Weakly infinite-dimensional compactifications and countable-dimensional compactifications

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Abstract. In this paper we give a characterization of a separable metrizable space having a metrizable S-weakly infinite-dimensional compactification in terms of a special metric. Moreover, we give two characterizations of a separable metrizable space having a metrizable countable-dimensional compactification.

Keywords: S-weakly infinite-dimensional, countable-dimensional, compactification

Classification: Primary 54D35, 54F45

1. Introduction

We assume that all spaces are separable and metrizable. By a compactification of a space X, we mean a compact metrizable space containing X as a dense subspace. We refer the reader to [3] for notions and terminology not explicitly given.

Borst [2] gave a characterization of spaces having a S-weakly infinite-dimensional compactification in terms of a special base. In [6], we obtained a result concerning C-spaces, which is similar to Borst's one.

On the other hand, in [1], Borst gave a characterization of spaces having a compactification which is a C-space in terms of a special metric. In this paper we give an alternative characterization of spaces having a S-weakly infinite-dimensional compactification in terms of a special metric, which is similar to Borst's one.

It is known that the class of C-spaces contains the class of countable-dimensional spaces.

Next, we give a characterization of spaces having a countable-dimensional compactification. The following theorem is well-known.

1.1 Theorem ([3, Theorem 7.2.21]). *A space X has a countable-dimensional compactification if and only if X has small transfinite dimension* trind.

However, by using Borst's method, we give two characterizations of spaces having a countable-dimensional compactification.

For a collection $\mathcal{A}$ of subsets of a space X and for $Y \subset X$ we write $\mathcal{A}|Y$ for $\{A \cap Y : A \in \mathcal{A}\}$, $\bigcup \mathcal{A}$ for $\bigcup\{A : A \in \mathcal{A}\}$, $\bigcap \mathcal{A}$ for $\bigcap\{A : A \in \mathcal{A}\}$ and $[\mathcal{A}]^{<\omega}$ for $\{\mathcal{B} : \mathcal{B}$ is a finite subcollection of $\mathcal{A}\}$.

We denote by $(\tilde{X}, \tilde{d})$ the completion of a metric space (X, d).

For a point x of a metric space (X, d) and for a positive number ε, the set $B(x; \varepsilon) = \{y \in X : \mathrm{d}(x, y) < \varepsilon\}$ is called the ε-ball about x. For a set $A \subset X$ and a positive number ε, by the ε-ball about A we mean $B(A; \varepsilon) = \bigcup\{B(x; \varepsilon) : x \in A\}$.

Let Γ be an index set. A collection $\tau = \{(A_i, B_i) : i \in \Gamma\}$ of pairs of disjoint closed subsets of X is called *essential* if for every $\{L_i : i \in \Gamma\}$, where L_i is a partition in X between A_i and B_i for every $i \in \Gamma$, we have $\bigcap_{i \in \Gamma} L_i \neq \emptyset$; if τ is not essential then it is called *inessential*.

A collection $\mathcal{A}$ of subsets of a space X is called *closure-distributive* if for every finite subcollection $\{A_1, A_2, \cdots, A_n\}$ of $\mathcal{A}$, the equality $\mathrm{Cl}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathrm{Cl}\, A_1 \cap \mathrm{Cl}\, A_2 \cap \cdots \cap \mathrm{Cl}\, A_n$ holds.

1.2 Lemma ([7, Lemma 3.2]). *Let $\mathcal{V}$ be a closure-distributive finite collection of open subsets of a space X and (F, U) be a pair of subsets of X such that F is closed, U is open and $F \subset U$. Then there exists an open subset V of X such that $F \subset V \subset \mathrm{Cl}\, V \subset U$ and $\mathcal{V} \cup \{V\}$ is closure-distributive.*

The following lemma will play an important role in the proof of our main theorem.

1.3 Lemma ([10], cf. [5]). *Every Čech-complete space X has a compactification αX such that $\alpha X - X$ is strongly countable-dimensional.*

2. Spaces having a S-weakly infinite-dimensional compactification

We consider a characterization of spaces having a S-weakly infinite-dimensional compactification in terms of a special metric.

2.1 Definition. A space X is *μ-S-weakly infinite-dimensional* if there exists a totally bounded metric d on X satisfying the following condition:

(∗) For every collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X with $d(A_i, B_i) > 0$ for every $i < \omega$, there exists a collection $\{L_i : i < \omega\}$ of subsets of X such that L_i is a partition in X between A_i and B_i for every $i < \omega$ and $\bigcap_{i<n} L_i = \emptyset$ for some $n < \omega$.

Obviously, every S-weakly infinite-dimensional space is μ-S-weakly infinite-dimensional and every μ-S-weakly infinite-dimensional compact space is S-weakly infinite-dimensional.

A space Y is a *Čech-complete extension* of a space X if Y contains X as a dense subspace and Y is Čech-complete.

2.2 Lemma. *Every μ-S-weakly infinite-dimensional space has a μ-S-weakly infinite-dimensional Čech-complete extension.*

Proof: Let X be a μ-S-weakly infinite-dimensional space and d be a totally bounded metric on X satisfying the condition (∗) in Definition 2.1.

Take an arbitrary countable base $\mathcal{U}$ for $\tilde{X}$ which is closed under finite unions. Note that $\tilde{X}$ is compact. Let us set

$$\mathcal{A} = \left\{ (U, U'; V, V') \ : \ \begin{array}{c} U, U', V, V' \in \mathcal{U}, \operatorname{Cl}_{\tilde{X}} U \subset U', \operatorname{Cl}_{\tilde{X}} V \subset V' \\ \text{and } \operatorname{Cl}_{\tilde{X}} U' \cap \operatorname{Cl}_{\tilde{X}} V' = \emptyset \end{array} \right\}.$$

We enumerate $\mathcal{A}$ as $\mathcal{A} = \{(U_i, U_i'; V_i, V_i') : i < \omega\}$. Let us set

$$\mathbb{D} = \{\Delta \in [\omega]^{<\omega} : \{(\operatorname{Cl}_{\tilde{X}} U_n' \cap X, \operatorname{Cl}_{\tilde{X}} V_n' \cap X) : n \in \Delta\} \ \text{ is inessential in } X\}.$$

Consider an element $\Delta \in \mathbb{D}$. We can take a partition $L(\Delta, n)$ in X between $\operatorname{Cl}_{\tilde{X}} U_n' \cap X$ and $\operatorname{Cl}_{\tilde{X}} V_n' \cap X$ for every $n \in \Delta$ such that $\bigcap_{n \in \Delta} L(\Delta, n) = \emptyset$. For every $n \in \Delta$, we take a partition $\tilde{L}(\Delta, n)$ in $\tilde{X}$ between $\operatorname{Cl}_{\tilde{X}} U_n$ and $\operatorname{Cl}_{\tilde{X}} V_n$ such that $\tilde{L}(\Delta, n) \cap X \subset L(\Delta, n)$. For every $\Delta \in \mathbb{D}$ the set

$$T_\Delta = \bigcap_{n \in \Delta} \tilde{L}(\Delta, n)$$

is closed in $\tilde{X}$ and disjoint from X. Thus the space

$$Y = \tilde{X} - \bigcup \{T_\Delta : \Delta \in \mathbb{D}\}$$

is a Čech-complete extension of X. Let d_Y be the restriction of $\tilde{d}$ to Y. It suffices to show that d_Y satisfies the condition $(*)$ in Definition 2.1. Consider a collection $\{(A_i, B_i) : i < \omega\}$ of pairs of closed subsets of Y with $d_Y(A_i, B_i) > 0$ for every $i < \omega$. Since $\tilde{d}(\operatorname{Cl}_{\tilde{X}} A_i, \operatorname{Cl}_{\tilde{X}} B_i) = d_Y(A_i, B_i) > 0$, we have $\operatorname{Cl}_{\tilde{X}} A_i \cap \operatorname{Cl}_{\tilde{X}} B_i = \emptyset$. Take $U^i, {U'}^i, V^i, {V'}^i \in \mathcal{U}$ such that $\operatorname{Cl}_{\tilde{X}} A_i \subset U^i \subset \operatorname{Cl}_{\tilde{X}} U^i \subset {U'}^i$, $\operatorname{Cl}_{\tilde{X}} B_i \subset V^i \subset \operatorname{Cl}_{\tilde{X}} V^i \subset {V'}^i$, and $\operatorname{Cl}_{\tilde{X}} {U'}^i \cap \operatorname{Cl}_{\tilde{X}} {V'}^i = \emptyset$ for every $i < \omega$. Since $\tilde{d}(\operatorname{Cl}_{\tilde{X}} {U'}^i, \operatorname{Cl}_{\tilde{X}} {V'}^i) > 0$, we have $d(\operatorname{Cl}_{\tilde{X}} {U'}^i \cap X, \operatorname{Cl}_{\tilde{X}} {V'}^i \cap X) > 0$. Thus there exists a partition L^i in X between $\operatorname{Cl}_{\tilde{X}} {U'}^i \cap X$ and $\operatorname{Cl}_{\tilde{X}} {V'}^i \cap X$ for every $i < \omega$ such that $\bigcap_{i<m} L^i = \emptyset$ for some $m < \omega$. Since $\{(\operatorname{Cl}_{\tilde{X}} {U'}^i \cap X, \operatorname{Cl}_{\tilde{X}} {V'}^i \cap X) : i < m\}$ is inessential in X, we have $\{(U^i, {U'}^i; V^i, {V'}^i) : i < m\} \in \mathcal{A}$; thus $(U^i, {U'}^i, V^i; {V'}^i) = (U_{n(i)}, U'_{n(i)}; V_{n(i)}, V'_{n(i)})$ for some $n(i) < \omega$. Letting

$$\Delta = \{n(i) : i < m\},$$

we have $\Delta \in \mathbb{D}$. For every $i < m$, letting

$$L_i = \tilde{L}(\Delta, n(i)) \cap Y,$$

L_i is a partition in Y between A_i and B_i. For every $i \geq m$ we take a partition L_i in Y between A_i and B_i. We have

$$\begin{aligned}\bigcap_{i<m} L_i = \bigcap_{i<m} (\,\tilde{L}(\Delta, n(i)) \cap Y\,) = \bigcap_{n(i)\in\Delta} \tilde{L}(\Delta, n(i)) \cap Y \\ = T_\Delta \cap Y \subset T_\Delta \cap (\tilde{X} - T_\Delta) = \emptyset,\end{aligned}$$

thus d_Y satisfies the condition $(*)$ in Definition 2.1. Hence Y is μ-S-weakly infinite-dimensional. $\square$

2.3 Lemma. *Every Čech-complete μ-S-weakly infinite-dimensional space X has a S-weakly infinite-dimensional compactification.*

Proof: Since X is Čech-complete, by Lemma 1.3, there exists a compactification αX such that the remainder $\alpha X - X$ is strongly countable-dimensional. We shall prove that αX is S-weakly infinite-dimensional.

Let $\{(A_i, B_i) : i < \omega\}$ be a collection of pairs of disjoint closed subsets of αX. For every $i < \omega$, we take two open subsets U_{2i+1} and V_{2i+1} of αX such that $A_{2i+1} \subset U_{2i+1}$, $B_{2i+1} \subset V_{2i+1}$ and $\mathrm{Cl}_{\alpha X} U_{2i+1} \cap \mathrm{Cl}_{\alpha X} V_{2i+1} = \emptyset$. Since $\alpha X - X$ is A-weakly infinite-dimensional, there exists a partition L_{2i+1} in $\alpha X - X$ between $\mathrm{Cl}_{\alpha X} U_{2i+1} \cap (\alpha X - X)$ and $\mathrm{Cl}_{\alpha X} V_{2i+1} \cap (\alpha X - X)$ for every $i < \omega$ such that $\bigcap_{i<\omega} L_{2i+1} = \emptyset$. For every $i < \omega$ we take a partition L'_{2i+1} in αX between A_{2i+1} and B_{2i+1} such that $L'_{2i+1} \cap (\alpha X - X) \subset L_{2i+1}$. Let us set $K = \bigcap_{i<\omega} L'_{2i+1}$. Since K is S-weakly infinite-dimensional, there exists a partition L_{2i} in K between $A_{2i} \cap K$ and $B_{2i} \cap K$ for every $i < \omega$ such that $\bigcap_{i<n} L_{2i} = \emptyset$ for some $n < \omega$. For every $i < \omega$ we take a partition L'_{2i} in αX between A_{2i} and B_{2i} such that $L'_{2i} \cap K \subset L_{2i}$. Obviously, we have $\bigcap_{i<\omega} L'_i = \emptyset$. This implies that αX is A-weakly infinite-dimensional and hence since αX is compact it is also S-weakly infinite-dimensional. $\square$

2.4 Lemma. *Every space X having a S-weakly infinite-dimensional compactification αX is μ-S-weakly infinite-dimensional.*

Proof: Take an arbitrary metric d on αX. Let d_X be the restriction d to X. It is easy to show that d_X satisfies the condition $(*)$ in Definition 2.1. Hence X is μ-S-weakly infinite-dimensional. $\square$

We now come to our main theorem.

2.5 Theorem. *A space X has a S-weakly infinite-dimensional compactification if and only if X is μ-S-weakly infinite-dimensional.*

Proof: The theorem follows from Lemmas 2.2, 2.3 and 2.4. $\square$

2.6 Problem. Does Lemma 2.2 remain true if we replace 'a totally bounded metric on X' in Definition 2.1 by 'a metric on X'?

3. Spaces having a countable-dimensional compactification

In this section we consider characterizations of spaces having a countable-dimensional compactification.

A collection $\mathcal{A}$ of subsets of a space X is *strongly point-finite* if for every infinite subcollection $\mathcal{A}'$ of $\mathcal{A}$ there exists $\mathcal{A}'' \in [\mathcal{A}']^{<\omega}$ such that $\cap\mathcal{A}'' = \emptyset$.

We need the following theorem to prove our main theorems.

3.1 Theorem ([5, Theorem 1]). *A space X has small transfinite dimension* trind *if and only if X has a base $\mathcal{B}$ such that $\{\operatorname{Bd} B : B \in \mathcal{B}\}$ is strongly point-finite.*

On the other hand, the following theorem is well-known.

3.2 Theorem ([8], [9]). *A space X is countable-dimensional if and only if for every collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X, there exists a collection $\{L_i : i < \omega\}$ of subsets of X such that L_i is a partition in X between A_i and B_i for every $i < \omega$ and $\{L_i : i < \omega\}$ is point-finite.*

A collection $\mathcal{A}$ of subsets of a space X is *separating* in X if for every $x \in X$ and every closed set $F \subset X$ with $x \notin F$ there exist $A_1, A_2 \in \mathcal{A}$ such that $x \in A_1$, $F \subset A_2$ and $A_1 \cap A_2 = \emptyset$. Obviously, every separating collection of open subsets of a space X is a base for X.

3.3 Definition. A space X is *small countable-dimensional* if there exists a countable separating collection $\mathcal{B}$ of open subsets of X satisfying the following condition:

($*$) For every collection $\{(B_{i1}, B_{i2}) : i < \omega\}$ of pairs of elements of $\mathcal{B}$ with $\operatorname{Cl} B_{i1} \cap \operatorname{Cl} B_{i2} = \emptyset$ for every $i < \omega$, there exists a collection $\{L_i : i < \omega\}$ of subsets of X such that L_i is a partition in X between $\operatorname{Cl} B_{i1}$ and $\operatorname{Cl} B_{i2}$ for every $i < \omega$ and $\{L_i : i < \omega\}$ is strongly point-finite.

We now come to our main theorem.

3.4 Theorem. *A space X has a countable-dimensional compactification if and only if X is small countable-dimensional.*

Proof: Let X be small countable-dimensional and $\mathcal{U}$ be a countable separating collection of open subsets of X satisfying the condition ($*$) in Definition 3.3. Let us set

$$\mathcal{A} = \{(U, U') : U, U' \in \mathcal{U} \text{ with } \operatorname{Cl}_{\tilde{X}} U \cap \operatorname{Cl}_{\tilde{X}} U' = \emptyset\}.$$

We enumerate $\mathcal{A}$ as $\mathcal{A} = \{(U_i, U_i') : i < \omega\}$. Take a partition L_i between $\operatorname{Cl} U_i$ and $\operatorname{Cl} U_i'$ for every $i < \omega$ such that $\{L_i : i < \omega\}$ is strongly point-finite. We can take disjoint open subsets B_i and B_i' such that $\operatorname{Cl} U_i \subset B_i, \operatorname{Cl} U_i' \subset B_i'$ and $X - L_i = B_i \cup B_i'$. It is easy to show that the set $\mathcal{B} = \{B_i : i < \omega\}$ is a base for X. Since $\{L_i : i < \omega\}$ is strongly point-finite, so is $\{\operatorname{Bd} B_i : i < \omega\}$. From Theorem 1.1, X has small transfinite dimension trind. By Theorem 3.1, X has a countable-dimensional compactification.

Now let αX be a countable-dimensional compactification of X and $\mathcal{U}$ be a countable base $\mathcal{U}$ for αX. Let us set

$$\mathcal{A} = \{(U, U') : U, U' \in \mathcal{U} \text{ with } \operatorname{Cl}_{\alpha X} U \subset U'\}.$$

We enumerate $\mathcal{A}$ as $\mathcal{A} = \{(U_i, U_i') : i < \omega\}$. For every $i < \omega$, inductively, we shall construct two open subsets V_i and V_i' of αX satisfying the following conditions:

$$\operatorname{Cl}_{\alpha X} U_i \subset V_i \subset \operatorname{Cl}_{\alpha X} V_i \subset \alpha X - \operatorname{Cl}_{\alpha X} V_i' \subset \alpha X - V_i' \subset U_i' \text{ and}$$
$$\mathcal{V} = \{V_i : i < \omega\} \cup \{V_i' : i < \omega\} \text{ is closure-distributive.}$$

Assume that for every $k < i$ (> 0) we have constructed two open subsets V_k and V_k' of αX satisfying the following conditions: $\operatorname{Cl}_{\alpha X} U_k \subset V_k \subset \operatorname{Cl}_{\alpha X} V_k \subset \alpha X - \operatorname{Cl}_{\alpha X} V_k' \subset \alpha X - V_k' \subset U_k'$ and $\mathcal{V}_i = \{V_k : k < i\} \cup \{V_k' : k < i\}$ is closure-distributive. By Lemma 1.2, there exists open subsets V_i' and V_i' of αX such that $\operatorname{Cl}_{\alpha X} U_i \subset V_i \subset \operatorname{Cl}_{\alpha X} V_i \subset \alpha X - \operatorname{Cl}_{\alpha X} V_i' \subset \alpha X - V_i' \subset U_i'$ and $\mathcal{V}_{i+1} = \mathcal{V}_i \cup \{V_i, V_i'\}$ is closure-distributive. It is easily seen that $\mathcal{V}$ is closure-distributive. Let us set

$$\mathcal{B} = \mathcal{V}|X.$$

We shall prove that $\mathcal{B}$ is a countable separating collection of open subsets of X satisfying the condition $(*)$ in Definition 3.3. First we shall show that $\mathcal{B}$ is separating. Consider a point $x \in X$ and a closed subset F of X with $x \notin F$. The collection $\mathcal{U}$ being a base for αX, we can take $U, U' \in \mathcal{U}$ such that $x \in U \subset \operatorname{Cl}_{\alpha X} U \subset U' \subset \operatorname{Cl}_{\alpha X} U' \subset \alpha X - \operatorname{Cl}_{\alpha X} F$. Since $(U, U') \in \mathcal{A}$, $(U, U') = (U_n, U_n')$ for some $n < \omega$. We have $x \in U_n \subset \operatorname{Cl}_{\alpha X} U_n \subset V_n$; thus $x \in V_n \cap X \in \mathcal{B}$. Since $\alpha X - V_n' \subset U_n' \subset \operatorname{Cl}_{\alpha X} U_n' \subset \alpha X - \operatorname{Cl}_{\alpha X} F$, we have $\operatorname{Cl}_{\alpha X} F \subset V_n'$; thus $F = \operatorname{Cl}_{\alpha X} F \cap X \subset V_n' \cap X \in \mathcal{B}$. Obviously, $(V_n \cap X) \cap (V_n' \cap X) = \emptyset$. Thus $\mathcal{B}$ is separating. Next, we shall show that $\mathcal{B}$ satisfies the condition $(*)$ in Definition 3.3. Consider a collection $\{(B_{i1}, B_{i2}) : i < \omega\}$ of pairs of elements of $\mathcal{B}$ with $\operatorname{Cl}_X B_{i1} \cap \operatorname{Cl}_X B_{i2} = \emptyset$ for every $i < \omega$. For every $i < \omega$ we can take $B_{i1}', B_{i2}' \in \mathcal{V}$ such that

$$B_{i1} = B_{i1}' \cap X \text{ and } B_{i2} = B_{i2}' \cap X.$$

Then we have

$$\begin{aligned}\operatorname{Cl}_{\alpha X} B_{i1}' \cap \operatorname{Cl}_{\alpha X} B_{i2}' &= \operatorname{Cl}_{\alpha X}(B_{i1}' \cap B_{i2}') = \operatorname{Cl}_{\alpha X}(B_{i1}' \cap B_{i2}' \cap X) \\ &= \operatorname{Cl}_{\alpha X}(B_{i1} \cap B_{i2}) \subset \operatorname{Cl}_{\alpha X}(\operatorname{Cl}_X B_{i1} \cap \operatorname{Cl}_X B_{i2}) = \emptyset.\end{aligned}$$

Since αX is countable-dimensional, we can take a collection $\{L_i' : i < \omega\}$ of subsets of αX such that L_i' is a partition in αX between $\operatorname{Cl}_{\alpha X} B_{i1}'$ and $\operatorname{Cl}_{\alpha X} B_{i2}'$, and $\{L_i' : i < \omega\}$ is strongly point-finite. Then $L_i = L_i' \cap X$ is a partition in X between $\operatorname{Cl}_X B_{i1}$ and $\operatorname{Cl}_X B_{i2}$. Obviously, $\{L_i : i < \omega\}$ is strongly point-finite; thus $\mathcal{B}$ satisfies the condition $(*)$ in Definition 3.3. Hence X is small countable-dimensional. $\square$

3.5 Problem. Does Theorem 3.4 remain true if we replace 'a countable separating collection of open subsets of a space X' in Definition 3.3 by 'a countable base for X'?

Next we consider a characterization of spaces having a countable-dimensional compactification in terms of a special metric.

3.6 Definition. A space X is *μ-countable-dimensional* if there exists a totally bounded metric d on X satisfying the following condition:

(∗) For every collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X with $d(A_i, B_i) > 0$ for every $i < \omega$, there exists a collection $\{L_i : i < \omega\}$ of subsets of X such that L_i is a partition in X between A_i and B_i for every $i < \omega$ and $\{L_i : i < \omega\}$ is strongly point-finite.

3.7 Theorem. *A space X has a countable-dimensional compactification if and only if X is μ-countable-dimensional.*

Proof: Let X be μ-countable-dimensional and d be a totally bounded metric on X satisfying the condition (∗) in Definition 3.6. The completion $(\tilde{X}, \tilde{d})$ of (X, d) is compact. Take an arbitrary countable base $\mathcal{U}$ for $\tilde{X}$. Let us set

$$\mathcal{A} = \{(U, U') : U, U' \in \mathcal{U} \ \text{ with } \ \mathrm{Cl}_{\tilde{X}}\, U \subset U'\}.$$

We enumerate $\mathcal{A}$ as $\mathcal{A} = \{(U_i, U_i') : i < \omega\}$. For every $i < \omega$, since $\mathrm{Cl}_{\tilde{X}}\, U_i \cap (\tilde{X} - U_i') = \emptyset$, $\varepsilon_i = \tilde{d}(\mathrm{Cl}_{\tilde{X}}\, U_i, \tilde{X} - U_i') > 0$. Thus we can take a partition L_i in X between $\mathrm{Cl}_{\tilde{X}}\, B(\mathrm{Cl}_{\tilde{X}}\, U_i; \varepsilon_i/3) \cap X$ and $\mathrm{Cl}_{\tilde{X}}\, B(\tilde{X} - U_i'; \varepsilon_i/3) \cap X$ for every $i < \omega$ such that $\{L_i : i < \omega\}$ is strongly point-finite. For every $i < \omega$ we take a partition $\tilde{L_i}$ in $\tilde{X}$ between $\mathrm{Cl}_{\tilde{X}}\, U_i$ and $\tilde{X} - U_i'$ such that $\tilde{L_i} \cap X \subset L_i$. Let us set

$$\mathbb{D} = \{\Delta \in [\omega]^{<\omega} : \bigcap_{n\in\Delta} L_n = \emptyset\}.$$

For every $\Delta \in \mathbb{D}$ the set

$$T_\Delta = \bigcap_{n\in\Delta} \tilde{L_n}$$

is closed in $\tilde{X}$ and disjoint from X. The set

$$Y = \tilde{X} - \bigcup\{T_\Delta : \Delta \in \mathbb{D}\}$$

is a Čech-complete extension of X. Now, for every $i < \omega$, we can take disjoint open subsets V_i and V_i' of Y such that $\mathrm{Cl}_{\tilde{X}}\, U_i \cap Y \subset V_i$, $(\tilde{X} - U_i') \cap Y \subset V_i'$ and $Y - (\tilde{L_i} \cap Y) = V_i \cup V_i'$. Let us set

$$\mathcal{V} = \{V_i : i < \omega\}.$$

It is easily seen that $\mathcal{V}$ is a base for Y. We shall show that $\{\mathrm{Bd}_Y V_i : i < \omega\}$ is strongly point-finite. Obviously, $\mathrm{Bd}_Y V_i \subset \tilde{L_i} \cap Y$ for every $i < \omega$. It suffices to show that $\{\tilde{L_i} \cap Y : i < \omega\}$ is strongly point-finite. Consider an infinite subset Λ of ω. The collection $\{L_i : i < \omega\}$ being strongly point-finite, we can take $\Delta \in [\Lambda]^{<\omega}$ such that $\bigcap_{n\in\Delta} L_n = \emptyset$; thus $\Delta \in \mathbb{D}$. We have $\bigcap_{n\in\Delta}(\tilde{L_n} \cap Y) = T_\Delta \cap Y \subset T_\Delta \cap (\tilde{X} - T_\Delta) = \emptyset$. Thus $\{\tilde{L_i} \cap Y : i < \omega\}$ is strongly point-finite. By Theorem 3.1, Y has a countable-dimensional compactification αY. Then αY is a compactification of X.

Now let αX be a countable-dimensional compactification of X. Take an arbitrary metric d on αX. Let d_X be the restriction of d to X. It is easy to show that d_X satisfies the condition $(*)$ in Definition 3.6. Hence X is μ-countable-dimensional.

□

3.8 Problem. Does Theorem 3.7 remain true if we replace 'a totally bounded metric on X' in Definition 3.6 by 'a metric on X'?

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(*Received* July 6, 2007)