Positive solutions for systems of generalized three-point nonlinear boundary value problems

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Abstract. Values of λ are determined for which there exist positive solutions of the system of three-point boundary value problems, $u'' + \lambda a(t)f(v) = 0$, $v'' + \lambda b(t)g(u) = 0$, for $0 < t < 1$, and satisfying, $u(0) = \beta u(\eta)$, $u(1) = \alpha u(\eta)$, $v(0) = \beta v(\eta)$, $v(1) = \alpha v(\eta)$. A Guo-Krasnosel'skii fixed point theorem is applied.

Keywords: generalized three-point boundary value problem, system of differential equations, eigenvalue problem

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1. Introduction

We are concerned with determining values of λ (eigenvalues) for which there exist positive solutions for the system of three-point boundary value problems,

$$\begin{cases} u''(t) + \lambda a(t) f(v(t)) = 0, & 0 < t < 1, \\ v''(t) + \lambda b(t) g(u(t)) = 0, & 0 < t < 1, \end{cases} \tag{1}$$

$$\begin{cases} u(0) = \beta u(\eta), & u(1) = \alpha u(\eta), \\ v(0) = \beta v(\eta), & v(1) = \alpha v(\eta), \end{cases} \tag{2}$$

where $0 < \eta < 1$, $0 < \alpha < 1/\eta$, $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$ and

(A) $f, g \in C([0,\infty), [0,\infty))$,
(B) $a, b \in C([0,1], [0,\infty))$, and each does not vanish identically on any subinterval,
(C) all of

$$f_0 := \lim_{x\to 0^+} \frac{f(x)}{x}, \quad g_0 := \lim_{x\to 0^+} \frac{g(x)}{x},$$
$$f_\infty := \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{and} \quad g_\infty := \lim_{x\to\infty} \frac{g(x)}{x}$$

exist as positive real numbers.

For several years now, there has been a great deal of activity in studying positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from both a theoretical sense [3], [5], [8], [11], [18] and as applications for which only positive solutions are meaningful [1], [4], [12], [13]. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [9], [10], [15], [17], [19]. The existence of positive solutions for three-point boundary value problems has been studied extensively in recent years. For some appropriate references we refer the reader to [15], [16]. Recently in [14], the existence of positive solutions was studied for the following generalized second order three-point boundary value problem

$$y''(t) + a(t)f(y(t)) = 0, \quad 0 < t < T, \tag{3}$$

$$y(0) = \beta y(\eta),\ y(T) = \alpha y(\eta). \tag{4}$$

When $\beta = 0$, the conditions (4) reduce to the usual three-point boundary conditions

$$y(0) = 0,\ y(T) = \alpha y(\eta). \tag{5}$$

Recently Benchohra *et al.* [2] and Henderson and Ntouyas [6] studied the existence of positive solutions for systems of nonlinear eigenvalue problems. Also Henderson and Ntouyas [7] studied the existence of positive solutions for systems of nonlinear eigenvalue problems for three-point boundary conditions of the form (5) with $T = 1$. Here we extend these results to eigenvalue problems for the systems of generalized three-point boundary value problems (1), (2). The main tool in this paper is an application of the Guo-Krasnosel'skii fixed point theorem for operators leaving a Banach space cone invariant [5]. A Green's function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. Some preliminaries

In this section, we state some preliminary lemmas and the well-known Guo-Krasnosel'skii fixed point theorem.

Lemma 2.1 ([14]). *Let $\beta \neq \frac{1-\alpha\eta}{1-\eta}$; then for any $y \in C[0,1]$, the boundary value problem*

$$u''(t) + y(t) = 0, \quad 0 < t < 1 \tag{6}$$

$$u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta), \tag{7}$$

has the unique solution

$$u(t) = \int_0^1 k(t,s)y(s)\,ds$$

where $k(t,s):[0,1]\times[0,1]\to\mathbb{R}^+$ *is defined by*

$$(8)\quad k(t,s)=\begin{cases}\frac{[(1-\beta)t+\beta\eta](1-s)}{1-\alpha\eta-\beta(1-\eta)} \\ \quad +\frac{[(\beta-\alpha)t-\beta](\eta-s)}{1-\alpha\eta-\beta(1-\eta)}-(t-s), & 0\le s\le t\le 1 \text{ and } s\le\eta,\\ \frac{[(1-\beta)t+\beta\eta](1-s)}{1-\alpha\eta-\beta(1-\eta)}+\frac{[(\beta-\alpha)t-\beta](\eta-s)}{1-\alpha\eta-\beta(1-\eta)}, & 0\le t\le s\le\eta,\\ \frac{[(1-\beta)t+\beta\eta](1-s)}{1-\alpha\eta-\beta(1-\eta)}, & 0\le t\le s\le 1 \text{ and } s\ge\eta,\\ \frac{[(1-\beta)t+\beta\eta](1-s)}{1-\alpha\eta-\beta(1-\eta)}-(t-s), & \eta\le s\le t\le 1.\end{cases}$$

Notice that by Lemma 2.1 it follows that

$$(9)\quad \begin{aligned}u(t)&=\frac{(1-\beta)t+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-s)y(s)\,ds\\&\quad+\frac{(\beta-\alpha)t-\beta}{1-\alpha\eta-\beta(1-\eta)}\int_0^\eta(\eta-s)y(s)\,ds-\int_0^t(t-s)y(s)\,ds.\end{aligned}$$

If $y\ge 0$ and $0<\beta<\frac{1-\alpha\eta}{1-\eta}$, from (9) we have that

$$(10)\qquad u(t)\le\frac{(1-\beta)t+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-s)y(s)\,ds,$$

and

$$(11)\qquad u(\eta)\ge\frac{\eta}{1-\alpha\eta-\beta(1-\eta)}\int_\eta^1(1-s)y(s)\,ds.$$

Lemma 2.2 ([14])**.** *Let* $0<\alpha<1/\eta$, $0<\beta<\frac{1-\alpha\eta}{1-\eta}$ *and assume that* (A) *and* (B) *hold. Then, the unique solution of* (1)–(2) *satisfies*

$$\inf_{t\in[0,1]}u(t)\ge\gamma\|u\|,$$

where $\gamma=\min\left\{\alpha\eta,\frac{\alpha(1-\eta)}{1-\alpha\eta},\beta\eta,\beta(1-\eta)\right\}$.

We note that a pair $(u(t),v(t))$ is a solution of the eigenvalue problem (1), (2) if, and only if,

$$u(t)=\lambda\int_0^1k(t,s)a(s)f\left(\lambda\int_0^1k(s,r)b(r)g(u(r))\,dr\right)ds,\quad 0\le t\le 1,$$

and

$$v(t)=\lambda\int_0^1k(t,s)b(s)g(u(s))\,ds,\quad 0\le t\le 1.$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1), (2) will be determined via applications of the following fixed point theorem, which is now commonly called the Guo-Krasnosel'skii fixed point theorem.

Theorem 1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Positive solutions in a cone

In this section, we apply Theorem 1 to obtain positive solution pairs of (1), (2). For our construction, let $\mathcal{B} = C[0,1]$ be equipped with the usual supremum norm, $\|\cdot\|$, and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \geq 0 \ \text{ on } \ [0,1], \ \text{and} \ \min_{t \in [\eta,\, 1]} x(t) \geq \gamma \|x\| \right\}.$$

For our first result, we define the positive numbers L_1 and L_2 by

$$L_1 := \max\left\{ \left[\frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)a(r)f_\infty \, dr \right]^{-1}, \left[\frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)g_\infty \, dr \right]^{-1} \right\},$$

and

$$L_2 := \min\left\{ \left[\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)} \int_0^1 (1-r)a(r)f_0 \, dr \right]^{-1}, \left[\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)} \int_0^1 (1-r)b(r)g_0 \, dr \right]^{-1} \right\}.$$

Theorem 2. *Assume that conditions* (A), (B) *and* (C) *hold. Then, for each λ satisfying*

$$L_1 < \lambda < L_2, \tag{12}$$

there exists a pair (u,v) satisfying (1), (2) *such that $u(x) > 0$ and $v(x) > 0$ on $(0,1)$.*

PROOF: Let λ be as in (12), and let $\epsilon > 0$ be chosen such that

$$\max\left\{\left[\frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)}\int_\eta^1(1-r)a(r)(f_\infty-\epsilon)\,dr\right]^{-1},\right.$$
$$\left.\left[\frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)}\int_\eta^1(1-r)b(r)(g_\infty-\epsilon)\,dr\right]^{-1}\right\}\le\lambda$$

and

$$\lambda\le\min\left\{\left[\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-r)a(r)(f_0+\epsilon)\,dr\right]^{-1},\right.$$
$$\left.\left[\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-r)b(r)(g_0+\epsilon)\,dr\right]^{-1}\right\}.$$

Define an integral operator $T:\mathcal{P}\to\mathcal{B}$ by

$$Tu(t):=\lambda\int_0^1 k(t,s)a(s)f\left(\lambda\int_0^1 k(s,r)b(r)g(u(r))\,dr\right)ds,\quad u\in\mathcal{P}. \tag{13}$$

We seek suitable fixed points of T in the cone $\mathcal{P}$. By Lemma 2.2, $T\mathcal{P}\subset\mathcal{P}$. In addition, standard arguments show that T is completely continuous. Now, from the definitions of f_0 and g_0, there exists an $H_1>0$ such that

$$f(x)\le(f_0+\epsilon)x\ \text{ and }\ g(x)\le(g_0+\epsilon)x,\quad 0<x\le H_1.$$

Let $u\in\mathcal{P}$ with $\|u\|=H_1$. First, from (10) and the choice of ϵ, we have

$$\begin{aligned}\lambda\int_0^1 k(s,r)b(r)g(u(r))\,dr&\le\lambda\frac{(1-\beta)t+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-r)b(r)g(u(r))\,dr\\&\le\lambda\frac{(1-\beta)t+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-r)b(r)(g_0+\epsilon)u(r)\,dr\\&\le\lambda\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-r)b(r)\,dr(g_0+\epsilon)\|u\|\\&\le\|u\|\\&=H_1.\end{aligned}$$

As a consequence, in view of (10), and the choice of ϵ, we obtain

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 k(t,s)a(s)f\left(\lambda \int_0^1 k(s,r)b(r)g(u(r))\,dr\right)ds \\
&\leq \lambda \frac{(1-\beta)t+\beta\eta}{1-\alpha\eta-\beta(1-\eta)} \int_0^1 (1-s)a(s)f\left(\lambda \int_0^1 k(s,r)b(r)g(u(r))\,dr\right)ds \\
&\leq \lambda \frac{(1-\beta)t+\beta\eta}{1-\alpha\eta-\beta(1-\eta)} \int_0^1 (1-s)a(s)(f_0+\epsilon)\lambda \int_0^1 k(s,r)b(r)g(u(r))\,dr\,ds \\
&\leq \lambda \frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)} \int_0^1 (1-s)a(s)(f_0+\epsilon)H_1\,ds \\
&\leq H_1 \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \leq \|u\|$ for every $u \in \mathcal{P}$ with $\|u\| = H_1$. Hence if we set

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \leq \|u\|, \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_1. \tag{14}$$

Next, by the definitions of f_∞ and g_∞, there exists an $\overline{H}_2 > 0$ such that

$$f(x) \geq (f_\infty - \epsilon)x \quad \text{and} \quad g(x) \geq (g_\infty - \epsilon)x, \quad x \geq \overline{H}_2.$$

Let

$$H_2 = \max\left\{2H_1, \frac{\overline{H}_2}{\gamma}\right\}.$$

Then, for $u \in \mathcal{P}$ and $\|u\| = H_2$,

$$\min_{t\in[\eta,1]} u(t) \geq \gamma\|u\| \geq \overline{H}_2.$$

Consequently, from (11) and the choice of ϵ, we find

$$\begin{aligned}
\lambda \int_0^1 k(s,r)b(r)g(u(r))\,dr &\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)g(u(r))\,dr \\
&\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)g(u(r))\,dr \\
&\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)(g_\infty-\epsilon)u(r)\,dr \\
&\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)(g_\infty-\epsilon)\,dr\gamma\|u\| \\
&\geq \|u\| \\
&= H_2.
\end{aligned}$$

And so, we have from (11) and the choice of ϵ,

$$\begin{aligned}
Tu(\eta) &\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s) f\left(\lambda \int_0^1 k(s,r)b(r)g(u(r))\,dr\right)\,ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s)(f_\infty-\epsilon)\lambda \int_0^1 k(s,r)b(r)g(u(r))\,dr\,ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s)(f_\infty-\epsilon)H_2\,ds \\
&\geq \lambda \frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s)(f_\infty-\epsilon)H_2\,ds \\
&\geq H_2 \\
&= \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{15}$$

In view of (14) and (15), applying Theorem 1 we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, and with v defined by

$$v(t) = \lambda \int_0^1 k(t,s)b(s)g(u(s))\,ds,$$

the pair (u,v) is a desired solution of (1), (2) for the given λ. The proof is complete. □

Prior to our next result, we define positive numbers L_3 and L_4 by

$$\begin{aligned}
L_3 := \max\Bigg\{ &\left[\frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)a(r)f_0\,dr\right]^{-1}, \\
&\left[\frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)g_0\,dr\right]^{-1} \Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
L_4 := \min\Bigg\{ &\left[\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)} \int_0^1 (1-r)a(r)f_\infty\,dr\right]^{-1}, \\
&\left[\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)} \int_0^1 (1-r)b(r)g_\infty\,dr\right]^{-1} \Bigg\}.
\end{aligned}$$

Theorem 3. *Assume that conditions* (A)–(C) *hold. Then, for each* λ *satisfying*

$$L_3 < \lambda < L_4, \tag{16}$$

there exists a pair (u, v) *satisfying* (1), (2) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(0, 1)$.

PROOF: Let λ be as in (16) and $\epsilon > 0$ be chosen such that

$$\max\left\{\left[\frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)}\int_\eta^1 (1-r)a(r)(f_0-\epsilon)\,dr\right]^{-1}, \left[\frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)}\int_\eta^1 (1-r)b(r)(g_0-\epsilon)\,dr\right]^{-1}\right\} \le \lambda$$

and

$$\lambda \le \min\left\{\left[\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1 (1-r)a(r)(f_\infty+\epsilon)\,dr\right]^{-1}, \left[\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1 (1-r)b(r)(g_\infty+\epsilon)\,dr\right]^{-1}\right\}.$$

Let T be the cone preserving, completely continuous operator defined by (13). By the definitions of f_0 and g_0, there exists an $\overline{H}_3 > 0$ such that

$$f(x) \ge (f_0-\epsilon)x \quad \text{and} \quad g(x) \ge (g_0-\epsilon)x, \quad 0 < x \le \overline{H}_3.$$

Also, from the definition of g_0 it follows that $g(0) = 0$ and so there exists $0 < H_3 < \overline{H}_3$ such that

$$\lambda g(x) \le \frac{\overline{H}_3}{\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1 (1-r)b(r)\,dr}, \quad 0 \le x \le H_3.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_3$. Then

$$\begin{aligned}
\lambda\int_0^1 k(s,r)b(r)g(u(r))\,dr &\le \lambda\frac{(1-\beta)t+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1 (1-r)b(r)g(u(r))\,dr \\
&\le \lambda\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1 (1-r)b(r)g(u(r))\,dr \\
&\le \frac{\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1 (1-r)b(r)\overline{H}_3\,dr}{\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1 (1-s)b(s)\,ds} \\
&\le \overline{H}_3.
\end{aligned}$$

Then, by (11)

$$
\begin{aligned}
Tu(\eta) &\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s)\times \\
&\qquad \times f\left(\lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)g(u(r))\,dr\right) ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s)\times \\
&\qquad \times (f_0-\epsilon)\lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)g(u(r))\,dr\,ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s)\times \\
&\qquad \times (f_0-\epsilon)\lambda \frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-r)b(r)(g_0-\epsilon)\|u\|\,dr\,ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s)(f_0-\epsilon)\|u\|\,ds \\
&\geq \lambda \frac{\gamma\eta}{1-\alpha\eta-\beta(1-\eta)} \int_\eta^1 (1-s)a(s)(f_0-\epsilon)\|u\|\,ds \\
&\geq \|u\|.
\end{aligned}
$$

So, $\|Tu\| \geq \|u\|$. If we put

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_3\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_3. \tag{17}$$

Next, by the definitions of f_∞ and g_∞, there exists an $\overline{H}_4$ such that

$$f(x) \leq (f_\infty+\epsilon)x \;\text{ and }\; g(x) \leq (g_\infty+\epsilon)x, \quad x \geq \overline{H}_4.$$

Clearly, since g_∞ is assumed to be a positive real number, it follows that g is unbounded at ∞, and so, there exists an $\widetilde{H}_4 > \max\{2H_3, \overline{H}_4\}$ such that $g(x) \leq g(\widetilde{H}_4)$, for $0 < x \leq \widetilde{H}_4$.

Set

$$f^*(t) = \sup_{0\leq s\leq t} f(s), \quad g^*(t) = \sup_{0\leq s\leq t} g(s), \qquad \text{for} \quad t \geq 0.$$

Clearly f^* and g^* are nondecreasing real valued function for which it holds

$$\lim_{x\to\infty}\frac{f^*(x)}{x}=f_\infty,\quad \lim_{x\to\infty}\frac{g^*(x)}{x}=g_\infty.$$

Hence, there exists an H_4 such that $f^*(x)\le f^*(H_4)$, $g^*(x)\le g^*(H_4)$ for $0<x\le H_4$. For $u\in\mathcal{P}$ with $\|u\|=H_4$, we have

$$\begin{aligned}
Tu(t) &= \lambda\int_0^1 k(t,s)a(s)f\left(\lambda\int_0^1 k(s,r)b(r)g(u(r))\,dr\right)\,ds\\
&\le \lambda\int_0^1 k(t,s)a(s)f^*\left(\lambda\int_0^1 k(s,r)b(r)g(u(r))\,dr\right)\,ds\\
&\le \lambda\int_0^1 k(t,s)a(s)f^*\left(\lambda\int_0^1 k(s,r)b(r)g^*(u(r))\,dr\right)\,ds\\
&\le \lambda\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-s)a(s)\times\\
&\qquad\times f^*\left(\lambda\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-r)b(r)g^*(H_4)\,dr\right)\,ds\\
&\le \lambda\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-s)a(s)\times\\
&\qquad\times f^*\left(\lambda\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-r)b(r)(g_\infty+\epsilon)H_4\,dr\right)\,ds\\
&\le \lambda\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-s)a(s)f^*(H_4)\,ds\\
&\le \lambda\frac{1-\beta+\beta\eta}{1-\alpha\eta-\beta(1-\eta)}\int_0^1(1-s)a(s)\,ds(f_\infty+\epsilon)H_4\\
&\le H_4\\
&=\|u\|,
\end{aligned}$$

and so $\|Tu\|\le\|u\|$. For this case, if we set

$$\Omega_2=\{x\in\mathcal{B}\mid \|x\|<H_4\},$$

then

$$\|Tu\|\le\|u\|,\ \text{for } u\in\mathcal{P}\cap\partial\Omega_4. \tag{18}$$

Application of part (ii) of Theorem 1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which in turn yields a pair (u, v) satisfying (1), (2) for the chosen value of λ. The proof is complete. $\square$

4. Examples

In this section we give some examples illustrating our results. For the sake of simplicity we take $a(t) = b(t)$ and $f(t) = g(t)$.

Example 1. Consider the three-point boundary value problem

$$\begin{aligned} u''(t) + \frac{1}{10}\lambda t \frac{kve^{2v}}{c + e^v + e^{2v}} &= 0, \quad 0 < t < 1, \\ v''(t) + \frac{1}{10}\lambda t \frac{kue^{2u}}{c + e^u + e^{2u}} &= 0, \quad 0 < t < 1, \\ u(0) = \frac{1}{4}u\left(\frac{1}{3}\right), \quad u(1) &= 2u\left(\frac{1}{3}\right), \\ v(0) = \frac{1}{4}v\left(\frac{1}{3}\right), \quad v(1) &= 2v\left(\frac{1}{3}\right). \end{aligned}$$

Here: $a(t) = b(t) = \frac{1}{10}t$, $k = 500$, $c = 1000$, $\alpha = 2$, $\beta = \frac{1}{4}$, $\eta = \frac{1}{3}$, $f(v) = \frac{kve^{2v}}{c+e^v+e^{2v}}$, $f(u) = \frac{kue^{2u}}{c+e^u+e^{2u}}$. By simple calculations we find: $\gamma = \frac{1}{12}$, $f_0 = g_0 = \frac{k}{c+2} = \frac{500}{1002}$, $f_\infty = g_\infty = k = 500$, $L_1 = \frac{486}{500} \simeq 0.972$, $L_2 = \frac{12024}{500} = 24.048$. By Theorem 2 it follows that for every λ such that $0.972 < \lambda < 24.048$ the three-point boundary value problem has at least one positive solution.

Example 2. Consider the system of three-point boundary value problem

$$\begin{aligned} u''(t) + \lambda tv\left(1 + \frac{c}{1+v^2}\right) &= 0, \quad 0 < t < 1, \\ v''(t) + \lambda tu\left(1 + \frac{c}{1+u^2}\right) &= 0, \quad 0 < t < 1, \\ u(0) = \frac{1}{2}u\left(\frac{1}{4}\right), \quad u(1) &= 2u\left(\frac{1}{4}\right), \\ v(0) = \frac{1}{2}v\left(\frac{1}{4}\right), \quad v(1) &= 2v\left(\frac{1}{4}\right). \end{aligned}$$

Here: $a(t) = b(t) = t$, $c = 100$, $\alpha = 2$, $\beta = \frac{1}{2}$, $\eta = \frac{1}{4}$, $f(v) = v\left(1 + \frac{c}{1+v^2}\right)$, $f(u) = u\left(1 + \frac{c}{1+u^2}\right)$. We find: $\gamma = \frac{1}{8}$, $f_0 = g_0 = 1 + c$, $f_\infty = g_\infty = 1$, $L_3 = \frac{768}{2727} \simeq 0.28$, $L_4 = \frac{6}{5} = 1.2$. Therefore Theorem 3 holds for every λ such that $0.28 < \lambda < 1.2$.

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