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# Ultrafilter-limit points in metric dynamical systems

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*Abstract.* Given a free ultrafilter $p$ on $\mathbb{N}$ and a space $X$, we say that $x \in X$ is the *p-limit* point of a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ (in symbols, $x = p\text{-}\lim_{n\to\infty} x_n$) if for every neighborhood $V$ of $x$, $\{n \in \mathbb{N} : x_n \in V\} \in p$. By using $p$-limit points from a suitable metric space, we characterize the selective ultrafilters on $\mathbb{N}$ and the $P$-points of $\mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N}$. In this paper, we only consider dynamical systems $(X, f)$, where $X$ is a compact metric space. For a free ultrafilter $p$ on $\mathbb{N}^*$, the function $f^p : X \to X$ is defined by $f^p(x) = p\text{-}\lim_{n\to\infty} f^n(x)$ for each $x \in X$. These functions are not continuous in general. For a dynamical system $(X, f)$, where $X$ is a compact metric space, the following statements are shown:

**1.** If $X$ is countable, $p \in \mathbb{N}^*$ is a $P$-point and $f^p$ is continuous at $x \in X$, then there is $A \in p$ such that $f^q$ is continuous at $x$, for every $q \in A^*$.

**2.** Let $p \in \mathbb{N}^*$. If the family $\{f^{p+n} : n \in \mathbb{N}\}$ is uniformly equicontinuous at $x \in X$, then $f^{p+q}$ is continuous at $x$, for all $q \in \beta(\mathbb{N})$.

**3.** Let us consider the function $F : \mathbb{N}^* \times X \to X$ given by $F(p, x) = f^p(x)$, for every $(p, x) \in \mathbb{N}^* \times X$. Then, the following conditions are equivalent.

(1) $f^p$ is continuous on $X$, for every $p \in \mathbb{N}^*$.
(2) There is a dense $G_\delta$-subset $D$ of $\mathbb{N}^*$ such that $F|_{D\times X}$ is continuous.
(3) There is a dense subset $D$ of $\mathbb{N}^*$ such that $F|_{D\times X}$ is continuous.

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## 1. Preliminaries and notation

All the spaces are assumed to be Tychonoff (= completely regular and Hausdorff). If $f : X \to Y$ is a continuous function, then $\overline{f} : \beta(X) \to \beta(Y)$ will stand for the Stone extension of $f$. For a metric space $X$ and $\epsilon > 0$, $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. For short, $x_n \to x$ means that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x$. The Stone-Čech compactification $\beta(\mathbb{N})$ of the natural numbers $\mathbb{N}$ with the discrete topology will be identified with the set of all ultrafilters on $\mathbb{N}$, and its remainder $\mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N}$ with the set of all free ultrafilters on $\mathbb{N}$. If $A \subseteq \mathbb{N}$,

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then $\hat{A} = \mathrm{cl}_{\beta(\mathbb{N})} A = \{p \in \beta(\mathbb{N}) : A \in p\}$ is a basic clopen subset of $\beta(\mathbb{N})$, and $A^* = \hat{A} \setminus A = \{p \in \mathbb{N}^* : A \in p\}$ is a basic clopen subset of $\mathbb{N}^*$. If $A, B \subseteq \mathbb{N}$, then $A \subseteq^* B$ means that $A \setminus B$ is finite. In this paper, we shall use the following fact: If $\{A_n : n \in \mathbb{N}\}$ is a family of subsets of $\mathbb{N}$ with the infinite finite intersection property, then there is an infinite subset $B$ of $\mathbb{N}$ such that $B \subseteq^* A_n$, for every $n \in \mathbb{N}$. The set of real numbers will be denoted by $\mathbb{R}$ and the set of positive integers will be denoted by $\mathbb{N}^+$. A pair $(X, f)$ is called a *dynamical system* if $X$ is a Tychonoff space and $f : X \to X$ is a continuous function. If $(X, f)$ is a dynamical system, then the *orbit* of a point $x \in X$ is the set $\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{N}\}$. For an infinite set $X$, we let $[X]^\omega = \{A \subseteq X : |A| = \omega\}$.

Let $X$ be space. Given $p \in \mathbb{N}^*$, a point $x \in X$ is said to be the *p-limit* point of a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ ($x = p\text{-}\lim_{n\to\infty} x_n$) if for every neighborhood $V$ of $x$, $\{n \in \mathbb{N} : x_n \in V\} \in p$. The notion of $p$-limit point was introduced, in the context of non-standard analysis, by R.A. Bernstein [4]. H. Furstenberg [9, p. 179] and E. Atkin [1, p. 5, 61] considered the $\mathcal{F}$-limit points in Dynamical Systems, where $\mathcal{F}$ is a family of nonempty sets with the finite intersection property (for the definition of a $\mathcal{F}$-limit point of a sequence we replace $p$ by $\mathcal{F}$). The $p$-limit points play a very important role in the study of countably compact spaces. In this paper, we will give some of their applications to Dynamical Systems.

Observe that a point $x \in X$ is an adherent point of a countable set $\{x_n : n \in \mathbb{N}\}$ iff there is $p \in \beta(\mathbb{N})$ such that $x = p\text{-}\lim_{n\to\infty} x_n$. In other words, $x$ is an adherent point of a countable set $\{x_n : n \in \mathbb{N}\}$ iff the set $\{\{n \in \mathbb{N} : x_n \in V\} : V \in \mathcal{N}(x)\}$ is a filter base on $\mathbb{N}$. Notice that $x_n \to x$ iff $x = \mathcal{F}_r\text{-}\lim_{n\to\infty} x_n$, where $\mathcal{F}_r$ is the Frechét filter $\{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$. Hence, we see that $x_n \to x$ iff $x = p\text{-}\lim_{n\to\infty} x_n$ for all $p \in \mathbb{N}^*$. It is not hard to prove that in a compact space the $p$-limit point of a sequence always exists and is unique (for Hausdorff spaces), for every $p \in \mathbb{N}^*$.

By using $p$-limit points in metric spaces, we characterize the $P$-points of $\mathbb{N}^*$ and the selective ultrafilters on $\mathbb{N}$. In the second section, we study the continuity of the functions $f^p$ (for the definition of this function see the abstract) when $(X, f)$ is a dynamical system in which $X$ is a compact metric space. These functions have been also studied in [5], where the author establishes the connection between the algebra of $\beta(\mathbb{N})$ and an arbitrary dynamical system. We consider the particular case when $p$ is a $P$-point of $\mathbb{N}^*$ and analyze the continuity of the corresponding function $f^p$. The functions $f^p$'s are very useful to study the limiting behavior of the iterates of the original function $f$ when $X$ is a metric compact space. The fourth section is concerning with some applications to actions of compact metrizable semigroups.

## 2. $p$-limit points in metric spaces

Suppose that $X$ is a metric space and $p \in \mathbb{N}^*$. If $x = p\text{-}\lim_{n\to\infty} x_n$, then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \to x$. In general, $x_n \not\to x$

and $\{n_k : k \in \mathbb{N}\} \notin p$. Our first task is to use this remark to characterize the $P$-points of $\mathbb{N}^*$ and the selective ultrafilters on $\mathbb{N}$. Let us recall a combinatorial definition of a $P$-point of $\mathbb{N}^*$:

An ultrafilter $p \in \mathbb{N}^*$ is called *$P$-point* iff for every partition $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ with $A_n \notin p$, for each $n \in \mathbb{N}$, there is $A \in p$ such that $A \cap A_n$ is finite for every $n \in \mathbb{N}$.

W. Rudin [13] proved that CH implies the existence of $2^{\mathfrak{c}}$-many $P$-points in $\mathbb{N}^*$, and years later S. Shelah [6] found a model of ZFC in which $\mathbb{N}^*$ does not have any $P$-point.

**Lemma 2.1.** *Let $p \in \mathbb{N}^*$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a space $X$. If there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\lim_{k\to\infty} x_{n_k} = x$, then $x = p\text{-}\lim_{n\to\infty} x_n$.*

PROOF: Let $V \in \mathcal{N}(x)$. By assumption, we know that $\{n_k : k \in \mathbb{N}\} \subseteq^* \{n \in \mathbb{N} : x_n \in V\}$. Hence, we deduce that $\{n \in \mathbb{N} : x_n \in V\} \in p$. This shows that $x = p\text{-}\lim_{n\to\infty} x_n$. □

The next lemma was suggested by the referee and simplifies the original proofs of our main results of this section.

**Lemma 2.2.** *Let $p \in \mathbb{N}^*$ and let $\{A_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets such that $A_n \notin p$, for all $n \in \mathbb{N}$. Let $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection such that $\sigma[A_n] = \{n\} \times \mathbb{N}$, for every $n \in \mathbb{N}$. If for every $k \in \mathbb{N}$ we have that $x_k = \frac{1}{n} + \frac{1}{a_n+m}$, where $\sigma(k) = (n,m)$ and $n \leq a_n \in \mathbb{N}$, then $0 = p\text{-}\lim_{k\to\infty} x_k$.*

PROOF: Let $\epsilon > 0$ and assume that $A = \{k \in \mathbb{N} : x_k > \epsilon\} \in p$. Since $A_n \notin p$, for each $n \in \mathbb{N}$, we must have that $\{n \in \mathbb{N} : A \cap A_n \neq \emptyset\}$ is infinite. Hence, we can find $n > \frac{2}{\epsilon}$ such that $A \cap A_n \neq \emptyset$. Pick $k \in A \cap A_n$. Then, $\sigma(k) = (n,m)$ for some $m \in \mathbb{N}$ and we have that $x_k = \frac{1}{n} + \frac{1}{a_n+m} < \frac{2}{n} < \epsilon$, but this is a contradiction. □

**Theorem 2.3.** *For a point $p \in \mathbb{N}^*$, the following are equivalent.*

1. *$p$ is a $P$-point of $\mathbb{N}^*$.*
2. *In every metric space $X$, for every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ and every $x \in X$, we have that $x = p\text{-}\lim_{n\to\infty} x_n$ iff there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\lim_{k\to\infty} x_{n_k} = x$.*
3. *For every sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers and for every $x \in \mathbb{R}$, we have that $x = p\text{-}\lim_{n\to\infty} x_n$ iff there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\lim_{k\to\infty} x_{n_k} = x$.*

PROOF: (1) ⇒ (2). Necessity. Let $A_n = \{i \in \mathbb{N} : x_i \in B(x, \frac{1}{n})\}$. By assumption, $A_n \in p$ for every $n \in \mathbb{N}$. Then, we can find $A \in p$ so that $A \subseteq^* A_k$ for every $k \in \mathbb{N}$. If we enumerate $A$ as $\{x_{n_k} : k \in \mathbb{N}\}$, then $(x_{n_k})_{k\in\mathbb{N}}$ is the desired subsequence.

Sufficiency. This follows directly from Lemma 2.1.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Suppose that $p$ is not a $P$-point of $\mathbb{N}^*$. Then, there is a partition $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $A_n \notin p$, for every $n \in \mathbb{N}$, and for every $A \in p$ there is $n \in \mathbb{N}$ for which $A \cap A_n$ is infinite. Fix a bijection $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ so that $\sigma[A_n] = \{n\} \times \mathbb{N}$, for all $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define $x_k = \frac{1}{n} + \frac{1}{n+m}$ provided that $\sigma(k) = (n, m)$. Then, by Lemma 2.2, we know that $0 = p\text{-}\lim_{k\to\infty} x_k$. By assumption, we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $B = \{n_k : k \in \mathbb{N}\} \in p$ and $0 = \lim_{k\to\infty} x_{n_k}$. Pick $l \in \mathbb{N}$ so that $B \cap A_l$ is infinite. Then, the sequence $(x_n)_{n\in B\cap A_l}$ must converge to $\frac{1}{l}$ and as a subsequence of $(x_{n_k})_{k\in\mathbb{N}}$ it must converge to 0, which is impossible. $\square$

An ultrafilter $p \in \mathbb{N}^*$ is called *selective* if for every partition $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ with $A_n \notin p$, for each $n \in \mathbb{N}$, there is $A \in p$ such that $|A \cap A_n| \leq 1$, for every $n \in \mathbb{N}$. Every selective ultrafilter is a $P$-point and under CH we can find $2^{\mathfrak{c}}$-many selective ultrafilters (see [7]).

**Theorem 2.4.** *For a point $p \in \mathbb{N}^*$, the following are equivalent.*

(1) *$p$ is selective.*

(2) *In every metric space $X$, for every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ and every $x \in X \setminus \{x_n : n \in \mathbb{N}\}$, we have that $x = p\text{-}\lim_{n\to\infty} x_n$ iff there are a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and an increasing sequence of integers $(m_k)_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\frac{1}{m_{k+1}} \leq d(x_{n_k}, x) < \frac{1}{m_k}$, for every $k \in \mathbb{N}$.*

(3) *For every sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ and every $x \in \mathbb{R} \setminus \{x_n : n \in \mathbb{N}\}$, we have that $x = p\text{-}\lim_{n\to\infty} x_n$ iff there are a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and an increasing sequence of integers $(m_k)_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\frac{1}{m_{k+1}} \leq |x_{n_k} - x| < \frac{1}{m_k}$, for every $k \in \mathbb{N}$.*

Proof: (1) $\Rightarrow$ (2). Necessity. Define $A_0 = \{i \in \mathbb{N} : 1 \leq d(x_i, x)\}$ and for every $1 \leq n \in \mathbb{N}$, we let $A_n = \{i \in \mathbb{N} : \frac{1}{n+1} \leq d(x_i, x) < \frac{1}{n}\}$. It is evident that $A_n \notin p$, for each $n \in \mathbb{N}$. Then, we can find $A \in p$ so that $|A \cap A_n| \leq 1$, for every $n \in \mathbb{N}$. Enumerate $A$ as $\{x_{n_k} : k \in \mathbb{N}\}$. Then, for every $k \in \mathbb{N}$ there is a unique $m_k \in \mathbb{N}$ such that $n_k \in A_{m_k}$. Without loss of generality we may assume that the sequence $(m_k)_{k\in\mathbb{N}}$ is increasing. It clear that$(x_{n_k})_{k\in\mathbb{N}}$ is the desired subsequence.

Sufficiency. It is a consequence of Lemma 2.1.

(2) $\Rightarrow$ (3). It is evident.

(3) $\Rightarrow$ (1). Assume that $p$ is not selective. Then there is a partition $\{A_n : n \in \mathbb{N}^+\}$ of $\mathbb{N}$ such that for all $n \in \mathbb{N}^+$, $A_n \notin p$ and for every $A \in p$ there is $n \in \mathbb{N}^+$ with $|A \cap A_n| \geq 2$. Let $\sigma : \mathbb{N}^+ \to \mathbb{N}^+ \times \mathbb{N}^+$ be a bijection such that $\sigma[A_n] = \{n\} \times \mathbb{N}^+$, for each $n \in \mathbb{N}^+$. Put $a_1 = 1$ and for $n > 1$, we let $a_n = n^2 - n$. Observe that if $n > 1$, then $\frac{1}{n} + \frac{1}{a_n} = \frac{1}{n-1}$. Now, for each $k \in \mathbb{N}^+$, we define $x_k = \frac{1}{n} + \frac{1}{a_n+m}$ provided that $\sigma(k) = (n, m)$. By Lemma 2.2, we know that $0 = p\text{-}\lim_{k\to\infty} x_k$. Then, by hypothesis, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$

and an increasing sequence of integers $(m_k)_{k\in\mathbb{N}}$ such that $B = \{n_k : k \in \mathbb{N}\} \in p$ and $\frac{1}{m_{k+1}} \le x_{n_k} < \frac{1}{m_k}$, for every $k \in \mathbb{N}$. Notice that $B \setminus A_1 \in p$. We know that there is $r \in \mathbb{N}$ with $r > 1$ such that $|B \cap A_r| = |(B \setminus A_1) \cap A_r| \ge 2$. Choose $k, l \in \mathbb{N}$ such that $k < l$ and $n_k, n_l \in B \cap A_r$. Put $\sigma(n_k) = (r, s)$ and $\sigma(n_l) = (r, t)$ for some $s, t \in \mathbb{N}^+$. Then, we have that $\frac{1}{r} < x_{n_l} = \frac{1}{r} + \frac{1}{a_r+t} < \frac{1}{m_l}$ and $\frac{1}{r-1} = \frac{1}{r} + \frac{1}{a_r} > \frac{1}{r} + \frac{1}{a_r+s} = x_{n_k} \ge \frac{1}{m_{k+1}}$. Hence, $r - 1 < m_{k+1} \le m_l < r$, which is impossible since $r$ and $m_l$ are natural numbers. □

## 3. $p$-limit points and dynamical systems

This section is devoted to study the continuity and discontinuity of the function $f^p : X \to X$, for $p \in \mathbb{N}^*$.

**Definition 3.1.** Let $(X, f)$ be a dynamical system, where $X$ is a compact space. For a free ultrafilter $p$ on $\mathbb{N}$, the function $f^p : X \to X$ is defined by $f^p(x) = p\text{-}\lim_{n\to\infty} f^n(x)$, for every $x \in X$. For a point $x \in X$, the function $f_x := p \mapsto f^p(x) : \beta(\mathbb{N}) \to X$ is the Stone extension of the continuous function $n \mapsto f^n(x) : \mathbb{N} \to X$.

We remark that the function $f_x : \beta(\mathbb{N}) \to X$ is continuous for every $x \in X$. Observe that $f_x[\beta(\mathbb{N})] = \mathrm{cl}_X(\mathcal{O}_f(x))$. But, the functions $f^p$ are not always continuous as we shall see in the next example:

**Example 3.2.** Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$ and define $f : X \to X$ as follows:

$$f(x) = \begin{cases} x & \text{if } \ x \in \{0, 1\} \\ \frac{1}{n} & \text{if } \ x = \frac{1}{n+1} \ \text{ and } \ 1 \le n \in \mathbb{N}. \end{cases}$$

It is easy to see that if $p \in \mathbb{N}^*$, then $f^p(x) = 1$ for every $x > 0$ and $f^p(0) = 0$. Thus, $f^p$ is discontinuous at 0, for all $p \in \mathbb{N}^*$. For a connected example, take $X = [0, 1]$ and define $f : [0, 1] \to [0, 1]$ as follows:

$$f(x) = \begin{cases} x & \text{if } \ x \in [0, \frac{1}{2}] \\ \frac{(n+1)(n+2)x-(2n+1)}{n(n+1)} & \text{if } \ x \in [\frac{n}{n+1}, \frac{n+1}{n+2}] \text{ and } 1 \le n \in \mathbb{N} \\ 1 & \text{if } \ x = 1. \end{cases}$$

Observe that $f$ is a homeomorphism between the closed intervals $[\frac{n}{n+1}, \frac{n+1}{n+2}]$ and $[\frac{n-1}{n}, \frac{n}{n+1}]$, for each $1 \le n \in \mathbb{N}$. Then, we have that $f^p[[0, 1)] = [0, \frac{1}{2}]$ and $f^p(1) = 1$, for every $p \in \mathbb{N}^*$. This implies that $f^p$ is discontinuous at 1, for all $p \in \mathbb{N}^*$.

Let us explain one way to extend the ordinary addition on the set of natural numbers to the whole $\beta(\mathbb{N})$ and how to apply this extension to the Theory of Dynamical Systems:

For $p \in \beta(\mathbb{N})$ and $n \in \mathbb{N}$, we define $p+n = p\text{-}\lim_{m\to\infty}(m+n)$ and if $p, q \in \beta(\mathbb{N})$, then we define $p + q = q\text{-}\lim_{n\to\infty} p + n$.

The following theorem is taken from [5].

**Theorem 3.3.** *Let $(X, f)$ be a dynamical system where $X$ is a compact space. Then*

$$f^p \circ f^q(x) = f^{q+p}(x),$$

*for every $x \in X$ and for every $p, q \in \beta(\mathbb{N})$.*

Thus, if $f^q$ is continuous at $x$ and $f^p$ is continuous at $f^q(x)$, then $f^{q+p}$ is continuous at $x$, for $p, q \in \beta(\mathbb{N})$.

The following two theorems are characterizations of the continuity of the function $f^p$ at some point of the given space.

**Theorem 3.4.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and let $p \in \mathbb{N}^*$. For a point $x \in X$, the following are equivalent.*

(1) *$f^p$ is continuous at $x$.*
(2) *For all $\epsilon > 0$ there is $\delta > 0$ such that for all $y \in X$ if $d(x, y) < \delta$, then $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$.*

Proof: (1) $\Rightarrow$ (2). Let $\epsilon > 0$. So, there is $\delta > 0$ such that if $y \in X$ and $d(x, y) < \delta$, then $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$. Suppose that $y \in X$ satisfies that $d(x, y) < \delta$. By definition, we have that $A = \{n \in \mathbb{N} : d(f^n(x), f^p(x)) < \frac{\epsilon}{3}\} \cap \{n \in \mathbb{N} : d(f^n(y), f^p(y)) < \frac{\epsilon}{3}\} \in p$. Hence,

$$\begin{aligned} d(f^n(x), f^n(y)) &\le d(f^n(x), f^p(x)) + d(f^p(x), f^p(y)) + d(f^p(y), f^n(y)) \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

for every $n \in A$.

(2) $\Rightarrow$ (1). Let $\epsilon > 0$ and let $\delta > 0$ be satisfy the conditions of our hypothesis. Fix $y \in X$ with $d(x, y) < \delta$. Then, we have that $A = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \frac{\epsilon}{3}\} \in p$. Thus,

$$\begin{aligned} d(f^p(x), f^p(y)) &\le d(f^p(x), f^n(x)) + d(f^n(x), f^n(y)) + d(f^n(y), f^p(y)) \\ &\le d(f^p(x), f^n(x)) + \frac{\epsilon}{3} + d(f^n(y), f^p(y)). \end{aligned}$$

We know that $n \in A$ can be chosen so that $d(f^p(x), f^n(x)) < \frac{\epsilon}{3}$, $d(f^n(x), f^n(y)) < \frac{\epsilon}{3}$ and $d(f^n(y), f^p(y)) < \frac{\epsilon}{3}$. Therefore, $d(f^p(x), f^p(y)) < \epsilon$. This shows the continuity of $f^p$ at $x$. $\square$

**Definition 3.5.** Let $(X, f)$ be a dynamical system, where $X$ is a metric space, and let $p \in \mathbb{N}^*$. We say that a sequence $(x_k)_{k\in\mathbb{N}}$ in $X$ is *$p$-proximal* to a point $x$ if $\lim_{k\to\infty} x_k = x$ and for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) < \epsilon\} \in p$, for every $k \in \mathbb{N}$ with $k \geq N$. Two points $x, y \in X$ are said to be *$p$-proximal* if for every $\epsilon > 0$, $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$.

**Theorem 3.6.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and let $p \in \mathbb{N}^*$. For a point $x \in X$ the following are equivalent.*

(1) *$f^p$ is continuous at $x$.*
(2) *Every sequence $(x_k)_{k\in\mathbb{N}}$ that converges to $x$ is $p$-proximal to $x$.*

PROOF: (1) $\Rightarrow$ (2). Let $(x_k)_{k\in\mathbb{N}}$ be a sequence converging to $x$. Given $\epsilon > 0$, by Theorem 3.4, we can find $\delta > 0$ such that for all $y \in X$, if $d(x, y) < \delta$, then $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$. Let $N \in \mathbb{N}$ such that $d(x_k, x) < \delta$ for every $N \leq k \in \mathbb{N}$. Then, we have that $\{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) < \epsilon\} \in p$ for every $k \in \mathbb{N}$ with $k \geq N$.

(2) $\Rightarrow$ (1). Let us assume that $f^p$ is not continuous at $x$. Then, by Theorem 3.4, there is $\epsilon > 0$ such that for every $k \in \mathbb{N}$ there is $x_k \in X$ such that $d(x, x_k) < \frac{1}{k+1}$ and $\{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) < \epsilon\} \notin p$. It is evident that the sequence $(x_k)_{k\in\mathbb{N}}$ converges to $x$ and it is not $p$-proximal to $x$. □

Next we state a classical notion in Dynamical Systems and establish its relation with the concept introduced in Definition 3.5.

**Definition 3.7.** Let $(X, f)$ be a dynamical system where $X$ is a metric space. We say that two points $x, y \in X$ are *proximal* if for every $\epsilon > 0$, $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\}$ is infinite.

The following result shows that the standard notion "proximal" is included in Definition 3.5.

**Theorem 3.8.** *Let $(X, f)$ be a dynamical system, where $X$ is a metric space, and let $x, y \in X$. The following conditions are equivalent.*

(1) *$x$ and $y$ are proximal.*
(2) *There is $p \in \mathbb{N}^*$ such that $f^p(x) = f^p(y)$.*
(3) *$x$ and $y$ are $p$-proximal for some $p \in \mathbb{N}^*$.*

PROOF: The equivalence (1) $\Leftrightarrow$ (2) is stated, for a general case, in [3] and it is proved in [5]. The implication (3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3). For every $\epsilon > 0$, we define $A_\epsilon = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\}$. Since the family $\{A_\epsilon : \epsilon > 0\}$ has the finite intersection property, we can find $p \in \mathbb{N}^*$ such that $\{A_\epsilon : \epsilon > 0\} \subseteq p$. It is then evident that $x$ and $y$ are $p$-proximal. □

The equivalence (2) $\Leftrightarrow$ (3) of the previous theorem can be rewritten as follows.

**Theorem 3.9.** *Let $(X, f)$ be a dynamical system, where $X$ is a metric space, let $x, y \in X$ and let $p \in \mathbb{N}^*$. The following conditions are equivalent.*

(1) *$x$ and $y$ are $p$-proximal.*
(2) *$f^p(x) = f^p(y)$.*

It follows from Theorem 3.9 that if $p \in \mathbb{N}^*$ is an idempotent (that is, $p+p = p$), then every $x \in X$ is $p$-proximal to $f^p(x)$. Indeed, $f^p(x) = f^{p+p}(x) = f^p(f^p(x))$.

**Theorem 3.10.** *Let $(X, f)$ be a dynamical system, where $X$ is a metric space, and let $x, y \in X$. Then, $\{p \in \mathbb{N}^* : x$ and $y$ are $p$-proximal$\}$ is a closed subset of $\mathbb{N}^*$.*

Proof: Put $D = \{p \in \mathbb{N}^* : x$ and $y$ are $p$-proximal$\}$ and let $q \in \mathrm{cl}_{\mathbb{N}^*} D$. Suppose that $x$ and $y$ are not $q$-proximal. Then, there is $\epsilon > 0$ such that $A = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) \geq \epsilon\} \in q$. Choose $p \in A^* \cap D$. By assumption, $B = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$. But this is impossible since $A \cap B = \emptyset$. Therefore, $D = \mathrm{cl}_{\mathbb{N}^*} D$. $\square$

We remark that the points $x$ and $y$ are $p$-proximal, for all $p \in \mathbb{N}^*$, iff

$$\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

The next example shows that the notion of $p$-proximally could distinguish, in some sense, two proximal points.

**Example 3.11.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$, $a_0 = 1$ and $a_{n+1} < a_n$, for each $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, choose a strictly decreasing sequence $(a_{n,m})_{m\in\mathbb{N}}$ such that

(1) $\lim_{m\to\infty} a_{n,m} = a_n$, for each $n \in \mathbb{N}$, and
(2) $a_n < a_{n,m} < a_{n-1}$, for all $n, m \in \mathbb{N}$; here, $a_{-1} = 2$.

Consider the subspace $X = \{0\} \cup \{a_n : n \in \mathbb{N}\} \cup \{a_{n,m} : n, m \in \mathbb{N}\}$ of $\mathbb{R}$. Then, $X$ is a compact metric space. Now, we shall define a function $f : X \to X$ as follows.

a. $f(a_0) = 0$ and $f(0) = 0$.
b. $f(a_n) = a_{n-1}$, for each $n \in \mathbb{N}$.
c. $f(a_{n,0}) = a_{n+1,0}$, for each $n \in \mathbb{N}$.
d. $f(a_{0,n}) = a_{n,1}$, for each $1 \leq n \in \mathbb{N}$.
e. $f(a_{n-m,m+1}) = a_{n-m-1,m+2}$, for each $m < n \in \mathbb{N}$.

It is not difficult to prove that $f$ is continuous. Let $x = a_{0,0}$ and $y = a_{0,1}$. We define $i_0 = 1$, $j_0 = 2$, $i_1 = 3$, $j_1 = 5$ and if $2 \leq k \in \mathbb{N}$, then we define $i_k = j_{k-1}+1$ and $j_k = j_{k-1} + k + 2$. We know from the definition that $f^{i_0}(a_{0,1}) = a_{1,1}$, $f^{j_0}(a_{0,1}) = a_{0,2}$, $f^{i_1}(a_{0,1}) = a_{2,1}$ y $f^{j_1}(a_{0,1}) = a_{0,3}$. By induction, we can establish that

$$f^{i_k}(a_{0,1}) = f^{j_{k-1}+1}(a_{0,1}) = f(f^{j_{k-1}}(a_{0,1})) = f(a_{0,k+1}) = a_{k+1,1},$$

$$f^{j_k}(a_{0,1}) = f^{j_{k-1}+k+2}(a_{0,1}) = f^{k+1}(f^{i_k}(a_{0,1})) = f^{k+1}(a_{k+1,1}) = a_{0,k+2},$$

and

$$f^i(a_{k,1}) = a_{k-i,i+1},$$

for every $k \in \mathbb{N}$ and for each $1 \leq i \leq k$. Let us define $A = \{i_k : k \in \mathbb{N}\}$ and $B = \{j_k : k \in \mathbb{N}\}$. Then, we have that

$$\lim_{k\to\infty} |f^{i_k}(a_{0,0}) - f^{i_k}(a_{0,1})| = \lim_{k\to\infty} |a_{i_k+1,0} - a_{k+1,0}| = 0.$$

On the other hand,

$$\lim_{k\to\infty} |f^{j_k}(a_{0,0}) - f^{j_k}(a_{0,1})| = \lim_{k\to\infty} |a_{j_k+1,0} - a_{0,k+1}| = 1.$$

These two conditions imply that $x$ and $y$ are $p$-proximal for all $p \in A^*$ and they are not $q$-proximal for any $q \in B^*$.

When the function $f^p$ is continuous on the whole space we have the following uniform property:

**Theorem 3.12.** *Let $(X, f)$ be a dynamical system where $X$ is a compact metric space and let $p \in \mathbb{N}^*$. Then, $f^p$ is continuous iff for every $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$, if $d(x, y) < \delta$, then $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$.*

Proof: Necessity. If $f^p$ is continuous on $X$, then $f^p$ is uniformly continuous on $X$ and then we follow the proof of Theorem 3.4.

Sufficiency. This follows directly from Theorem 3.4. □

Now, let us study the behavior of the function $f_x$ around a $P$-point of $\mathbb{N}^*$.

**Theorem 3.13.** *Let $(X, f)$ be a dynamical system and let $x \in X$, where $X$ is a compact metric space. If $p \in \mathbb{N}^*$ is a $P$-point, then there is $A \in p$ such that $f_x(p) = f_x(q)$, for every $q \in A^*$.*

Proof: By the continuity of $f_x$, for every $k \in \mathbb{N}$ there is $A_k \in p$ such that

$$d(f_x(p), f_x(q)) < \frac{1}{k+1},$$

for all $q \in A_k^*$. Since $p$ is a $P$-point there is $A \in p$ such that $A \subseteq^* A_k$, for each $k \in \mathbb{N}$. Thus, if $q \in A^*$ and $k \in \mathbb{N}$, then $q \in A_k^*$ and hence $d(f_x(p), f_x(q)) < \frac{1}{k+1}$. This implies that $f_x(p) = f_x(q)$, for every $q \in A^*$. □

For an arbitrary free ultrafilter $p$ on $\mathbb{N}$ we have the following property.

**Theorem 3.14.** *Let $(X, f)$ be a dynamical system and let $x \in X$, where $X$ is a compact metric space. Then, for every $p \in \mathbb{N}^*$, there is $A \in [\mathbb{N}]^\omega$ such that $f_x(p) = f_x(q)$ for every $q \in A^*$.*

Proof: We know that $f_x(p) \in \mathrm{cl}_X(\{f^n(x) : n \in \mathbb{N}\})$. First suppose that $f_x(p)$ is not an accumulation point of $\mathcal{O}_f(x)$. Then, $f_x(p) = f^p(x) = f^n(x)$ for some $n \in \mathbb{N}$ and there is $\epsilon > 0$ such that $B(f^n(x), \epsilon) \cap \mathcal{O}_f(x) = \{f^n(x)\}$. Since $f_x$ is continuous, there is $A \in p$ such that $f_x(q) \in B(f^n(x), \epsilon)$ for all $q \in A^*$. That is, $f_x(p) = f_x(q) = f^n(x)$ for every $q \in A^*$. Now, assume that there is a non-trivial sequence $(f^{n_k}(x))_{k\in\mathbb{N}}$ for which $\lim_{k\to\infty} f^{n_k}(x) = f_x(p)$ and we also assume that $f^{n_i}(x) \neq f^{n_j}(x)$ for distinct $i, j \in \mathbb{N}$. Put $A = \{n_k : k \in \mathbb{N}\}$ and fix $q \in A^*$. According to Lemma 2.1, we obtain that $f_x(p) = f_x(q)$. □

The proof of Theorem 3.14 with small changes establishes the next result.

**Theorem 3.15.** *Let $(X, f)$ be a dynamical system and let $x \in X$, where $X$ is a compact metric space. Then, for every $A \in [\mathbb{N}]^\omega$, there is $B \in [A]^\omega$ such that $f_x(p) = f_x(q)$, for every $p, q \in B^*$.*

Now, let us study the continuity of the function $f^p$ when $p$ is a $P$-point of $\mathbb{N}^*$ and $X$ is a countable metric space.

**Theorem 3.16.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric countable space. If $f^p$ is continuous at $x \in X$, for some $P$-point $p \in \mathbb{N}^*$, then for every $\epsilon > 0$ there are $\delta > 0$ and $A \in p$ so that for $y \in X$ if $d(x, y) < \delta$, then $d(f^p(y), f^n(y)) < \epsilon$, for all $n \in A$ except finitely many.*

Proof: By definition, we know that $f^p(x) = p\text{-}\lim_{n\to\infty} f^n(x)$. Since $X$ is a metric space, by Theorem 2.3, there is a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that $f^p(x) = \lim_{k\to\infty} f^{n_k}(x)$ and $B = \{n_k : k \in \mathbb{N}\} \in p$. Given $\epsilon > 0$, by Theorem 3.4, we may find $\delta > 0$ such that $C_y = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \frac{\epsilon}{3}\} \in p$ and $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$, provided that $d(x, y) < \delta$. As $p$ is a $P$-point, there is $A \in p$ such that $A \subseteq^* C_y \cap B$ for all $y \in X$ with $d(x, y) < \delta$. Fix $y \in X$ with $d(x, y) < \delta$ and $m \in \mathbb{N}$ such that $A \setminus \{0, 1, \ldots, m\} \subseteq C_y$ and $d(f^n(x), f^p(x)) < \frac{\epsilon}{3}$, for every $n \in A \setminus \{0, 1, \ldots, m\}$. Then, for $n \in A \setminus \{0, 1, \ldots, m\}$ we have that

$$\begin{aligned} d(f^p(y), f^n(y)) &< d(f^p(y), f^p(x)) + d(f^p(x), f^n(x)) + d(f^n(x), f^n(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

as required. □

**Lemma 3.17.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric countable space. Suppose that $f^p$ is continuous at $x \in X$ for a $P$-point $p$ of $\mathbb{N}^*$. Then, for every $\epsilon > 0$ there are $\delta > 0$ and $A \in p$ such that if $y \in X$ satisfies that $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in A$ except finitely many.*

Proof: According to Theorem 3.16, we can find $\delta > 0$ and $B \in p$ so that if $y \in X$ and $d(x, y) < \delta$, then $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$ and $d(f^p(y), f^n(y)) < \frac{\epsilon}{3}$ for all $n \in B$ except finitely many. Put $A = \{n \in B : d(f^p(x), f^n(x)) < \frac{\epsilon}{3}\}$. Assume that $y \in X$ satisfies the inequality $d(x,y) < \delta$. By assumption, we can find $m \in \mathbb{N}$ such that $d(f^p(y), f^n(y)) < \frac{\epsilon}{3}$, for each $n \in A \setminus m$. Thus, if $n \in A \setminus m$, then we obtain that

$$\begin{aligned} d(f^n(x), f^n(y)) &\leq d(f^n(x), f^p(x)) + d(f^p(x), f^p(y)) + d(f^p(y), f^n(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

**Theorem 3.18.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric countable space, and let $x \in X$. Suppose that $f^p$ is continuous at $x \in X$ for a $P$-point $p$ of $\mathbb{N}^*$. Then, there is $A \in p$ such that $f^q$ is continuous at $x$, for every $q \in A^*$.*

Proof: By Theorem 3.13, we know that there is $B \in p$ such that $f^p(x) = f^q(x)$ for each $q \in B^*$. From the previous lemma, for every $n \in \mathbb{N}$, we can find $\delta_n > 0$ and $A_n \subseteq B$ such that if $d(x,y) < \delta_n$, then $d(f^k(x), f^k(y)) < \frac{1}{n+1}$ for all $k \in A_n$ except finitely many. For every $n \in \mathbb{N}$, let $C_n = \{k \in \mathbb{N} : d(f^p(x), f^k(x)) < \frac{1}{n+1}\}$. We know that $C_n \in p$ for all $n \in \mathbb{N}$. Since $p$ is a $P$-point, we can find $A \in p$ so that $A \subseteq^* A_n \cap C_n$, for each $n \in \mathbb{N}$. Now, fix $q \in A^*$ and let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \frac{\epsilon}{3}$. Suppose that $y \in X$ satisfies that $d(x,y) < \delta_n$. Since $D = \{i \in \mathbb{N} : d(f^i(y), f^q(y)) < \frac{1}{n+1}\} \in q$, we can find $k \in D \cap C_n \cap A_n$ for which $d(f^k(x), f^k(y)) < \frac{1}{n+1}$. Then, we have that

$$\begin{aligned} d(f^q(x), f^q(y)) &= d(f^p(x), f^q(y)) \\ &\leq d(f^p(x), f^k(x)) + d(f^k(x), f^k(y)) + d(f^k(y), f^q(y)) \\ &< \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, $f^q$ is continuous at $x$. □

The next corollary is a direct application of Theorem 3.18.

**Corollary 3.19.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric countable space. If $p \in \mathbb{N}^*$ is a $P$-point and $f^p$ is continuous on $X$, then there is $A \in p$ such that $f^q$ is continuous on $X$, for every $q \in A^*$.*

Proof: According to Theorem 3.18, for every $x \in X$, there is $A_x \in p$ such that $f^q$ is continuous at $x$, for every $q \in A_x^*$. Choose $A \in p$ so that $A \subseteq^* A_x$, for all $x \in X$. Then, it is evident that $f^q$ is continuous on $X$, for each $q \in A^*$. □

In the general case, we have the following statement:

**Theorem 3.20.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and let $p \in \mathbb{N}^*$. Suppose that there exist $A \in p$ and $x \in X$ such that*

(1) *$f_x(s) = f_x(t)$ for each $s, t \in A^*$; and*
(2) *$f^p$ is continuous at $x$.*

*If $x = \lim_{n\to\infty} x_n$, then there is $B \in [A]^\omega$ such that $f^q(x) = \lim_{n\to\infty} f^q(x_n)$ for every $q \in B^*$.*

Proof: Since $f^p$ is continuous at $x$, by Theorem 3.4, for every $i \in \mathbb{N}$ there is $K_i \in \mathbb{N}$ such that $B_{k,i} = \{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) < \frac{1}{i+1}\} \in p$, for all $k \geq K_i$. For each $i \in \mathbb{N}$, let $C_i = \{n \in \mathbb{N} : d(f^n(x), f^p(x)) < \frac{1}{i+1}\}$. By definition, we know that $C_i \in p$ for each $i \in \mathbb{N}$. Choose $B \in [A]^\omega$ so that $B \subseteq^* B_{k,i} \cap C_i$, for every $i \in \mathbb{N}$ and for every $k \geq K_i$. Let $q \in B^*$ and let $\epsilon > 0$. Pick $j \in \mathbb{N}$ such that $\frac{1}{j+1} < \frac{\epsilon}{3}$. Fix $k \geq K_j$. We know that $D = \{n \in \mathbb{N} : d(f^n(x_k), f^q(x_k)) < \frac{1}{j+1}\} \in q$. Let $h \in D \cap B_{k,j} \cap C_j$. Then, we have that

$$
\begin{aligned}
d(f^q(x_k), f^q(x)) &= d(f^q(x_k), f^p(x)) \\
&\leq d(f^q(x_k), f^h(x_k)) + d(f^h(x_k), f^h(x)) + d(f^h(x), f^p(x)) \\
&< \frac{1}{j+1} + \frac{1}{j+1} + \frac{1}{j+1} < \epsilon.
\end{aligned}
$$

□

Next, we shall study the continuity properties of various functions $f^p$'s at the same time.

**Lemma 3.21.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric, $x, y \in X$ and $p \in \mathbb{N}^*$. If $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$ for some $\epsilon > 0$, then $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$.*

Proof: We know that $A = \{n \in \mathbb{N} : d(f^p(x), f^n(x)) < \frac{\epsilon}{3}\} \in p$ and $B = \{n \in \mathbb{N} : d(f^p(y), f^n(y)) < \frac{\epsilon}{3}\} \in p$. Then, we have that $A \cap B \in p$ and if $n \in A \cap B$, then

$$
\begin{aligned}
d(f^n(x), f^n(y)) &\leq d(f^n(x), f^p(x)) + d(f^p(x), f^p(y)) + d(f^p(y), f^n(y)) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

□

**Theorem 3.22.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and let $x \in X$. Let $\{p_n : n \in \mathbb{N}\} \subseteq \beta(\mathbb{N})$ and assume that the family $\{f^{p_n} : n \in \mathbb{N}\}$ is uniformly equicontinuous at $x$. Then, $f^q$ is continuous at $x$, for each $q \in \mathrm{cl}_{\mathbb{N}^*}(\{p_n : n \in \mathbb{N}\})$.*

Proof: Fix $q \in \mathrm{cl}_{\mathbb{N}^*}(\{p_n : n \in \mathbb{N}\})$. We know that $q = p\text{-}\lim_{n\to\infty} p_n$ for some $p \in \mathbb{N}^*$. Suppose that $f^q$ is not continuous at $x$. According to Theorem 3.4, there is $\epsilon > 0$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in $X$ converging to $x$ such that $A_k = \{m \in \mathbb{N} : d(f^m(x), f^m(x_k)) \geq \epsilon\} \in q$, for each $k \in \mathbb{N}$. We know that $B_k = \{n \in \mathbb{N} : A_k \in p_n\} \in p$, for all $k \in \mathbb{N}$. By assumption, there is $\delta > 0$ such that if $y \in X$ and $d(x,y) < \delta$, then $d(f^{p_n}(x), f^{p_n}(y)) < \frac{\epsilon}{3}$, for all $n \in \mathbb{N}$. Choose $l \in \mathbb{N}$ such that $d(x, x_k) < \delta$ for each $k \in \mathbb{N}$ with $l \leq k$. Fix $k \in \mathbb{N}$ with $l \leq k$. So, $d(f^{p_n}(x), f^{p_n}(x_k)) < \frac{\epsilon}{3}$ for all $n \in \mathbb{N}$. By Lemma 3.21, we have that

$$C_n = \{m \in \mathbb{N} : d(f^m(x), f^m(x_k)) < \epsilon\} \in p_n,$$

for every $n \in \mathbb{N}$. Pick $n \in B_k$. It then follows that $A_k \cap C_n \in p_n$, which is impossible. □

**Corollary 3.23.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and let $p \in \mathbb{N}^*$. If $\{f^{p+n} : n \in \mathbb{N}\}$ is uniformly equicontinuous at $x \in X$, then $f^{p+q}$ is continuous at $x$, for all $q \in \beta(\mathbb{N})$.*

Proof: Let $p \in \mathbb{N}^*$. We know that the function $\lambda_p : \beta(\mathbb{N}) \to \beta(\mathbb{N})$ given by $\lambda_p(q) = p + q$ is continuous (see [11]). Hence, we obtain that $\lambda_p[\mathrm{cl}_{\beta(\mathbb{N})}\mathbb{N}] = \{p + q : q \in \beta(\mathbb{N})\} = \mathrm{cl}_{\beta(\mathbb{N})}(\lambda_p[\mathbb{N}])$. By Theorem 3.22, we conclude that $f^{p+q}$ is continuous at $x$, for each $q \in \beta(\mathbb{N})$. □

**Theorem 3.24.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and $x \in X$. If $\{q \in \mathbb{N}^* : f^q$ is continuous at $x\}$ is dense in $\mathbb{N}^*$, then $f^p$ is continuous at $x$ for all $p \in \mathbb{N}^*$.*

Proof: Put $D = \{q \in \mathbb{N}^* : f^q$ is continuous at $x\}$. Suppose that $f^p$ is not continuous at $x$ for some $p \in \mathbb{N}^* \setminus D$. Then, by Theorem 3.4, there is $\epsilon > 0$ and for every $k \in \mathbb{N}$ there is $x_k \in X$ such that $x = \lim_{k\to\infty} x_k$ and $A_k = \{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) \geq \epsilon\} \in p$, for each $k \in \mathbb{N}$. We can find $A \in [\mathbb{N}]^\omega$ such that $A \subseteq^* A_k$ for all $k \in \mathbb{N}$. By assumption, there is $q \in A^* \cap D$ for which $f^q$ is continuous at $x$. Hence, we may chose $N \in \mathbb{N}$ such that $d(f^q(x), f^q(x_k)) < \frac{\epsilon}{3}$, for all $k \in \mathbb{N}$ with $k \geq N$. It then follows from Lemma 3.21 that

$$B_k = \{m \in A : d(f^m(x), f^m(x_k)) < \epsilon\} \in q,$$

for all $k \geq N$. Fix $N \leq i \in \mathbb{N}$. We know that $B_i \subseteq^* A_i$. So, if $m \in B_i \cap A_i$, then $d(f^m(x), f^m(x_i)) < \epsilon$ and $d(f^m(x), f^m(x_i)) \geq \epsilon$, but this is impossible. Therefore, $f^p$ is continuous at $x$, for all $p \in \mathbb{N}^*$. □

**Theorem 3.25.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and $x \in X$. Let $1 < k \in \mathbb{N}$. For $i < k$, we define $A_i = \{n \in \mathbb{N} : n \cong i \bmod(k)\}$. If there is $j < k$ such that $f^q$ is continuous at $x$ for all $q \in A_j^*$, then $f^p$ is continuous at $x$ for every $p \in \mathbb{N}^*$.*

Proof: First, observe that $\mathbb{N}^* = \bigcup_{i<k} A_i^*$. Let $j \neq i < k$. We define $\phi_i : \mathbb{N} \to \mathbb{N}$ by $\phi_i(n) = |n + i - j|$ for every $n \in \mathbb{N}$. It is not hard to see that $\phi_i$ is a bijection between $A_j$ and $A_i$ module a finite set. Hence, if $p \in A_i^*$, then there is $q \in A_j^*$ such that $\overline{\phi_i}(q) = q + i - j = p$. Thus, if $i > j$ and $f^q$ is continuous at $x$, then $f^{q+i-j} = f^p = f^{i-j} \circ f^q$ is continuous at $x$. If $i < j$, then we consider the function $\phi_{k+i}$ which is also a bijection between $A_j$ and $A_i$ module a finite set. Thus, for a given $p \in A_i^*$ there is $q \in A_j^*$ such that $\overline{\phi_{k+i}}(q) = q + k + i - j = p$ and then $f^{q+k+i-j} = f^p = f^{k+i-j} \circ f^q$ is continuous at $x$ whenever $f^q$ is continuous at $x$. $\square$

Let $(X, f)$ be a dynamical system, where $X$ is a metric compact space, and let $x \in X$. The previous corollary assures that if $f^p$ is continuous at $x$, for all $p \in \{an : n \in \mathbb{N}\}^*$, where $a \in \mathbb{N}$, then $f^p$ is continuous at $x$, for all $p \in \mathbb{N}^*$.

**Lemma 3.26.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and let $x \in X$ be a fixed point of $f$. Suppose that there is $\epsilon > 0$ such that for every $k \in \mathbb{N}$ there are $x_k, y_k \in X$ such that $d(x, x_k) < \frac{1}{k+1}$, $\mathcal{O}_f(y_k) \cap B(x, \epsilon) = \emptyset$ and $\mathcal{O}_f(y_k) \cap \mathcal{O}_f(x_k) \neq \emptyset$. Then, $f^p$ is discontinuous at $x$ for every $p \in \mathbb{N}^*$.*

PROOF: Fix $k \in \mathbb{N}$. We know that $f^l(x_k) = f^m(y_k)$, for some $l, m \in \mathbb{N}$. Then, $f^{l+a}(x_k) = f^{m+a}(y_k) \in \mathcal{O}_f(y_k)$, for all $a \in \mathbb{N}$. Hence, $\{n \in \mathbb{N} : d(f^n(x_k), x) \geq \epsilon\}$ is a cofinite subset of $\mathbb{N}$ and so

$$\{n \in \mathbb{N} : d(f^n(x_k), f^n(x)) \geq \epsilon\} = \{n \in \mathbb{N} : d(f^n(x_k), x) \geq \epsilon\} \in p,$$

for each $p \in \mathbb{N}^*$. Therefore, $f^p$ is discontinuous at $x$ for every $p \in \mathbb{N}^*$. $\square$

**Theorem 3.27.** *Let $(X, f)$ be a dynamical system such that $X$ is a compact metric space with only one non-isolated point. Then, either $f^p$ is continuous for all $p \in \mathbb{N}^*$ or $f^p$ is discontinuous for all $p \in \mathbb{N}^*$.*

PROOF: Let $x$ be the unique non-isolated point of $X$. First, suppose that $f(x) \neq x$. Then, we have that $A = \{y \in X : f(y) = f(x)\}$ is cofinite. If $y \in A$ and $n \in \mathbb{N}$, then $f^n(y) = f^n(x)$; hence, we deduce that $f^p(y) = f^p(x)$ for all $y \in A$ and for all $p \in \mathbb{N}^*$. Thus, $f^p$ is continuous, for all $p \in \mathbb{N}^*$. Now, we assume that $f(x) = x$. Let $\epsilon > 0$ and let $X \setminus B(x, \epsilon) = \{x_0, \ldots, x_m\}$. Put $F = \{i \leq m : \mathcal{O}_f(x_i)$ is finite$\}$ and $I = m \setminus F$. We may also assume that $x \notin \mathcal{O}_f(x_i)$ for every $i \in F$. Suppose that the conditions of the previous lemma fail. Then, we can find $\delta > 0$ such that $B(x, \delta) \cap \mathcal{O}_f(x_i) = \emptyset$, for each $i \leq F$, and if $d(x, y) < \delta$, then $\mathcal{O}_f(y) \cap \mathcal{O}_f(z) = \emptyset$, whenever $\mathcal{O}_f(z) \cap B(x, \epsilon) = \emptyset$. Let $y \in X$ such that $d(x, y) < \delta$. If $\mathcal{O}_f(y) \cap \mathcal{O}_f(x_i) \neq \emptyset$ for some $i \in I$, then $\lim_{n \to \infty} f^n(y) = x$ and hence $\{n \in \mathbb{N} : d(x, f^n(y)) < \epsilon\} \in p$ for all $p \in \mathbb{N}^*$. Suppose that $\mathcal{O}_f(y)$ does not intersect any $\mathcal{O}_f(x_i)$, for all $i \leq m$. Then, $\mathcal{O}_f(y) \subseteq B(x, \epsilon)$. So, $\mathbb{N} = \{n \in \mathbb{N} : d(x, f^n(y)) < \epsilon\} \in p$ for all $p \in \mathbb{N}^*$. This shows that $f^p : X \to X$ is continuous, for each $p \in \mathbb{N}^*$. $\square$

**Theorem 3.28.** *Let $(X, f)$ be a dynamical system such that $X$ is a compact metric space and let $x \in X$ be a fixed point of $f$. Suppose that there is $m \in \mathbb{N}$ such that $|\mathcal{O}_f(y)| \leq m$, for all $y \in X$. Then, either $f^p$ is continuous at $x$ for all $p \in \mathbb{N}^*$, or $f^p$ is discontinuous at $x$ for all $p \in \mathbb{N}^*$.*

PROOF: Suppose that $f^p$ is continuous at $x$ and $f^q$ is discontinuous at $x$, for some $p, q \in \mathbb{N}^*$. Then there are $\epsilon > 0$ and a sequence $(x_k)_{k \in \mathbb{N}}$ in $X$ converging to $x$ such that $\{n \in \mathbb{N} : d(x, f^n(x_k)) \geq \epsilon\} \in q$, for all $k \in \mathbb{N}$. By the continuity of $f^p$ and Theorem 3.4, there is $\delta > 0$ such that $\delta < \epsilon$ and if $y \in X$ and $d(x, y) < \delta$, then $\{n \in \mathbb{N} : d(x, f^n(y)) < \epsilon\} \in p$. We know that there is $M \in \mathbb{N}$ such that $d(x, x_k) < \delta$ for all $M \leq k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$ with $k \geq M$, there is $0 < m_k \leq m$ so that $d(x, f^{m_k+1}(x_k)) \geq \epsilon$ and $m_k$ is the minimum positive integer with this property. Without loss of generality, we may assume that there is $l \leq m$ for which $m_k = l$, for each $k \in \mathbb{N} \setminus M$. Since $f$ is continuous we can find $0 < \delta_l < \delta_{l-1} < \cdots < \delta_0 < \epsilon$ such that if $d(x, y) < \delta_i$, then $d(x, f(y)) < \delta_{i-1}$, for every $0 \leq i < l$, and if $d(x, y) < \delta_0$, then $d(x, f(x)) < \epsilon$. Choose $N \in \mathbb{N}$ such that $M < N$ and $d(x, x_k) < \delta_l$, for every $N \leq k \in \mathbb{N}$. Then, we have that $d(x, f^l(x_k)) < \delta_0$, for each $N \leq k \in \mathbb{N}$. But, this is impossible since $d(x, f^{l+1}(x_k)) \geq \epsilon$, for every $N \leq k \in \mathbb{N}$. $\square$

We finish this section with some conditions that are equivalent to the continuity of all functions $f^p$'s.

**Theorem 3.29.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space. Let us consider the function $F^* : \mathbb{N}^* \times X \to X$ given by $F^*(p, x) = f^p(x)$, for every $(p, x) \in \mathbb{N}^p \times X$. Then, the following conditions are equivalent.*

(1) *$f^p$ is continuous on $X$, for every $p \in \mathbb{N}^*$ (that is, $F^*$ is separately continuous).*
(2) *There is a dense $G_\delta$-subset $D$ of $\mathbb{N}^*$ such that $F^*|_{D \times X}$ is continuous.*
(3) *There is a dense subset $D$ of $\mathbb{N}^*$ such that $F^*|_{D \times X}$ is continuous.*

PROOF: The implication (1) $\Rightarrow$ (2) follows directly from Namioka's Theorem ([2, Theorem III.5.5], [12]), the implication (2) $\Rightarrow$ (3) is trivial and the implication (3) $\Rightarrow$ (1) follows directly from Theorem 3.24. $\square$

## 4. Dynamical systems and actions of metrizable semigroups

Throughout this section, $(X, f)$ will stand for a dynamical system where $X$ is a compact metric space. From now on to avoid trivial situations we assume that $X$ is infinite and that for every couple of natural numbers $(n, m)$ there exists $x \in X$ such that $f^n(x) \neq f^m(x)$. Our main goal is to establish that the action $F : \beta(\mathbb{N}) \times X \to X$ induced by $(X, f)$ is (in some sense) equivalent to the action of a metrizable semigroup on $X$. To do this, let us define an equivalent relation $\sim$ on $\beta(\mathbb{N})$ by letting $p \sim q$ if and only if $f^p(x) = f^q(x)$ for every $x \in X$. If $d$ is a compatible metric on $X$, the real-valued function on $\beta(\mathbb{N}) \times \beta(\mathbb{N})$ defined by

$$\bar{d}(p,q) = \sup_{x \in X} d(f^p(x), f^q(x)) \qquad p, q \in \beta(\mathbb{N}),$$

is a pseudometric on $\beta(\mathbb{N})$ (notice that being $X$ compact, $d$ is bounded). It is clear that $\bar{d}$ induces a metric (also denoted by $\bar{d}$) on the quotient space $\beta(\mathbb{N})/\sim$. The following result follows from Theorem 3.3.

**Proposition 4.1.** *$\beta(\mathbb{N})/\sim$ is a semigroup with the addition $+$ defined as*

$$[\,p\,] + [\,q\,] = [\,p + q\,],$$

*for each $p, q \in \beta(\mathbb{N})$.*

As we deal with actions on metrizable semigroups, a natural question is when the semigroup $(\beta(\mathbb{N})/\sim, +)$ equipped with the topology induced by the metric $\bar{d}$ is a topological semigroup; that is, when the operation defined in Proposition 4.1 is continuous. A useful sufficient condition is given in Theorem 4.3 below. Before the statement of this theorem, we prove a lemma.

**Lemma 4.2.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space. If the family of functions $\{f^n : n \in \mathbb{N}\}$ is uniformly equicontinuous, then the family $\{f^p : p \in \mathbb{N}^*\}$ is also uniformly equicontinuous.*

Proof: By assumption, given $\epsilon > 0$ we can find $\delta > 0$ such that if $x, y \in X$ satisfy that $d(x,y) < \delta$, then $d(f^n(x), f^n(y)) < \frac{\epsilon}{3}$ for all $n \in \mathbb{N}$. Let $x, y \in X$. Assume that $d(x,y) < \delta$ and fix $p \in \mathbb{N}^*$. Choose $n \in \mathbb{N}$ so that $d(f^p(x), f^n(x)) < \frac{\epsilon}{3}$ and $d(f^p(y), f^n(y)) < \frac{\epsilon}{3}$. Then, we obtain that

$$\begin{aligned} d(f^p(x), f^p(y)) &\le d(f^p(x), f^n(x)) + d(f^n(x), f^n(y)) + d(f^n(x), f^p(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, the family $\{f^p : p \in \mathbb{N}^*\}$ is also uniformly equicontinuous. $\square$

**Theorem 4.3.** *Assume that the family $\{f^n : n \in \mathbb{N}\}$ is uniformly equicontinuous, then $\beta(\mathbb{N})/\sim$ is a topological semigroup with the topology induced by the metric $\bar{d}$.*

Proof: Let $[\,p\,], [\,q\,] \in \beta(\mathbb{N})/\sim$. We know from Lemma 4.2 that the family of functions $\{f^t : t \in \beta(\mathbb{N})\}$ is also uniformly equicontinuous. Hence, given $\epsilon > 0$ there is $\delta > 0$ such that $\delta < \frac{\epsilon}{2}$ and if $x, y \in X$ and $d(x,y) < \delta$, then $d(f^t(x), f^t(y)) < \frac{\epsilon}{2}$ for all $t \in \beta(\mathbb{N})$. Suppose that $r, s \in \beta(\mathbb{N})$ satisfy that

$$\bar{d}(p,r) = \sup\{d(f^p(x), f^r(x)) : x \in X\} < \delta$$

and

$$\bar{d}(q,s) = \sup\{d(f^q(x), f^s(x)) : x \in X\} < \delta.$$

Then, $d(f^s(f^p(x)), f^s(f^r(x))) < \frac{\epsilon}{2}$ and $d(f^q(f^p(x)), f^s(f^p(x))) < \frac{\epsilon}{2}$, for all $x \in X$. Thus,

$$\begin{aligned} d(f^q(f^p(x)), f^s(f^r(x))) &\leq d(f^q(f^p(x)), f^s(f^p(x))) + d(f^s(f^p(x)), f^s(f^r(x))) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for all $x \in X$. Therefore,

$$\bar{d}(p+q, r+s) = \sup\{d(f^q(f^p(x)), f^s(f^r(x))) : x \in X\} \leq \epsilon.$$

This shows the theorem. □

Given a dynamical system $(X, f)$ where $X$ is a compact metric space, and an ultrafilter $p \in \beta(\mathbb{N})$, $f^{[p]}$ stands for the function from $X$ into itself defined by $f^{[p]}(x) = f^p(x)$, for every $x \in X$. Let $F : \beta(\mathbb{N}) \times X \longrightarrow X$ be defined by $F(p, x) = f^p(x)$, for all $(p, x) \in \beta(\mathbb{N}) \times X$. We observe that this action $F$ induces a natural action $\widehat{F} : (\beta(\mathbb{N})/\sim) \times X \longrightarrow X$ defined as

$$\widehat{F}([p], x) = f^{[p]}(x) \qquad \text{for each } ([p], x) \in (\beta(\mathbb{N})/\sim) \times X.$$

Although the authors could not find a specific reference, the following result is probably well known. We include a proof for reader convenience. Given a function $f: X \times Y \to Z$ we shall denote by $f_x$ (respectively, by $f^y$) the function $f_x: Y \to Z$ defined by the rule $f_x(y) = f(x, y)$ for every $y \in Y$ (respectively, by the rule $f^y(x) = f(x, y)$ for every $x \in X$). We recall that, if $X$, $Y$ and $Z$ are topological spaces, then $f$ is said to be *separately continuous* if every $f_x$ and every $f^y$ are continuous functions.

**Theorem 4.4.** *Let $(X, d^1)$, $(Y, d^2)$ and $(Z, d^3)$ be three compact metric spaces. If $f : X \times Y \to Z$ is a separately continuous function, then the following conditions are equivalent.*

(1) *$f$ is continuous.*
(2) *The family $\{f_x \mid x \in X\}$ is uniformly equicontinuous.*
(3) *The family $\{f^y \mid y \in Y\}$ is uniformly equicontinuous.*

Proof: Obviously we only need to prove that the clauses (1) and (2) are equivalent.

(1) ⇒ (2) Consider the space $(C(Y, Z), \|\cdot\|)$ where $\|\cdot\|$ stands for the supremum norm. It is a well-known fact that $f$ continuous implies that the function $g : X \to (C(Y, Z), \|\cdot\|)$ defined as $g(x) = f_x$ is continuous (for a more general result the reader can consult [14, Theorem 3.3]). Let $\varepsilon > 0$. Since $X$ is compact, the

family $g(X) = \{f_x \mid x \in X\}$ is compact so that there exists a finite subfamily $\{f_{x_1}, f_{x_2}, \dots, f_{x_n}\}$ such that

$$\{f_x \mid x \in X\} \subseteq \bigcup_{i=1}^{n} B(f_{x_i}, \varepsilon/3).$$

Moreover, since each $f_{x_i}$ is uniformly continuous, we can choose $\delta > 0$ such that $d_3(f_{x_i}(y_1), f_{x_i}(y_2)) < \frac{\varepsilon}{3}$ whenever $d_2(y_1, y_2) < \delta$, $i = 1, 2, \dots, n$.

Now let $x \in X$ and consider $f_x$. If $f_{x_i}$ satisfies that $f_x \in B(f_{x_i}, \varepsilon/3)$, then

$$\begin{aligned} d_3(f_x(y_1), f_x(y_2)) &\leq d_3(f_x(y_1), f_{x_i}(y_1)) \\ &\quad + d_3(f_{x_i}(y_1), f_{x_i}(y_2)) + d_3(f_{x_i}(y_2), f_x(y_2)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

whenever $d_2(y_1, y_2) < \delta$. Thus, the family $\{f_x \mid x \in X\}$ is uniformly equicontinuous.

(2) $\Rightarrow$ (1). Since the family $\{f_x \mid x \in X\}$ is uniformly equicontinuous it is apparent that the function $g : Y \to (C(X, Z), \|\cdot\|)$ defined as $g(y) = f^y$ is continuous. Now to see that $f$ is continuous, consider a point $(x_0, y_0) \in X \times Y$ and $\varepsilon > 0$. Since both $g$ and $f^{y_0}$ are continuous we can choose $\delta > 0$ such that

$$d_3(f(x, y), f(x, y_0)) < \frac{\varepsilon}{2} \quad \text{and} \quad d_3(f(x, y_0), f(x_0, y_0)) < \frac{\varepsilon}{2}$$

whenever $d_1(x, x_0) < \delta$ and $d_2(y, y_0) < \delta$, that is

$$\begin{aligned} d_3(f(x, y), f(x_0, y_0)) &\leq d_3(f(x, y), f(x, y_0)) + d_3(f(x, y_0), f(x_0, y_0)) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

whenever $d_1(x, x_0) < \delta$ and $d_2(y, y_0) < \delta$. Thus, $f$ is continuous at the point $(x_0, y_0) \in X \times Y$. This completes the proof. □

The proof of the following theorem is straightforward.

**Theorem 4.5.** *Let $(X, f)$ be a dynamical system, where $X$ is a compact metric space, and let $x \in X$. For every $p \in \beta(\mathbb{N})$, the following conditions are equivalent.*

(1) *$f^p$ is continuous at $x$.*
(2) *$f^{[p]}$ is continuous at $x$.*

**Theorem 4.6.** *For a compact metric dynamical system $(X, f)$, the following are equivalent.*

(1) *The set $\{f^n : n \in \mathbb{N}\}$ is uniformly equicontinuous on $X$.*
(2) *$\bar{d}$ induces the quotient topology on $\beta(\mathbb{N})/\sim$ and $F$ is continuous.*
(3) *The action $\widehat{F}$ is (jointly) continuous.*

Proof: The implication (3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) We shall prove that the quotient map $g : \beta(\mathbb{N}) \longrightarrow (\beta(\mathbb{N})/\sim, \bar{d})$ is continuous. Indeed, by Lemma 4.2, we deduce that the family $\{f^p : p \in \mathbb{N}^*\}$ is uniformly equicontinuous. Hence, given $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$ for all $p \in \beta(\mathbb{N})$, and Theorem 4.4 tells us that $F : \beta(\mathbb{N}) \times X \to X$ is continuous. It is clear that $g$ is continuous at any point of $\mathbb{N}$. Let $p \in \mathbb{N}^*$. Then, for every $x \in X$ there are $A_x \in p$ and $\delta_x < \delta$ such that if $(q, y) \in A_x^* \times B(x, \delta_x)$, then $d(f^p(x), f^q(y)) < \frac{\epsilon}{3}$. Since $X$ is compact, there are $x_0, \dots, x_k \in X$ such that $X = \bigcup_{i \le k} B(x_i, \delta_{x_i})$. Put $A = \bigcap_{i \le k} A_{x_i}$. Then, $A \in p$. Fix $q \in A^*$ and let $x \in X$. Then, $x \in B(x_j, \delta_{x_j})$, for some $j \le k$. Thus,

$$\begin{aligned} d(f^p(x), f^q(x)) &\le d(f^p(x), f^p(x_j)) + d(f^p(x_j), f^q(x_j)) + d(f^q(x_j), f^q(x)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

So, $\bar{d}([p], [q]) \le \epsilon$, whenever $q \in A^*$. This shows that $g$ is continuous.

(2) $\Rightarrow$ (3) Since $\beta(\mathbb{N})$ is compact, Whitehead's Theorem ([8, Theorem 3.3.17] and [15]) assures that the function $g \times \mathrm{id}_X : \beta(\mathbb{N}) \times X \to (\beta(\mathbb{N})/\sim) \times X$ is a quotient map. Since $F$ is continuous and $F = \widehat{F} \circ (g \times \mathrm{id}_X)$ is continuous, by Proposition 2.4.2 from [8], we get that the function $\widehat{F}$ is continuous. □

The previous theorem establishes a necessary and sufficient condition in order that the induced action $\widehat{F}$ be continuous. This can be applied to obtain that the action $F$ is equivalent to the action $\widehat{F}$ in the sense of Definition 4.7 below. If $F$ is a continuous action of a (compact) topological semigroup $S$ on a compact metric space, we say that $(S, X, F)$ is a *flow*.

**Definition 4.7.** Let $S, T$ be two compact topological semigroups. Two flows $(S, X, F)$ and $(T, Y, G)$ are said to be *topologically conjugate* (or *equivalent*) if there exists a continuous epimorphism $e : S \longrightarrow T$ and a homeomorphism $h : X \longrightarrow X$ such that the diagram

$$\begin{array}{ccc} S \times X & \xrightarrow{F} & X \\ e \downarrow \quad \downarrow h & & \downarrow h \\ T \times Y & \xrightarrow{G} & Y \end{array}$$

commutes, that is, $h(F(s,x)) = G((e \times h)(s,x))$ for each $(s,x) \in S \times X$.

From Theorem 4.6 we can see that a continuous action of $\beta(\mathbb{N})$ on a compact metric space $X$ is equivalent to a continuous action of a compact metrizable semigroup.

**Theorem 4.8.** *If $X$ is a compact metric space, then every flow $(\beta(\mathbb{N}), X, F)$ is equivalent to a flow $(S, X, G)$ where $S$ is compact metrizable semigroup.*

Proof: By density, the action $F$ is determined by its restriction to $\mathbb{N} \times X$. So, $F$ is the action induced by the dynamical system $(X, f)$ where $f$ is the function defines as $f(x) = F(1, x)$ for every $x \in X$. Since $F$ is continuous, Theorem 4.4 and Theorem 4.6 assert that $(\beta(\mathbb{N})/\!\sim, X, \widehat{F})$ is a flow. Hence, the diagram

$$\begin{tikzcd}
\beta(\mathbb{N}) \arrow[d, "g"'] & \times & X \arrow[d, "\mathrm{id}_X"] \arrow[r, "F"] & X \arrow[d, "\mathrm{id}_X"] \\
S & \times & X \arrow[r, "\widehat{F}"] & X
\end{tikzcd}$$

commutes, where $S = \beta(\mathbb{N})/\!\sim$ and $g$ is the quotient map. By Proposition 4.1, $g$ is an epimorphism. The proof is done by taking $(S, X, G) = (\beta(\mathbb{N})/\!\sim, X, \widehat{F})$. $\square$

## 5. Open questions

We end with some open questions that the authors were unable to solve.

**Question 5.1.** Given $p, q \in \mathbb{N}^*$ such that $p + n \neq q$, for all $n \in \mathbb{N}$, is there a dynamical system $(X, f)$ and a point $x \in X$ such that $X$ is a compact metric space, $f^p$ is continuous at $x$ and $f^q$ is discontinuous at $x$ ?

**Question 5.2.** Given $p, q \in \mathbb{N}^*$ such that $p + n \neq q$, for all $n \in \mathbb{N}$, is there a dynamical system $(X, f)$ and a point $x \in X$ such that $X$ is a connected, compact metric space, $f^p$ is continuous at $x$ and $f^q$ is discontinuous at $x$ ?

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