Some approximation properties of the Kantorovich variant of the Bleimann, Butzer and Hahn operators

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Abstract. For some classes of functions f locally integrable in the sense of Lebesgue or Denjoy-Perron on the interval $[0;\infty)$, the Kantorovich type modification of the Bleimann, Butzer and Hahn operators is considered. The rate of pointwise convergence of these operators at the Lebesgue or Lebesgue-Denjoy points of f is estimated.

Keywords: Bleimann, Butzer and Hahn operator, Lebesgue-Denjoy point, rate of convergence

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1. Introduction

In 1980 Bleimann, Butzer and Hahn [5] introduced a sequence of positive linear operators B_nf defined on the space $R([0;\infty))$ of real functions on the infinite interval $I=[0;\infty)$ by

$$B_nf(x)=\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right) f\left(\frac{k}{n+1-k}\right) \quad (x\in I,\ n\in\mathbb{N}),$$

where

$$p_{n,k}\left(\frac{x}{1+x}\right)=\binom{n}{k}\frac{x^k}{(1+x)^n}\,.$$

The approximation properties of those operators have been extensively studied in the literature [1], [2], [3], [5], [6], [7], [8], [9], [10], [11], [14]. For function f locally integrable in the Lebesgue or Denjoy-Perron sense, the n-th Kantorovich variant of the L_nf operators is defined as follows

$$M_nf(x)=(n+2)\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right)\int_{k/(n+2-k)}^{(k+1)/(n+1-k)} \frac{f(t)}{(1+t)^2}\,dt \quad (x\in I,\ n\in\mathbb{N}).$$

U. Abel and M. Ivan [3] found the rate of convergence by estimating $|M_nf(x)-f(x)|$ in terms of the modulus of the continuity of f, where f is assumed to be bounded and continuous on $[0;\infty)$.

The aim of this paper is to examine the rate of the convergence of operators $M_n f$, mainly, at those points $x \in I$ at which

$$\lim_{h\to 0} \frac{1}{h} \int_0^h (f(x+t) - f(x))\, dt = 0.$$

The general estimate is expressed in terms of the quantity

$$w_x(\delta; f) = \sup_{0<|h|\le\delta} \left| \frac{1}{h} \int_0^h (f(x+t) - f(x))\, dt \right| \quad (\delta > 0).$$

Clearly, if f is locally integrable in the Denjoy-Perron sense on I then

$$\lim_{\delta\to 0+} w_x(\delta; f) = 0 \ \text{ for almost every } x.$$

In view of this property, we deduce that for some classes of functions,

$$\lim_{n\to\infty} M_n f(x) = f(x) \ \text{ almost everywhere.}$$

Moreover, using some other properties of $w_x(\delta; f)$ we present a few estimates of the rate of the norm and pointwise convergence of $M_n f$ in terms of the weighted moduli of continuity. Throughout the paper, the symbol $K(\cdot)$, $K_j(\cdot)$, $(j = 1, 2, \dots)$ will mean some positive constants, not necessarily the same at each occurrence, depending only on the parameters indicated in parentheses.

2. Auxiliary estimates

As well-known, for every $x \in I$ and all integers $n \ge 1$,

$$\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right) = 1, \tag{1}$$

$$x p_{n,k-1}\left(\frac{x}{1+x}\right) = \frac{k}{n-k+1} p_{n,k}\left(\frac{x}{1+x}\right) \quad (k \in \{1, 2, \dots, n\}). \tag{2}$$

For $q \in \mathbb{N}$, $s \in \mathbb{N}$, $x \in I$ and $n \in \mathbb{N}$ we define

$$Q_{q,0}^{(n)}(x) = \sum_{k=0}^{n} \frac{1}{(n-k+q)\dots(n-k+1)} p_{n,k}\left(\frac{x}{1+x}\right),$$

$$Q_{q,s}^{(n)}(x) = \sum_{k=0}^{n} \frac{k\dots(k-s+1)}{(n-k+q)\dots(n-k+1)} p_{n,k}\left(\frac{x}{1+x}\right).$$

Lemma 1. *For $q \in \mathbb{N}$, $s \in \mathbb{N}_0$, $n \in \mathbb{N}$, $x \in [0;\infty)$ and $q \geq s$ we have*

$$Q_{q,s}^{(n)}(x) \leq \frac{x^s(1+x)^{q-s}}{(n+1)^{q-s}}\,. \tag{3}$$

(In the case where $x=0$ and $s=0$, the symbol x^s is equal to one).

Proof: In view of (1) and (2) we have

$$\begin{aligned} Q_{1,0}^{(n)}(x) &= \frac{x}{n+1}\sum_{k=1}^{n} p_{n,k-1}\left(\frac{x}{1+x}\right) + \frac{1}{n+1}\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right) \\ &= \frac{x}{n+1}\frac{1}{n+1} - \frac{x}{n+1}p_{n,n}\left(\frac{x}{1+x}\right) \\ &< \frac{1+x}{n+1}\,. \end{aligned}$$

Next, using (2), we have

$$\begin{aligned} xQ_{q,0}^{(n)}(x) = \sum_{k=0}^{n} &\frac{1}{(n-k+q+1)\dots(n-k+2)}\frac{n+1}{n-k+1}p_{n,k}\left(\frac{x}{1+x}\right) \\ &-\sum_{k=0}^{n}\frac{1}{(n-k+q+1)\dots(n-k+2)}p_{n,k}\left(\frac{x}{1+x}\right) + \frac{x}{q!}\left(\frac{x}{1+x}\right)^n. \end{aligned}$$

Therefore

$$(n+1)Q_{q+1,0}^{(n)}(x) \leq (x+1)Q_{q,0}^{(n)}(x).$$

Consequently, (3) follows for all $q \in \mathbb{N}$ and $s=0$ by induction.

For $s > 1$, (2) gives us

$$Q_{q+1,s+1}^{(n)}(x) = xQ_{q,s}^{(n)}(x) - \frac{n\dots(n+1-s)}{q!}\left(\frac{x}{1+x}\right)^n x < xQ_{q,s}^{(n)}(x).$$

Consequently, (3) follows for all $q \in \mathbb{N}$ and $s \in \mathbb{N}_0$ by induction. □

Remark 1. It is easy to see that for $q \in \mathbb{N}$, $s_1,\dots s_q \in \mathbb{N}$, $n \in \mathbb{N}$, $x \in [0;\infty)$

$$\sum_{k=0}^{n}\frac{1}{(n-k+s_1)\dots(n-k+s_q)}p_{n,k}\left(\frac{x}{1+x}\right) \leq q!Q_{q,0}^{(n)}(x). \tag{4}$$

For $i \in \mathbb{N}$, $q \in \mathbb{N}$, $n \in \mathbb{N}$, $x \in [0;\infty)$ we will use the notation

$$\begin{aligned} a_{k,j}^{(n)}(x) &= \frac{k+1-i}{n-k+i} - x \qquad (0 \leq k \leq n), \\ S_q^{(n)}(x) &= \sum_{k=0}^{n} a_{k,1}^{(n)}(x)\dots a_{k,q}^{(n)}(x)p_{n,k}\left(\frac{x}{1+x}\right). \end{aligned}$$

Lemma 2. *Let $x \in I$, $n \in \mathbb{N}$, $q \in \mathbb{N}$, $q \geq 2$. Then*

$$S_{q+1}^{(n)}(x) = \frac{q}{n+q+1}\left((x^2-1)S_q^{(n)}(x) + x(1+x)^2 S_{q-1}^{(n)}(x)\right) - R_q^{(n)}(x), \tag{5}$$

where

$$R_q^{(n)}(x) = \frac{x(n+1)^2}{q(n+q+1)} a_{n,1}^{(n)}(x) \dots a_{n,q-1}^{(n)}(x) p_{n,n}\left(\frac{x}{1+x}\right).$$

Proof: Simple calculations, (2) and identity $a_{k-1,i}^{(n)}(x) = a_{k,i+1}^{(n)}(x)$, give us

$$xS_q^{(n)}(x) = S_{q+1}^{(n)}(x) + \sum_{k=0}^{n} a_{k,2}^{(n)}(x) \dots a_{k,q+1}^{(n)}(x) p_{n,k}\left(\frac{x}{1+x}\right) + \widetilde{R}_q^{(n)}(x), \tag{6}$$

where

$$\widetilde{R}_q^{(n)}(x) = x a_{n,1}^{(n)}(x) \dots a_{n,q}^{(n)}(x) p_{n,n}\left(\frac{x}{1+x}\right).$$

Using the obvious equality

$$\frac{k}{n-k+1} - \frac{k-q+1}{n-k+q} = \frac{q}{n+1}\left(a_{k,1}^{(n)}(x) + 1 + x\right)\left(a_{k,q+1}^{(n)}(x) + 1 + x\right),$$

we have

$$\begin{aligned} S_{q+1}^{(n)}(x) = \frac{qx}{n+1}\Bigg(& S_{q+1}^{(n)}(x) + (1+x)S_q^{(n)}(x) \\ & + (1+x)\sum_{k=0}^{n} a_{k,2}^{(n)}(x) \dots a_{k,q+1}^{(n)}(x) p_{n,k}\left(\frac{x}{1+x}\right) \\ & + (1+x)^2 \sum_{k=0}^{n} a_{k,2}^{(n)}(x) \dots a_{k,q}^{(n)}(x) p_{n,k}\left(\frac{x}{1+x}\right)\Bigg) - \widetilde{R}_q^{(n)}(x). \end{aligned}$$

Applying (6), we obtain

$$\begin{aligned} S_{q+1}^{(n)}(x) = \frac{q}{n+1}\Bigg(& xS_{q+1}^{(n)}(x) + x(1+x)S_q^{(n)}(x) \\ & + (1+x)\left(xS_q^{(n)}(x) - S_{q+1}^{(n)}(x) - \widetilde{R}_q^{(n)}(x)\right) \\ & + (1+x)^2\left(xS_{q-1}^{(n)}(x) - S_q^{(n)}(x) - \widetilde{R}_{q-1}^{(n)}(x)\right)\Bigg) - \widetilde{R}_q^{(n)}(x). \end{aligned}$$

So (5) is now evident. □

Lemma 3. *Let $q \in \mathbb{N}$, $x \in I$, $n \in \mathbb{N}$. Then*

$$\left|S_q^{(n)}(x)\right| \le K(q)x(x+1)^{2q-2}\left(\frac{1}{n^{[(q+1)/2]}} + n^{q-1}p_{n,n}\left(\frac{x}{1+x}\right)\right). \tag{7}$$

Proof: In view of (1) and (2),

$$\left|S_1^{(n)}(x)\right| = \left|-xp_{n,n}\left(\frac{x}{1+x}\right)\right|.$$

The obvious identity

$$xS_1^{(n)}(x) = S_2^{(n)}(x) + x\sum_{k=0}^{n} a_{k,2}^{(n)}(x)p_{n,k}(\frac{x}{1+x}) + x(n-x)p_{n,n}(\frac{x}{1+x})$$

and (3) lead to

$$\begin{aligned}\left|S_2^{(n)}(x)\right| &= \left|x(n+1)\sum_{k=0}^{n}\frac{1}{(n-k+1)(n-k+2)}p_{n,k}\left(\frac{x}{1+x}\right)\right. \\ &\qquad \left. - x(n-x)p_{n,n}\left(\frac{x}{1+x}\right)\right| \\ &\le \frac{x(1+x)^2}{n+1} + x(x+1)np_{n,n}\left(\frac{x}{1+x}\right).\end{aligned}$$

Inequality (7) follows now immediately from the estimate

$$\left|R_q^{(n)}(x)\right| = 2^{q-1}\left((n+1)^q + (x+1)^{q-1}\right)p_{n,n}\left(\frac{x}{1+x}\right)$$

and (5) by induction. □

Let the symbol $\prod_{i=0}^{-1}$ be defined as one.

Lemma 4. *Let $n \in \mathbb{N}$, $x \in I$, $k \in \mathbb{N}_0$, $k \le n$. Given any numbers $r, q \in \mathbb{N}$, $s \in \mathbb{N}_0$, we have*

$$\begin{aligned}a_{k,r}^{(n)}(x) &= \sum_{j=0}^{s}\left(K_j(q,r,n,x) + \overline{K}_j(q,r,n,x)a_{k,q+j}^{(n)}(x)\right)\prod_{i=0}^{j-1}a_{k,q+i}^{(n)}(x) \\ &\quad + a_{k,r}^{(n)}(x)\overline{\overline{K}}_s(q,r,n,x)\prod_{i=0}^{s}a_{k,q+i}^{(n)}(x),\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\overline{\overline{K}}_j(q,r,n,x) &= \prod_{i=0}^{j} \frac{q+i-r}{n+1-(q+i-r)(x+1}\,,\ (j\in\mathbb{N}_0),\\
K_0(q,r,n,x) &= \frac{(q-r)(x+1)^2}{(n+1)-(q-r)(x+1)}\,,\\
K_j(q,r,n,x) &= K_0(q,r,n,x)\overline{\overline{K}}_{j-1}(q,r,n,x),\ (j\in\mathbb{N}),\\
\overline{K}_0(q,r,n,x) &= \frac{n+1+(q-r)(x+1)}{(n+1)-(q-r)(x+1)}\,,\\
\overline{K}_j(q,r,n,x) &= \overline{K}_0(q,r,n,x)\overline{\overline{K}}_{j-1}(q,r,n,x),\ (j\in\mathbb{N}).
\end{aligned}$$

Proof: It is easy to see that

$$a_{k,r}^{(n)}(x) = a_{k,q}^{(n)}(x) + \frac{q-r}{n+1}\left(a_{k,r}^{(n)}(x)+x+1\right)\left(a_{k,q}^{(n)}(x)+x+1\right).$$

Hence,

$$\begin{aligned}
a_{k,r}^{(n)}(x) &= \frac{(q-r)(x+1)^2}{n+1-(q-r)(x+1)} + \frac{n+1-(q-r)(x+1)}{(n+1)-(q-r)(x+1)}\ a_{k,q}^{(n)}(x)\\
&\quad+\frac{q-r}{n+1-(q-r)(x+1)}\ a_{k,q}^{(n)}(x)\ a_{k,r}^{(n)}(x).
\end{aligned}\tag{9}$$

Using (9) and the method of induction one can easily verify that for all $s\in\mathbb{N}_0$ (8) is true. □

Lemma 5. *Let $r\in\mathbb{N}$, $s_1,\dots s_r\in\mathbb{N}$, $n\in\mathbb{N}$, $x\in I$. Then*

$$\begin{aligned}
&\left|\sum_{k=0}^{n} a_{k,s_1}^{(n)}(x)\dots a_{k,s_r}^{(n)}(x)p_{n,k}\left(\frac{x}{1+x}\right)\right|\\
&\qquad\le Kx(x+1)^{2r-2}\left(\frac{1}{n^{[(r+1)/2]}}+n^{r-1}p_{n,n}\left(\frac{x}{1+x}\right)\right),
\end{aligned}\tag{10}$$

with a constant K depending only on $s_1,\dots s_r,r$.

Proof: First, we prove the estimate:

$$\begin{aligned}
&\left|\sum_{k=0}^{n} a_{k,1}^{(n)}(x)\dots a_{k,q-1}^{(n)}(x)a_{k,s_1}^{(n)}(x)\dots a_{k,s_r}^{(n)}(x)p_{n,k}\left(\frac{x}{1+x}\right)\right|\\
&\le Kx(x+1)^{2r+2q-4}\left(\frac{1}{n^{[(r+q)/2]}}+n^{r+q-2}p_{n,n}\left(\frac{x}{1+x}\right)\right)\ (r\in\mathbb{N},\ q\in\mathbb{N}).
\end{aligned}\tag{11}$$

For $r = 1$, by (8) we have

$$\left|\sum_{k=0}^{n} a_{k,1}^{(n)}(x)\dots a_{k,q-1}^{(n)}(x)a_{k,s_1}^{(n)}(x)p_{n,k}\left(\frac{x}{1+x}\right)\right|$$
$$\leq \sum_{j=0}^{s}\left(|K_j||S_{q+j-1}^{(n)}(x)| + |\overline{K}_j||S_{q+j}^{(n)}(x)\right)$$
$$+ \overline{\overline{K}}_s \sum_{k=0}^{n} a_{k,r}^{(n)}(x)\prod_{i=1}^{q+s} a_{k,i}^{(n)}(x)p_{n,k}\left(\frac{x}{1+x}\right).$$

Using (3) and (4) it is easy to see that

$$\sum_{k=0}^{n} a_{k,r}^{(n)}(x)\prod_{i=1}^{q+s} a_{k,i}^{(n)}(x)p_{n,k}(\frac{x}{1+x})$$

is bounded from above by $K(q,s,r)(x+1)^{q+s+1}$. Moreover

$$\left|K_j\right| \leq K(q,r,j)(x+1)^{j+2}\frac{1}{(n+1)^{j+1}},$$
$$\left|\overline{K}_j\right| \leq K(q,r,j)(x+1)^{j+1}\frac{1}{(n+1)^{j}},$$
$$\left|\overline{\overline{K}}_j\right| \leq K(q,r,j)(x+1)^{j+1}\frac{1}{(n+1)^{j+1}}.$$

These estimates and (7) for $s = [(q+1)/2]$ give us (11) for $r = 1$. Next (11) follows for all $r \in \mathbb{N}$ by induction. Choosing $q = 1$ in (11) we obtain (10). $\square$

Identity (1), estimate (10) and the Schwarz inequality lead to

Lemma 6. *Let $r \in \mathbb{N}$, $s_1, \dots s_r \in \mathbb{N}$, $n \in \mathbb{N}$, $x \in I$. Then*

$$\sum_{k=0}^{n}\left|a_{k,s_1}^{(n)}(x)\dots a_{k,s_r}^{(n)}(x)\right|p_{n,k}\left(\frac{x}{1+x}\right) \tag{12}$$
$$\leq K(r,s_1,\dots s_r)x(x+1)^{2r}\left(n^{-r/2} + n^{r-1}p_{n,n}\left(\frac{x}{1+x}\right)\right).$$

3. Main result

In this section we consider only the points $x \in [0;\infty)$ at which $w_x(\delta; f) < \infty$ for all $\delta > 0$.

Theorem. *Let $f : I \to R$ be integrable in the Lebesgue or Denjoy-Perron sense on every compact interval contained in I and let $n \in \mathbb{N}$, $x \in I$. Given any number $q \in \mathbb{N}$, we have*

$$(13)\quad |M_n f(x) - f(x)| \le K(q)(x+1)^{2q+4}\left(1 + n^{3q/2+2}\left(\frac{x}{1+x}\right)^n\right) \times \sum_{k=0}^{\mu} \frac{1}{(k+1)^q}\, w_x\left(\frac{k+1}{\sqrt{n}}\,; f\right),$$

where $\mu = [\sqrt{n}|n/2 - x|]$.

Proof: For the sake of brevity we will write $f(x+r) - f(x) = \varphi_x(t)$ and $w_x(\delta; f) = w_x(\delta)$. In view of (1) we have

$$M_n f(x) - f(x) = (n+2)\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right)\int_{k/(n+2-k)}^{(k+1)/(n+1-k)} \frac{f(t)-f(x)}{(1+t)^2}\,dt$$

$$= (n+2)\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right)\left(\int_0^{(k+1)/(n+1-k)-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt - \int_0^{k/(n+2-k)-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt\right)$$

$$= (n+2)p_{n,n}\left(\frac{x}{1+x}\right)\int_0^{n+1-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt - (n+2)p_{n,0}\left(\frac{x}{1+x}\right)\int_0^{-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt$$
$$+ (n+2)\sum_{k=1}^{n}\left(p_{n,k-1}\left(\frac{x}{1+x}\right) - p_{n,k}\left(\frac{x}{1+x}\right)\right)\int_0^{k/(n+2-k)-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt.$$

Consequently by (2)

$$x\,(M_n f(x) - f(x)) = x(n+2)p_{n,n}\left(\frac{x}{1+x}\right)\int_0^{n+1-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt$$
$$+ (n+2)\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right)\left(\frac{k}{n-k+1} - x\right)\int_0^{k/(n+2-k)-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt.$$

In view of the second mean value theorem

$$\left|\int_0^{k/(n+2-k)-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt\right| \le \left(\frac{1}{(1+x)^2}\right)\left|\int_0^{\xi_1} \varphi_x(t)\,dt\right| + \left(\frac{n+2-k}{n+2}\right)^2\left|\int_{-|k/(n+2-k)-x|}^{\xi_2} \varphi_x(t)\,dt\right|,$$

where $0 < \xi_1 < |k/(n+2-k) - x|, -|k/(n+2-k) - x| < \xi_2 < 0$.

Applying the obvious inequality $|\int_0^h \varphi_x(t)\,dt| \le |h| w_x(|h|)$, we obtain

$$\left| \int_0^{k/(n+2-k)-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt \right| \le 3 \left| \frac{k}{n+2-k} - x \right| w_x \left(\left| \frac{k}{n+2-k} - x \right| \right).$$

Therefore

$$\begin{aligned}
&x\,|M_n f(x) - f(x)| \\
&\le R_n(x) + 3(n+2) \sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right) \\
&\quad \times \left(\left| a_{k,1}^{(n)}(x) a_{k,2}^{(n)}(x) \right| + \frac{1}{n+1} \left| a_{k,1}^{(n)}(x) a_{k,2}^{(n)}(x) \right| + \frac{x+1}{n+1} \left| a_{k,1}^{(n)}(x) \right| \right) \\
&\quad \times w_x \left(\left| \frac{k}{n+2-k} - x \right| \right) \\
&\le R_n(x) + 3 \sum_{\nu=0}^{\mu} T_\nu^n(\lambda; x) w_x((\nu+1)\lambda),
\end{aligned}$$

where $\lambda \in (0;1)$, $\mu = [\frac{1}{\lambda} |\frac{n}{2} - x|]$,

$$\begin{aligned}
&T_\nu^{(n)}(\lambda; x) \\
&= \sum_{\nu\lambda < |k/(n-k+2) - x| \le (\nu+1)\lambda} (n+2) \left(2 \left| a_{k,1}^{(n)}(x) a_{k,2}^{(n)}(x) \right| + \frac{x+1}{n+1} \left| a_{k,1}^{(n)}(x) \right| \right) \\
&\quad \times p_{n,k}\left(\frac{x}{1+x}\right)
\end{aligned}$$

and

$$R_n(x) = x(n+2) p_{n,n}\left(\frac{x}{1+x}\right) \left| \int_0^{n+1-x} \frac{\varphi_x(t)}{(1+x+t)^2}\,dt \right|.$$

(If $k < 0$ or $k > n$, then $p_{n,k}(\frac{x}{1+x})$ is equal to zero.)

Applying (12) we obtain

$$\begin{aligned}
T_0^{(n)}(\lambda; x) &\le (n+2) \sum_{k=0}^{n} \left(2 \left| a_{k,1}^{(n)}(x) a_{k,2}^{(n)}(x) \right| + \frac{x+1}{n+1} \left| a_{k,1}^{(n)}(x) \right| \right) p_{n,k}\left(\frac{x}{1+x}\right) \\
&\le K(q) x (1+x)^4 \left(1 + n p_{n,n}\left(\frac{x}{1+x}\right) \right)
\end{aligned}$$

and, if $1 \le \nu \le \mu$

$$\begin{aligned} T_\nu^{(n)} &\le \frac{2n}{\nu^q \lambda^q} \sum_{k=0}^{n} \left(2 \left| a_{k,1}^{(n)}(x) \| a_{k,2}^{(n)}(x) \right| + \frac{x+1}{n+1} \left| a_{k,1}^{(n)}(x) \right| \right) \\ &\quad \times \left| \frac{k}{n-k+2} - x \right| p_{n,k} \left(\frac{x}{1+x} \right) \\ &\le \frac{4^{q+1} n}{\nu^q \lambda^q} \sum_{k=0}^{n} \left(\left| a_{k,1}^{(n)}(x) \| a_{k,2}^{(n)}(x) \right|^{q+1} + \frac{(x+1)^q}{(n+1)^q} \left| a_{k,1}^{(n)}(x) a_{k,2}^{(n)}(x) \right| \right. \\ &\quad \left. + \frac{(x+1)^{q+1}}{(n+1)^{q+1}} \left| a_{k,1}^{(n)}(x) \right| + \frac{x+1}{n+1} \left| a_{k,2}^{(n)}(x) \right|^q \right) p_{n,k} \left(\frac{x}{1+x} \right). \end{aligned}$$

Therefore using (12)

$$T_\nu^{(n)}(\lambda; x) \le K(q) x \frac{(x+1)^{2q+4}}{\nu^q \lambda^q} \left(n^{-q/2} + n^{q+2} \left(\frac{x}{1+x} \right)^n \right).$$

Collecting the results, choosing $\lambda = n^{-1/2}$ and estimating

$$|R_n(x)| \le 3x(x+1) n^2 w_x(|n+1-x|) p_{n,n} \left(\frac{x}{1+x} \right),$$

we get (13) immediately. $\square$

4. Special cases

Let $D^*_{\mathrm{loc}}(I)$ be the class of all functions integrable in the Denjoy-Perron sense on every compact interval contained in I. Clearly, if $f \in D^*_{\mathrm{loc}}(I)$, then the function

$$F(x) = \int_0^x f(t)\, dt$$

is ACG^* on every $[a; b] \subset I$ and $F'(x) = f(x)$ almost everywhere [13]. Consequently,

$$\lim_{\delta \to 0+} w_x(\delta; f) = 0 \quad \text{a.e. on } I.$$

Suppose that $f \in D^*_{\mathrm{loc}}(I)$ and that

$$\|f\| \equiv \sup_{0 \le \nu < \infty} \left(\left| \int_\nu^{\nu+\mu} f(t)\, dt \right| \right) < \infty.$$

The operators $M_n f$ are well-defined for all $n \in \mathbb{N}$.

As is known [12], for any $\varepsilon > 0$ there is a $\delta_0 > 0$ such that

$$w_x(\delta; f) \le \varepsilon + |f(x) + \frac{1}{\delta_0}(1+2\delta)\|f\| \quad \text{for all} \quad \delta > 0.$$

This inequality and the fact that $\lim_{\delta\to 0+} w_x(\delta; f) = 0$ ensure that the right-hand side of the estimate (13) (with arbitrary $q \ge 3$) converges almost everywhere to zero as $n \to \infty$.

Let $m \in \mathbb{N}_0$. Denote by $L_m(I)$ the class of all measurable functions f on I such that

$$\|f\|_m \equiv \sup_{x\in I} \frac{|f(x)|}{1+x^{2m}} < \infty.$$

It is easy to see that the operators $M_n f$ are well-defined for every function $f \in L_m(I)$. Moreover, for any $\delta > 0$, the inequality

$$w_x(\delta; f) \le \left\{2 + (1+2^m)\, x^{2m} + 2^m \delta^{2m}\right\} \|f\|_m,$$

(see [12]) assures the convergence of the sum

$$\sum_{k=0}^{[\sqrt{n}|\frac{n}{2}-x|]} \frac{1}{(k+1)^q}\, w_x\left(\frac{k+1}{\sqrt{n}}\,; f\right)$$

with an arbitrary $q \ge 2m+2$. Consequently, if x is a Lebesgue point of f, i.e. if $w_x(\delta; f) \to 0$ as $\delta \to 0+$, then the right-hand side of the inequality (13) (with $q \ge 2m+2$) converges to zero as $n \to \infty$.

Further, for continuous $f \in L_m(I)$, let us introduce the weighted modulus of continuity

$$\omega(\delta; f)_m = \sup_{|h|\le\delta} \|f(\cdot + h) - f(\cdot)\|_m \quad (\delta > 0).$$

Then Theorem (with $q = 2m+3$) and inequality

$$w_x(r\delta; f) \le \left\{1 + (2x)^{2m} + (2(r-1)\delta)^{2m}\right\} r\omega(\delta; f)_m, \quad (x \in I,\ \delta > 0,\ r \in \mathbb{N})$$

(see [12]) give us

Corollary 1. *If $f \in L_m(I)$ is continuous on I then, for all $n \in \mathbb{N}$,*

$$\|M_n f - f\|_m \le K(m)\omega\left(\frac{1}{\sqrt{n}}\,; f\right)_m.$$

Clearly, if f is such that $f(x)(1+x^{2m})^{-1} = o(1)$ as $x \to \infty$, then $\omega(\delta; f)_m \to 0$ as $\delta \to 0+$. Hence in this case $\|M_n f - f\|_m$ as $n \to \infty$.

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