Mapping theorems on $\aleph$-spaces

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Abstract. In this paper we improve some mapping theorems on $\aleph$-spaces. For instance we show that an $\aleph$-space is preserved by a closed and countably bi-quotient map. This is an improvement of Yun Ziqiu's theorem: an $\aleph$-space is preserved by a closed and open map.

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1. Preliminaries

In this paper all spaces are regular T_1 and all maps are continuous onto. For $A \subset X$ we denote by ∂A the boundary of A in X.

Definition 1.1. A cover $\mathcal{P}$ of subsets of a space X is a *k-network* for X [7] if whenever $K \subset U$ with K compact and U open in X, there is a finite subfamily $\mathcal{Q} \subset \mathcal{P}$ such that $K \subset \cup\mathcal{Q} \subset U$. A space is an *$\aleph$-space* [7] if it has a σ-locally finite k-network.

The notion of a k-network plays an important role in the theory of generalized metric spaces. For instance, a Fréchet $\aleph$-space is precisely the closed s-image of a metric space [2], [4].

Definition 1.2. A family $\{A_\alpha : \alpha \in I\}$ of subsets of a space X is *hereditarily closure-preserving* (simply, HCP) if $\bigcup\{\overline{B_\alpha} : \alpha \in J\} = \overline{\bigcup\{B_\alpha : \alpha \in J\}}$, whenever $J \subset I$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in J$.

Every locally finite family is hereditarily closure-preserving.

The space S_{ω_1} is the space obtained from the topological sum of ω_1 many convergent sequences by identifying all the limit points to a single point. The following is due to Junnila and Ziqiu [3].

Theorem 1.3. *Let X be a space with a σ-HCP k-network. Then X is an $\aleph$-space iff X contains no closed copy of S_{ω_1}.*

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2. Results

Definition 2.1. A subset A of a space Y is a *sequential neighborhood* of a point $y \in Y$ if any sequence converging to y is eventually in A. A map $\varphi : X \to Y$ satisfies *property* (ω_1) if, whenever $y \in Y$ and $\{U_\alpha : \alpha < \omega_1\}$ is an increasing open cover of X, then there is α such that $\varphi(U_\alpha)$ is a sequential neighborhood of y. A map $\varphi : X \to Y$ satisfies *property* (ω) if, whenever $y \in Y$ and $\{U_n : n \in \omega\}$ is an increasing open cover of X, then there is n such that $\varphi(U_n)$ is a sequential neighborhood of y.

Lemma 2.2. *Let A be a countably infinite subset of a space X such that every infinite subset of A is not closed in X. If $x \in \overline{A} \setminus A$ and $\{x\}$ is a G_δ-set, then there is a sequence in A converging to x.*

Proof: Let $\{G_n : n \in \omega\}$ be an open family in X satisfying $\{x\} = \bigcap\{G_n : n \in \omega\}$ and $\overline{G}_{n+1} \subset G_n$. For each $n \in \omega$, take a point $x_n \in A \cap G_n$. The set $\{x\} \cup \{x_n : n \in \omega\}$ is closed in X. For every open neighborhood U of x, $\{x_n : n \in \omega\} \setminus U$ is closed in X, hence $\{x_n : n \in \omega\} \setminus U$ is finite. Therefore $\{x_n : n \in \omega\}$ is a convergent sequence to x. □

Theorem 2.3. *The following hold respectively:*

(1) *an $\aleph$-space is preserved by a closed map with property (ω_1);*
(2) *an $\aleph$-space is preserved by a closed map with property (ω).*

Proof: Let $\varphi : X \to Y$ be a closed map with property (ω_1) (or property (ω)) and let X be an $\aleph$-space. Let $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$ be a σ-locally finite k-network for X. Without loss of generality, we may assume that each member of $\mathcal{P}$ is closed in X and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for $n \in \omega$. As noted in the proof of [10, Proposition 1.8(3)], the family $\{\varphi(P) : P \in \mathcal{P}\}$ is a σ-HCP k-network for Y.

Assume that Y is not an $\aleph$-space. Then by Theorem 1.3, Y has a closed copy of S_{ω_1}. Let

$$S_{\omega_1} = \{\infty\} \cup \big\{y_{\alpha,n} : \alpha < \omega_1, n \in \omega\big\} \subset Y,$$

where $\{y_{\alpha,n} : n \in \omega\}$ is the α-th sequence converging to ∞.

By induction we show that for each $\alpha < \omega_1$, there are $n_\alpha \in \omega$ and a finite subfamily $\mathcal{F}_\alpha \subset \mathcal{P}$ such that

(a) $\bigcup\{\varphi^{-1}(y_{\alpha,n}) : n \geq n_\alpha\} \subset \bigcup \mathcal{F}_\alpha$,
(b) for each $P \in \mathcal{F}_\alpha$, $P \cap (\bigcup\{\varphi^{-1}(y_{\alpha,n}) : n \geq n_\alpha\}) \neq \emptyset$,
(c) $\mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$ for $\alpha < \beta < \omega_1$.

Fix an arbitrary $\gamma < \omega_1$ and assume that for each $\alpha < \gamma$ we have already found $n_\alpha \in \omega$ and a finite subfamily $\mathcal{F}_\alpha \subset \mathcal{P}$. For each $\alpha < \gamma$ take a finite set $F_\alpha \subset \bigcup\{\varphi^{-1}(y_{\alpha,n}) : n \geq n_\alpha\}$ such that $F_\alpha \cap P \neq \emptyset$ for any $P \in \mathcal{F}_\alpha$. The set $F = \bigcup\{F_\alpha : \alpha < \gamma\}$ is closed in X. For each $n \in \omega$, let

$$\mathcal{Q}_n = \{P \in \mathcal{P}_n : P \cap F = \emptyset, P \cap \varphi^{-1}(y_{\gamma,k}) \neq \emptyset \text{ for infinitely many } k \in \omega\}.$$

Obviously $\mathcal{Q}_n \subset \mathcal{Q}_{n+1}$. Assume $P_i \in \mathcal{Q}_n$, $i \in \omega$ and $P_i \neq P_j$ for $i \neq j$. Then we can take a point $x_i \in P_i$ such that $\varphi(\{x_i\}_{i\in\omega})$ is a subsequence of $\{y_{\gamma,n} : n \in \omega\}$. Since $\mathcal{Q}_n$ is locally finite, $\{x_i\}_{i\in\omega}$ is closed in X. Since φ is closed, this is a contradiction. Therefore each $\mathcal{Q}_n$ is finite. Assume for each $n \in \omega$, there are infinitely many $k \in \omega$ with $\varphi^{-1}(y_{\gamma,k}) \setminus (\bigcup \mathcal{Q}_n) \neq \emptyset$. Then there are a sequence $k_0 < k_1 < \cdots$ and a point $x_n \in \varphi^{-1}(y_{\gamma,k_n}) \setminus (\bigcup \mathcal{Q}_n)$. Since φ is closed, no infinite subset of $\{x_n : n \in \omega\}$ is closed in X. Moreover every point of an ℵ-space is a G_δ-set. Hence by Lemma 2.2, $\{x_n : n \in \omega\}$ contains a convergent sequence to some point in $\varphi^{-1}(\infty)$. Since $\mathcal{P}$ is a k-network for X, there is $P \in \mathcal{P}$ such that $P \cap F = \emptyset$ and P contains infinitely many x_n's. Let $P \in \mathcal{P}_l$ for some $l \in \omega$. Then $P \in \mathcal{Q}_l$. Since P contains only finitely many x_n's, this is a contradiction. Consequently there is $n_\gamma \in \omega$ such that $\bigcup\{\varphi^{-1}(y_{\gamma,n}) : n \geq n_\gamma\} \subset \bigcup \mathcal{Q}_{n_\gamma}$. Let $\mathcal{F}_\gamma = \mathcal{Q}_{n_\gamma}$. The γ-th step of our induction is complete.

Since each $\mathcal{F}_\alpha$ is finite, there are $m \in \omega$ and an uncountable set $I \subset \omega_1$ such that $\mathcal{F}_\alpha \subset \mathcal{P}_m$ for any $\alpha \in I$. For each $\alpha \in I$, let $E_\alpha = \bigcup \mathcal{F}_\alpha$. Since $\mathcal{P}_m$ is locally finite, $\{E_\alpha : \alpha \in I\}$ is a locally finite closed family in X.

The case of property (ω_1). Consider the increasing open cover

$$\{X \setminus \bigcup_{\beta>\alpha} E_\beta : \alpha < \omega_1, \beta \in I\}$$

of X. By property (ω_1), there is α such that $\varphi(X \setminus \bigcup_{\beta>\alpha} E_\beta)$ is a sequential neighborhood of ∞. But the set obviously fails to be a sequential neighborhood of ∞. As a result, Y does not have any closed copy of S_{ω_1}, therefore Y is an ℵ-space.

The case of property (ω). The idea is the same as property (ω_1). Take an infinite subset $J = \{\alpha_n : n \in \omega\} \subset I$, and consider the increasing open cover $\{X \setminus \bigcup_{m>n} E_{\alpha_m} : n \in \omega\}$ of X. □

Definition 2.4. A map $\varphi : X \to Y$ is *countably bi-quotient* [9] if for each $y \in Y$ and each countable increasing open family $\{U_n : n \in \omega\}$ covering $\varphi^{-1}(y)$, there is $n \in \omega$ such that $\varphi(U_n)$ is a neighborhood of y.

S. Lin asked the author whether an ℵ-space is preserved by a closed and countably bi-quotient map. Since a countably bi-quotient map trivially satisfies property (ω), we have a positive answer to the question.

Corollary 2.5. *An ℵ-space is preserved by a closed and countably bi-quotient map.*

Corollary 2.6. (1) *An ℵ-space is preserved by a closed map satisfying that $\partial\varphi^{-1}(y)$ is Lindelöf for any $y \in Y$* [1], [4];

(2) *An ℵ-space is preserved by a closed and open map* [11].

Proof: (1) Let $\varphi : X \to Y$ be a closed map satisfying that $\partial\varphi^{-1}(y)$ is Lindelöf for any $y \in Y$. For each $y \in Y$, we define a set A_y as follows: if y is isolated in Y, take an arbitrary point $x_y \in \varphi^{-1}(y)$ and let $A_y = \{x_y\}$; otherwise let $A_y = \partial\varphi^{-1}(y)$. Let $A = \bigcup_{y\in Y} A_y$. Then the restricted map $\varphi|A : A \to Y$ is closed onto and each fiber of this map is Lindelöf. Since $\varphi|A$ satisfies property (ω_1), Y is an $\aleph$-space by Theorem 2.3.

(2) Every open map is obviously countably bi-quotient. Apply Corollary 2.5. □

Definition 2.7. A map $\varphi : X \to Y$ is *sequence-covering* in the sense of Siwiec [8] if, whenever $\{y_n\}_{n\in\omega}$ is a sequence in Y converging to a point $y \in Y$, there are a point $x \in \varphi^{-1}(y)$ and points $x_n \in \varphi^{-1}(y_n)$, $n \in \omega$, such that $\{x_n\}_{n\in\omega}$ converges to x.

C. Liu noted in [5] that an $\aleph$-space is preserved by a closed and sequence-covering map. This result follows from our theorem.

Proposition 2.8. *Let $\varphi : X \to Y$ be a closed and sequence-covering map. If X has a σ-HCP k-network, then φ satisfies property (ω_1).*

Proof: Let $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$ be a σ-HCP k-network for X. For each $n \in \omega$, the family $\{\overline{P} : P \in \mathcal{P}_n\}$ is also hereditarily closure-preserving. Therefore we may assume that each member of $\mathcal{P}$ is closed in X.

Assume that φ does not satisfy property (ω_1). Then there are a point $y \in Y$ and an increasing open cover $\{U_\alpha : \alpha < \omega_1\}$ of X such that each $\varphi(U_\alpha)$ fails to be a sequential neighborhood of y. For each $\alpha < \omega_1$, take a sequence L_α in Y such that L_α converges to y and $L_\alpha \cap \varphi(U_\alpha) = \emptyset$. Since φ is sequence-covering, for each $\alpha \geq 1$, there are a sequence K_α in X and a point $x_\alpha \in \varphi^{-1}(y)$ such that K_α converges to x_α and $\varphi(K_\alpha) = L_0 \cup L_\alpha$. Let $\{A_\alpha, B_\alpha\}$ be a decomposition of K_α with $\varphi(A_\alpha) = L_0$ and $\varphi(B_\alpha) = L_\alpha$. For each $\alpha \geq 1$, take $\gamma_\alpha < \omega_1$ with $\{x_\alpha\} \cup K_\alpha \subset U_{\gamma_\alpha}$, and take $P_\alpha \in \mathcal{P}$ such that $x_\alpha \in P_\alpha \subset U_{\gamma_\alpha}$ and P_α contains infinitely many points in A_α.

We note that the family $\{P_\alpha : \alpha \geq 1\}$ is uncountable. Since each P_α is contained in some member of the open cover, if the family is countable, $\bigcup\{P_\alpha : \alpha \geq 1\} \subset U_\delta$ for some $\delta < \omega_1$. Because of $L_\delta \cap \varphi(U_\delta) = \emptyset$, $x_\delta \notin U_\delta$. This is a contradiction. Thus the family is uncountable. Hence there are $m \in \omega$ and an uncountable set $I \subset \omega_1$ with $\{P_\alpha : \alpha \in I\} \subset \mathcal{P}_m$. Take a sequence $\alpha_0 < \alpha_1 < \cdots$ in I, and take a point $x_n \in P_{\alpha_n} \cap A_{\alpha_n}$ such that $\{\varphi(x_n)\}_{n\in\omega}$ converges to y. Since $\{P_{\alpha_n} : n \in \omega\}$ is hereditarily closure-preserving, $\{x_n\}_{n\in\omega}$ is closed in X. This is a contradiction, because φ is a closed map. Consequently φ satisfies property (ω_1). □

By the above proposition and Theorem 2.3, we have the following.

Corollary 2.9 ([5])**.** *An $\aleph$-space is preserved by a closed and sequence-covering map.*

It was proved in [6] that a topological group is an $\aleph$-space if it is the closed image of an $\aleph$-space.

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