On monotone Lindelöfness of countable spaces

Ronnie Levy, Mikhail Matveev

Abstract. A space is monotonically Lindelöf (mL) if one can assign to every open cover $\mathcal{U}$ a countable open refinement $r(\mathcal{U})$ so that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. We show that some countable spaces are not mL, and that, assuming CH, there are countable mL spaces that are not second countable.

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1. Introduction

When saying that a family of sets $\mathcal{A}$ refines a family of sets $\mathcal{B}$ (or that $\mathcal{B}$ is coarser than $\mathcal{A}$) we only mean that every element of $\mathcal{A}$ is a subset of an element of $\mathcal{B}$; we do not assume that $\bigcup\mathcal{A} = \bigcup\mathcal{B}$. If $\mathcal{A}$ refines a family of sets $\mathcal{B}$, we write $\mathcal{A} \prec \mathcal{B}$. A space is monotonically Lindelöf (or mL for short) [2], [9], [10] if there is a function r, henceforth called a mL operator, that assigns to every open cover $\mathcal{U}$ a countable open cover $r(\mathcal{U})$ which refines $\mathcal{U}$ in such a way that $r(\mathcal{U})$ is coarser than $r(\mathcal{V})$ whenever $\mathcal{U}$ is coarser than $\mathcal{V}$. There are two properties, countability and second countability, that trivially imply Lindelöfness. That all second countable spaces are monotonically Lindelöf is straightforward: given a countable base $\mathcal{B}$, put $r(\mathcal{U}) = \{O : O \in \mathcal{B}$ and there is a $U \in \mathcal{U}$ such that $O \subset U\}$. With countability, this is not so trivial. One can present examples of countable spaces which are easily seen not to be monotonically Lindelöf. (We do so in Section 2.) Then we discuss consistent examples of monotonically Lindelöf countable spaces that are not second countable. (The existence of ZFC examples remains an open question.) Then we show that having one such example one can get some others. But first, we discuss a reduction.

Let $(X, \mathcal{T})$ be a space, and $p \in X$. Denote by $\mathcal{T}_p$ the topology on X generated by $\mathcal{T} \cup \{\{q\} : q \neq p\}$.

Proposition 1. *A countable T_1 space $(X, \mathcal{T})$ is monotonically Lindelöf iff $(X, \mathcal{T}_p)$ is monotonically Lindelöf for every $p \in X$.*

Proof: Let r be a mL operator for $(X, \mathcal{T})$, and let $p \in X$. For an open cover $\mathcal{U}$ of $(X, \mathcal{T}_p)$, put $s(\mathcal{U}) = \{U \in \mathcal{U} : p \in U\} \cup \{X \setminus \{p\}\}$. Then $s(\mathcal{U})$ is an open cover of

$(X,\mathcal{T})$, and s is monotonic. Put $r_p(\mathcal{U}) = \{V \in s(\mathcal{U}) : p \in V\} \cup \{\{q\} : q \in X \setminus \{p\}\}$. Then r_p is a mL operator for $(X, \mathcal{T}_p)$.

On the other hand, for every $p \in X$, let r_p be a mL operator for $(X, \mathcal{T}_p)$, and let $\mathcal{U}$ be an open cover of $(X,\mathcal{T})$. Put $r(\mathcal{U}) = \bigcup\{\{V \in r_p(\mathcal{U}) : p \in V\} : p \in X\}$. Then r is a mL operator for $(X,\mathcal{T})$. □

Say that X is mL at $p \in X$ if there is an operator r_p assigning to every non empty family $\mathcal{F}$ of neighborhoods of p a non empty countable family $r_p(\mathcal{F})$ of neighborhoods of p so that $r_p(\mathcal{F})$ refines $\mathcal{F}$ and $r_p(\mathcal{F})$ refines $r_p(\mathcal{G})$ whenever $\mathcal{F}$ refines $\mathcal{G}$. It is clear that a T_1 mL space is mL at every point. Proposition 1 can now be restated as follows:

Proposition 2. *A countable T_1 space is mL iff it is mL at every point.*

The referee has observed that the axiom T_1 (omitted in the original version of the paper) is essential in Propositions 1 and 2. The counterexample suggested by the referee is the following: let X be a countable space which is not mL, and let $y \notin X$. Topologize $Y = X \cup \{y\}$ declaring X (with its original topology) open while the only neighborhood of y is Y. Then Y is mL while for some $p \in X$, X is not mL at p, and hence Y is not mL at p.

Propositions 1 and 2 show that the problem we consider can be reduced to spaces with unique non isolated point.

2. ZFC counterexamples

Here we show that some countable spaces are not monotonically Lindelöf. By a subbase of neighborhoods of a point p we mean a family $\mathcal{B}$ of neighborhoods of p such that finite intersections of $\mathcal{B}$ form a base of neighborhoods of p. The proof of the next proposition is similar to an argument from [8].

Proposition 3. *Let $\mathcal{B}$ be a subbase of neighborhoods of a point p in a space X. Suppose there is a cardinal κ such that the following two conditions hold:*

(1) *for every neighborhood U of p, $|\{B \in \mathcal{B} : U \subset B\}| < \kappa$;*
(2) *every subfamily of $\mathcal{B}$ which is still a subbase at p has cardinality $> \kappa$.*

Then X is not mL at p.

Proof: Let r be a mL operator for X at p. We construct a sequence of covers $\mathcal{U}_\alpha = \mathcal{C}_\alpha \cup \{X \setminus \{p\}\}$ (where $\mathcal{C}_\alpha$ is a family of neighborhoods of p) and subfamilies $\mathcal{B}_\alpha \subset \mathcal{B}$ for $\alpha < \kappa^+$. Put $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{C}_0 = \{U : U$ is an open neighborhood of p and there is $B \in \mathcal{B}_0$ such that $U \subset B\}$. If $\mathcal{B}_\beta$, $\mathcal{C}_\beta$ have been constructed for all $\beta < \alpha$, put $\mathcal{B}_\alpha = \mathcal{B} \setminus \{B \in \mathcal{B} :$ there are $\beta < \alpha$ and $U \in r(\mathcal{U}_\beta)$ such that $U \subset B\}$ and $\mathcal{C}_\alpha = \{U : U$ is an open neighborhood of p and there is $B \in \mathcal{B}_\alpha$ such that $U \subset B\}$. By condition (2), $\mathcal{B}_\alpha$ and $\mathcal{C}_\alpha$ are well defined for all $\alpha < \kappa^+$, however, at step $\alpha = k$ we get a contradiction with (1). □

Example 1. Let $\mathcal{A}$ be an uncountable almost disjoint family of subsets of ω, $\bigcup\mathcal{A} = \omega$. Topologize $X = \omega \cup \{p\}$, where $p \notin \omega$, as follows: the points of ω are isolated while a set $U \subset X$ that contains p is open if and only if $X \setminus U$ is contained in the union of a finite subfamily of $\mathcal{A}$.

X is not mL at p: to apply Proposition 3, take $\kappa = \omega$, and $\mathcal{B} = \{X \setminus A : A \in \mathcal{A}\}$.

Example 2. Let $\omega_1 \leq \kappa \leq c$, and let X be a dense countable subspace in 2^κ.

To observe that X is not mL at any $p \in X$, put $\kappa = \omega$ and $\mathcal{B} = \{\{x \in X : x(i) = p(i)\} : i \in \kappa\}$, and apply Proposition 3.

Remark. Proposition 3 can be applied to uncountable spaces as well. Thus, it is easy to see, for example, that the one point compactification of an uncountable discrete space is not mL, and neither is the one point Lindelöfication of a discrete space of cardinality $\geq \omega_2$.

3. A consistent example from (↑)

Henceforward by an example we mean a non metrizable countable mL space. We use the notation from [3] for small uncountable cardinals, $\supset^*$, and so on.

Let $\kappa > 0$ be a cardinal. Say that a sequence $\{T_\alpha : \alpha < \kappa\}$ of infinite subsets of ω is a *pretower* if $T_\alpha \supseteq^* T_\beta$ and $T_\alpha \neq^* T_\beta$ whenever $\alpha < \beta < \kappa$. A pretower is a *tower* [3] when it does not have infinite pseudointersection. Naturally, κ is called the height of the pretower. Let $p \notin \omega$ and let $T = \{T_\alpha : \alpha < k\}$ be a pretower. Denote by X_T the set $\omega \cup \{p\}$ with the topology $\mathcal{T}_T$ generated by the base $\{\{n\} : n \in \omega\} \cup \{\{p\} \cup (T_\alpha \setminus A) : \alpha < \kappa \text{ and } |A| < \omega\}$.

For an open cover $\mathcal{U}$ of X_T and $\alpha < \kappa$, put $s_\alpha(\mathcal{U}) = \{\{p\} \cup (T_\alpha \setminus A) : |A| < \omega, \{\{p\} \cup (T_\alpha \setminus A)\} \prec \mathcal{U} \text{ and } \{\{p\} \cup (T_\alpha \setminus A)\} \not\prec \bigcup\{s_\beta(\mathcal{U}) : \beta < \alpha\}\}$. Set

$$r(\mathcal{U}) = \bigcup\{s_\alpha(\mathcal{U}) : \alpha < \kappa\} \cup \{\{n\} : n \in \omega\}.$$

Say that a pretower $T = \{T_\alpha : \alpha < \kappa\}$ is *good* if every cofinal subsequence of T contains a pair of elements related with respect to "real" inclusion (i.e. for every cofinal subsequence of T, there are $\alpha < \beta < \kappa$ such that T_α and T_β are in this subsequence, and $T_\alpha \supset T_\beta$).

Proposition 4. (0) *For every pretower T, r is a monotonic operator, that is, $r(\mathcal{U})$ is coarser than $r(\mathcal{V})$ whenever $\mathcal{U}$ is coarser than $\mathcal{V}$.*

(1) *If $\mathrm{cf}(\kappa) > \omega$, and T is a good pretower, then for every open cover $\mathcal{U}$ the families $s_\alpha(\mathcal{U})$ are eventually empty.*

(2) *If $\kappa \leq \omega_1$ and T is a good pretower, then r is a mL operator on X.*

Proof: (0) Let $\mathcal{U}$ and $\mathcal{V}$ be two open covers of X_T, $\mathcal{U}$ being coarser than $\mathcal{V}$, and let $V \in r(\mathcal{V})$. We have to find an $U \in r(\mathcal{U})$ such that $U \supset V$. If $V = \{n\}$ for $n \in \omega$, then $U = V = \{n\} \in r(\mathcal{U})$, so let $V = \{p\} \cup (T_\alpha \setminus A)$ for some $\alpha < k$ and some finite $A \subset \omega$. If $V \in s_\alpha(\mathcal{U})$, put $U = V$. If $V \notin s_\alpha(\mathcal{U})$, then by definition of s_α, there are $\beta < \alpha$ and an $U \in s_\beta(\mathcal{U})$ such that $V \subset U$.

(1) Assume the contrary. Then there is a $\mathcal{U}$ such that the set $B = \{\alpha < \kappa : s_\alpha(\mathcal{U}) \neq \emptyset\}$ is cofinal in k. For each $\alpha \in B$, pick $O_\alpha = T_\alpha \setminus A_\alpha \in s_\alpha(\mathcal{U})$ where A_α is a finite subset of ω. Since $\mathrm{cf}(\kappa) > \omega$, there are a finite $A \subset \omega$ and a cofinal subset $C \subset B$ such that $A_\alpha = A$ for all $\alpha \in C$. Since T is good, there are $\alpha, \beta \in C$, $\alpha < \beta$ such that $T_\alpha \supset T_\beta$. Then we have $O_\alpha \supset O_\beta$ which contradicts the definition of the operators s_α.

(3) If k is countable, then X_T is second countable, hence mL. If $\kappa = \omega_1$, then it follows from (1) that $r(\mathcal{U})$ is countable for every $\mathcal{U}$. So it follows from (0) that r is a mL operator. □

It is shown in [5] that the following combinatorial principle follows from CH:

$$(\uparrow) \quad : \quad \text{There is a good pretower of height } \omega_1.$$

Together with the trivial observation that $\chi(X_T, p) = \mathrm{cf}(\kappa)$, this implies the following:

Corollary 1. ($\uparrow$) *There is a non metrizable countable mL space.*

4. The countable fan as an example

Recall that the countable fan is the space $V_\omega = (\omega \times \omega) \cup \{p\}$ in which the points of $\omega \times \omega$ are isolated while a basic neighborhood of p is of the form $U_f = \{p\} \cup \{(m, n) : m \in \omega \text{ and } n > f(m)\}$ where $f \in \omega^\omega$.

Lemma 1 ([11]). *Let P denote the collection of non-decreasing elements of ${}^\omega\omega$. Suppose F is cofinal in P the $<^*$ ordering of P. Then there exist $d, f \in F$ such that $d < f$.*

Then identification of $\omega \times \omega$ and ω provides

Corollary 2. (CH) *V_ω is mL.*

5. Getting more examples from existing examples

First of all, it is clear that the discrete sum of countably many examples is again an example. The next step is to consider the simplest possible quotient spaces.

Question 1. *Let Z_1 and Z_2 be countable mL spaces with unique non isolated points p_1 and p_2 respectively, and Z the quotient space obtain by identifying p_1 and p_2. Must Z be mL?*

Question 2. *Must the product of two countable mL spaces be mL?*

We give only partial answers.

Proposition 5. *If X is countable and mL, and Y is countable and second countable, then $X \times Y$ is mL.*

Proof: Following Proposition 1, it is enough to consider $X \times (\omega+1)$ with unique non isolated point $\langle p, \omega\rangle$ where p is the unique non isolated point of X. Let r be a mL operator for $(X, \mathcal{T}_p)$, and let $\mathcal{U}$ be an open cover of $X \times (\omega + 1)$. For $n \in \omega$, let $\mathcal{V}_n = \{V \subset X : V \ni p, V$ is open in X, and there is $U \in \mathcal{U}$ such that $V \times [n, \omega] \subset U\} \cup \{\{q\} : q \in X \setminus \{p\}\}$. Put $s(\mathcal{U}) = \{W \times [n, \omega] : p \in W \in r(\mathcal{V}_n), n \in \omega\} \cup \{\langle x, y\rangle \in X \times (\omega + 1) : x \neq p$ or $y \neq \omega\}$. Then s is a mL operator for $X \times (\omega + 1)$. □

Remark. It follows from Proposition 5 that, consistently, a countable mL space need not be monotonically normal. Indeed, if $X \times (\omega+1)$ is monotonically normal, then X is stratifiable [G].

Proposition 6. *Let $Z = \omega \cup \{p\}$ be a mL space with unique non isolated point p. Then all finite powers of X are mL.*

Proof: We give the proof for Z^2. Let r be a mL operator for Z. Let $\mathcal{U}$ be an open cover of Z^2. Let $s_{p,p}(\mathcal{U}) = \{U : U$ is open in Z, $p \in U$ and $U \times U \prec \mathcal{U}\} \cup \{\{n\} : n \in \omega\}$. Then $s_{p,p}(\mathcal{U})$ is an open cover of Z and $s_{p,p}$ is a monotonic operation. Put $R_{p,p}(\mathcal{U}) = \{V \times V : V \in r(s_{p,p}(\mathcal{U}))$ and $p \in V\}$. For $n \in \omega$, let $s_{p,n}(\mathcal{U}) = \{U : U$ is open in Z, $U \ni p$ and $\{p\} \times U \prec \mathcal{U}\} \cup \{\{n\} : n \in \omega\}$ and $s_{n,p}(\mathcal{U}) = \{U : U$ is open in Z, $U \ni p$ and $U \times \{p\} \prec \mathcal{U}\} \cup \{\{n\} : n \in \omega\}$. Put $R_{p,n}(\mathcal{U}) = \{\{p\} \times V : V \in r(s_{p,n}(\mathcal{U}))$ and $V \ni p\}$ and $R_{n,p}(\mathcal{U}) = \{V \times \{p\} : V \in r(s_{n,p}(\mathcal{U}))$ and $V \ni p\}$. Finally, put $R(\mathcal{U}) = R_{p,p}(\mathcal{U}) \cup \bigcup\{R_{n,p}(\mathcal{U}) \cup R_{p,n}(\mathcal{U}) : n \in \omega\} \cup \{\{(n, n)\} : n \in \omega\}$. □

Proposition 7. *Suppose $\{X_a : a \in A\}$ where $|A| = \omega$, be a family of countable spaces, and all finite subproducts in the product $P = \prod\{X_a : a \in A\}$ are mL. Then a σ-product in P is mL.*

Proof: Let S be a σ-product in P with base point $x = (x_a : a \in A)$. It is enough to show that S is mL at x. Moreover, it is enough to show that P is mL at x.

For $B \subset A$, we denote $X_B = \prod\{X_a : a \in B\}$ and $x_B = (x_a : a \in B) = \pi_B(x)$. For finite B, let r_B be operators witnessing that X_B is mL at x_B. Let us say that a subset $U \subset P$ is B-standard, where $B \subset A$, if $U = \pi_B^{-1}(\pi_B(U))$. Let $\mathcal{U}$ be a nonempty family of neighborhoods of x in P. For finite $B \subset A$, let $s_B(\mathcal{U}) = \{U : U$ is B-standard and $x \in U \prec \mathcal{U}\}$ and $t_B(\mathcal{U}) = \{\pi_B(O) : O \in s_B(\mathcal{U})\}$. Put $\mathcal{B}(\mathcal{U}) = \{B \subset A : |B| < \omega$ and $s_B(\mathcal{U}) \neq \emptyset\}$ and $r(\mathcal{U}) = \{\pi_B^{-1}(V) : V \in r_B(t_B(s_B(\mathcal{U})))$, $B \in \mathcal{B}(\mathcal{U})\}$. Then r witness that P is mL at x. □

Corollary 3. *Let $Z = \omega \cup \{p\}$ be a mL space with unique non isolated point p. Then a σ-product in Z^ω is mL.*

Consistently, Corollary 3 gives an example without isolated points. Alternatively, one can get such an example using the Pixley-Roy exponent. Recall that the Pixley-Roy space $\mathrm{PR}(X)$ over a topological space $(X, \mathcal{T})$ is the set of all nonempty finite subsets of X with basic open sets of the form

$$[F, V] = \{G \in \mathrm{PR}(X) : F \subseteq G \subseteq V\}$$

where $F \in \mathrm{PR}(X)$ and $F \subset V \in \mathcal{T}$ (see [3]).

Proposition 8. *If X is a countable mL space, then $\mathrm{PR}(X)$ is mL.*

Proof: Let r be a mL operator for X. We construct a mL operator R for $\mathrm{PR}(X)$ in the form

$$R(\mathcal{U}) = \bigcup\{R_F(\mathcal{U}) : F \in \mathrm{PR}(X)\}$$

where $\mathcal{U}$ is an open cover of $\mathrm{PR}(X)$. Let $F \in \mathrm{PR}(X)$. Put

$$\mathcal{U}_F = \{[F, V] : V \in \mathcal{T}, F \subset V, [F, V] \prec \mathcal{U}\},$$
$$\mathcal{V}_F = \{V : [F, V] \in \mathcal{U}_F\}, \qquad \mathcal{O}_F = \mathcal{V}_F \cup \{X \setminus F\}$$

(the latter is an open cover of X),

$$\mathcal{W}_F = \{W \in r(\mathcal{O}_F) : W \cap F \neq \emptyset\}, \qquad \widetilde{\mathcal{W}_F} = \{\bigcap \mathcal{A} : \mathcal{A} \in [\mathcal{W}_F]^{<\omega}\},$$

$$\widetilde{\widetilde{\mathcal{W}_F}} = \{\bigcup \mathcal{B} : \mathcal{B} \in [\widetilde{\mathcal{W}_F}]^{<\omega} \text{ and } \bigcup \mathcal{B} \supset F \text{ and } (\exists V \in \mathcal{V}_F)(\bigcup \mathcal{B} \subset V)\},$$

$$R_F(\mathcal{U}) = \{[F, O] : O \in \widetilde{\widetilde{\mathcal{W}_F}}\}.$$

□

6. Final questions

Question 3. *Is it consistent that every countable mL space is metrizable?*

Question 4. *Does the existence of a non-metrizable countable mL space imply the existence of a good tower (of uncountable height)?*

Question 5. *How many pairwise non homeomorphic countable mL spaces with unique non isolated point are there?*

Question 6. *Let X be a countable mL space with unique non isolated point p. Consider X embedded into $2^{w(X)}$ so that $p = \overline{0}$. Denote G the subgroup of $2^{w(X)}$ generated by X. Is G mL?*

Question 7. *Let X be a countable mL space with unique non isolated point. Are the free topological group $F(X)$ and free Abelian topological group $A(X)$ mL?*

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Department of Mathematical Sciences, George Mason University, 4400 University Drive, Fairfax, Virginia 22030, USA

E-mail: rlevy@gmu.edu
misha_matveev@hotmail.com, mmatveev@gmu.edu

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