\big) \times P_{\vartheta_n} \times \prod_{n<i<\omega} C_{\vartheta_i} \\ &\subseteq \prod_{i<n} C_{\vartheta_i} \times C_{\xi_n} \times \prod_{n<i<\omega} C_{\vartheta_i} = C_\xi\ (1 < \xi < \omega_1,\ n < \omega). \end{aligned} \tag{17}$$

Proof: It is obvious that $(C_\xi, \tau_{C_\xi})$ is homeomorphic to $(C, \tau_C)$. We prove the other statements by induction on $\xi$. For $\xi = 1$, $P_1$ is a single point so it is $\tau$-nowhere dense in $C_1$, as stated. Remember that by Definition 17, $\tau_P = \tau_P^< = \tau$.

Let now $1 < \xi < \omega_1$ and suppose that the statements are true for $\vartheta < \xi$. Then by definition,

$$\tau_{P_\xi}^< = \bigvee_{i<\omega} \tau_{P^\xi_{(i)}} = \prod_{i<\omega} \tau_{P_{\vartheta_i}},$$

as stated.

Let now $(D, G) \in \mathcal{C}_\xi(C_\xi, \tau_{C_\xi}, P_\xi)$. If $D = X \setminus P_\xi$ and $G \in \tau_{P_\xi}^<$ is nonempty then $G$ is nontrivial only on finitely many coordinates so it intersects $X \setminus P_\xi = \bigcup_{i<\omega} P^\xi_{(i)}$. If for some $i < \omega$,

$$(D, G) \in \mathcal{C}_{\vartheta_i}\left(C_\xi, \bigvee_{j<\omega, j\neq i} \tau_{P^\xi_{(j)}}, P^\xi_{(i)}\right)$$

then by Proposition 26 for $P = P^\xi_{(i)}$, $(X, \tau) = (C_{\vartheta_i}, \tau_{C_{\vartheta_i}})$ and

$$(Y, \sigma) = \left(\prod_{j<\omega, j\neq i} C_{\vartheta_j}, \prod_{j<\omega, j\neq i} \tau_{P_{\vartheta_j}}\right),$$

$D$ is nontrivial only on the $C_{\vartheta_i}$ coordinate and $G = \prod_{j<\omega} G_j$ where $G_j = C_{\vartheta_j}$ except for finitely many $j < \omega$, $G_j$ is basic $\tau_{P_{\vartheta_j}}$-open ($j \in \omega \setminus \{i\}$) while $G_i$ is basic $\tau_{C_{\vartheta_i}}$-open. Since $[P_{\vartheta_i}, (P_t^{\vartheta_i})_{t\in T_{\vartheta_i}}]$ satisfies $\mathsf{C}_{\vartheta_i}$ in $(C_{\vartheta_i}, \tau_{C_{\vartheta_i}})$ by the induction hypothesis, we have $\mathrm{Pr}_{C_{\vartheta_i}}(D) \cap \mathrm{Pr}_{C_{\vartheta_i}}(G) \neq \emptyset$, which implies $D \cap G \neq \emptyset$. So $[P_\xi, (P_t^\xi)_{t\in T_\xi}]$ indeed satisfies $\mathsf{C}_\xi$ in $(C_\xi, \tau_{C_\xi})$.

Finally we have $P^\xi_{(n)} \cap \bigcap_{i<n}(C_\xi \setminus P^\xi_{(i)}) = U_{\xi,n}$, so by Definition 17,

$$\tau_{P_\xi} = \tau_{P_\xi}^<[\{U_{\xi,n} : n < \omega\}].$$

This completes the proof. $\square$

Now we have everything to give the proof of Theorem 1.

Proof of Theorem 1 for $\xi > 2$: Fix our $\xi$, say $\xi = \xi' + 1$. First we construct $\Phi = \Phi_\xi$ for $(X, \tau) = (C_\xi, \tau_{C_\xi})$ and $P = P_\xi$; note that this is a valid setting since by Proposition 36, $[P_\xi, (P_t^\xi)_{t\in T_\xi}]$ satisfies $\mathsf{C}_\xi$ in $(C_\xi, \tau_{C_\xi})$ so by Proposition 25, Corollary 23.5 holds and gives that $P_\xi$ is a proper $\mathbf{\Pi}^0_\xi(\tau)$ set. For every $B \in \mathcal{S}^0_\xi(P_\xi)$ fix a decomposition $B = \bigcup_{j<\omega} B_j$ where $B_j$ is $\mathbf{\Pi}^0_{\xi'}(\tau_{C_\xi})$ ($j < \omega$). Since the class

$\mathbf{\Pi}^0_{\xi'}(\tau_{C_\xi})$ has the separation property (see e.g. [2, (22.16) Theorem]) we can take a sequence $(\Delta_n(B))_{n<\omega} \subseteq \mathbf{\Delta}^0_{\xi'}(\tau_{C_\xi})$ such that

$$\bigcup_{i\le n} B_i \subseteq \Delta_n(B) \subseteq C_\xi \setminus \bigcup_{i\le n} P^\xi_{(i)} \ (n<\omega). \tag{18}$$

Set

$$\Phi_\xi(B) = \bigcap_{m<\omega} \bigcup_{m\le n<\omega} \Delta_n(B). \tag{19}$$

It is clear that $\Phi_\xi(B)$ is $\mathbf{\Pi}^0_\xi(\tau_{C_\xi})$ and (18) implies $B \subseteq \Phi_\xi(B) \subseteq P_\xi$. It remains to show that if $B^i \in \mathcal{S}^0_\xi(P_\xi)$ with its fixed decomposition $B^i = \bigcup_{j<\omega} B^i_j$ $(i<\omega)$ then we can find a point in $P_\xi \setminus \bigcup_{i<\omega} \Phi_\xi(B^i)$.

We apply Proposition 34 for $A^n = \bigcup_{i\le n} \Delta_n(B^i)$ $(n<\omega)$. We obtain $P_\xi \setminus \bigcup_{n<\omega} A^n \ne \emptyset$. Since

$$\Phi_\xi(B^i) \subseteq \bigcup_{i\le n<\omega} \Delta_n(B^i) \subseteq \bigcup_{n<\omega} A^n \ (i<\omega),$$

we have $P_\xi \setminus \bigcup_{i<\omega} \Phi_\xi(B^i) \ne \emptyset$, which completes the proof of the special case.

Let now $(X,\tau)$ and $P$ be arbitrary. By Proposition 36, $(C_\xi, \tau_{C_\xi})$ is homeomorphic to $(C, \tau_C)$ so by Theorem 10 we can take a continuous one-to-one map $\varphi\colon (C_\xi, \tau_{C_\xi}) \to (X,\tau)$ such that $\varphi^{-1}(P) = P_\xi$. For $B \in \mathcal{S}^0_\xi(P)$ let

$$\Phi(B) = (P \setminus \varphi(P_\xi)) \cup \varphi(\Phi_\xi(\varphi^{-1}(B))).$$

Since $P \setminus \varphi(P_\xi) = P \cap (X \setminus \varphi(C_\xi))$ is a $\mathbf{\Pi}^0_\xi(\tau)$ set and homeomorphisms preserve the Borel class of sets this definition makes sense and fulfills the requirements. □

As we mentioned in the introduction we think that it is independent whether Theorem 1 holds for limit ordinals or not. For limit $\xi$ the argument of the proof above brakes down in (19) since the $\Phi_\xi(B)$ defined there is merely $\mathbf{\Pi}^0_{\xi+1}(\tau_{C_\xi})$. To have a $\mathbf{\Pi}^0_\xi(\tau_{C_\xi})$ cover instead, the sets $\Delta_n(B)$ should be of lower Borel class than what we get from the separation property. So we get back to the problem whether Theorem 6 can be extended to the entire Borel hierarchy or the answer to Question 4 can be consistently positive. As we mentioned above these problems seem to be open.

Finally we would like to draw the attention of the reader to one more aspect of Theorem 1. One could say that Theorem 1 is trivial if the $\mathbf{\Sigma}^0_\xi(\tau_{C_\xi})$ subsets of $P_\xi$ (which are all $\tau_{P_\xi}|_{P_\xi}$-meager by Corollary 23.2, Proposition 19.3 and Definition 15.3) could be covered by a $\tau_{P_\xi}|_{P_\xi}$-meager $\mathbf{\Pi}^0_\xi(\tau_{C_\xi})$ subset of $P_\xi$. Then a category argument would give that

$$P_\xi \setminus \bigcup_{i<\omega} \Phi(B^i) \neq \emptyset \ (B^i \in \mathcal{S}^0_\xi(P_\xi) \ (i < \omega)).$$

However, this is not the case even for $\xi = 3$ and $(X, \tau) = (C, \tau_C)$ as illustrated by the following result (see [4, Proposition 19]).

**Proposition 37.** *There is a $\mathbf{\Pi}^0_3(\tau_C)$ topological Hurewicz test pair $\{P_L, \tau_{P_L}\}$ and a $\mathbf{\Sigma}^0_3(\tau_C)$ set $A \subseteq P_L$ such that if $B$ is $\mathbf{\Pi}^0_3(\tau_C)$ and $A \subseteq B$ then $B \cap P_L$ is $\tau_{P_L}|_{P_L}$-residual in $P_L$.*

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