# Cellularity of a space of subgroups of a discrete group

A. Leiderman, I.V. Protasov

*Abstract.* Given a discrete group $G$, we consider the set $\mathcal{L}(G)$ of all subgroups of $G$ endowed with topology of pointwise convergence arising from the standard embedding of $\mathcal{L}(G)$ into the Cantor cube $\{0,1\}^G$. We show that the cellularity $c(\mathcal{L}(G)) \le \aleph_0$ for every abelian group $G$, and, for every infinite cardinal $\tau$, we construct a group $G$ with $c(\mathcal{L}(G)) = \tau$.

*Keywords:* space of subgroups, cellularity, Shanin number

*Classification:* 54B20, 54A25

Let $G$ be a discrete group and $\exp G$ the set of all subsets of $G$. We identify $\exp G$ with the Cantor cube $\{0,1\}^G$ and consider the set $\mathcal{L}(G)$ of all subgroups of $G$ as a subspace of $\{0,1\}^G$. Let $\mathcal{F}(G)$ be the family of all finite subsets of $G$. Given any $F, H \in \mathcal{F}(G)$, we put

$$[F, G \setminus H] = \{A \in \mathcal{L}(G) : F \subseteq A \subseteq G \setminus H\}.$$

Then the family $\{[F, G \setminus H] : F, H \in \mathcal{F}(G)\}$ forms a base of the topology in $\mathcal{L}(G)$. It is easy to see that $\mathcal{L}(G)$ is a zero-dimensional compact space.

We denote by $c(X)$ the *cellularity* of a topological space $X$. Remind that $c(X)$ is the supremum of cardinalities of disjoint families of open subsets of $X$.

We say that a topological space $X$ has *Shanin number* $\omega$ (see [4] or [1, Problem 2.7.11]) if any uncountable family $\mathcal{U}$ of non-empty open subsets of $X$ has an uncountable subfamily $\mathcal{V} \subseteq \mathcal{U}$ with $\bigcap \mathcal{V} \neq \emptyset$. Evidently, if a space $X$ has Shanin number $\omega$ then $c(X) \le \aleph_0$.

Given any topological group $G$, we denote by $L(G)$ the space of all closed subgroups of $G$ endowed with the Vietoris topology. By [3, Theorem 2], $c(L(G)) \le \aleph_0$ for every compact group $G$.

**Theorem 1.** *For every discrete abelian group $G$, the space $\mathcal{L}(G)$ has Shanin number $\omega$, in particular, $c(\mathcal{L}(G)) \le \aleph_0$.*

Proof: By [6, Theorem 24.1], $G$ is a subgroup of some divisible group $D$. By [6, Theorem 23.1], $D$ is a direct sum $\oplus_{\alpha \in I} G_\alpha$ of some family of countable groups.

The first author was partially supported by the Israel Science Foundation Grant no. 508/06.

Thus we suppose that $G$ is a subgroup of $\oplus_{\alpha\in I}G_\alpha$. Let $\{U_\beta : \beta \in J\}$ be an uncountable family of non-empty open subsets of $\mathcal{L}(G)$. For every $\beta \in J$, we pick finite subsets $F_\beta$, $H_\beta$ of $G$ such that $\emptyset \neq [F_\beta, G \setminus H_\beta] \subseteq U_\beta$, and put

$$K_\beta = \{\alpha \in I : \mathrm{pr}_\alpha\, g \neq 0 \ \text{ for some } \ g \in F_\beta \cup H_\beta\}.$$

Then $\{K_\beta : \beta \in J\}$ is an uncountable family of finite subsets of $I$. By the $\Delta$-Lemma, there exist an uncountable subset $J_1 \subseteq J$ and a finite subset $K \subseteq I$ such that $K_\beta \cap K_\gamma = K$ for all distinct $\beta, \gamma \in J_1$. We put

$$G_K = \{g \in G : \mathrm{pr}_\alpha\, g = 0 \ \text{ for every } \ \alpha \notin K\}.$$

For every subset $X$ of $G$, we denote by $\langle X\rangle$ the subgroup of $G$ generated by $X$. Every subgroup of a finitely generated abelian group is finitely generated, hence it follows that for every $\beta \in J_1$, the subgroup $\langle F_\beta\rangle \cap G_K$ is finitely generated. Since $G_K$ is a countable group, there are countably many finitely generated subgroups of $G_K$, and therefore there exist an uncountable subset $J_2 \subseteq J_1$ and a subgroup $A \subseteq G_K$ such that $\langle F_\beta\rangle \cap G_K = A$ for every $\beta \in J_2$. We put $S = \langle \bigcup_{\beta\in J_2} F_\beta\rangle$ and prove that $S \in \bigcap_{\beta\in J_2}[F_\beta, G \setminus H_\beta]$. It suffices to show that $H_\beta \cap S = \emptyset$ for every $\beta \in J_2$.

Pick any element $g \in S$ and fix $\beta \in J_2$. Clearly, if $g = 0$ then $g \notin H_\beta$ because of $0 \in \langle F_\beta\rangle$ and $\emptyset \neq [F_\beta, G \setminus H_\beta]$. If $g \neq 0$, we can write $g$ as

$$g = g_{\alpha_1} + \cdots + g_{\alpha_n},$$

where $g_{\alpha_i} \in G_{\alpha_i} \setminus \{0\}$ for every $i \in \{1, \ldots, n\}$. If $\alpha_i \notin K_\beta$ for some $i \in \{1, \ldots, n\}$ then, by definition of $K_\beta$, we have $g \notin H_\beta$.

Now consider the possibility of $\{\alpha_1, \ldots, \alpha_n\} \subseteq K_\beta$. We claim that in this case $g \in \langle F_\beta\rangle$. Let us assume the contrary, $g \notin \langle F_\beta\rangle$.

By the choice of $J_2$, we have $\langle F_\beta\rangle \cap G_K = \langle F_\gamma\rangle \cap G_K$ for every distinct $\beta, \gamma \in J_2$, therefore by definition of $S$, there exist $\gamma \in J_2 \setminus \{\beta\}$ and $j \in \{1, \ldots, n\}$ such that $\alpha_j \in K_\gamma \setminus K$. On the other hand, $K_\beta \cap K_\gamma = K$, hence $\alpha_j \notin K_\beta$, and we get a contradiction with $\{\alpha_1, \ldots, \alpha_n\} \subseteq K_\beta$. So, $g \in \langle F_\beta\rangle$ which implies that $g \notin H_\beta$. □

Remind that a Hausdorff compact space $X$ is called *dyadic* if $X$ is a continuous image of some Cantor cube. If $G$ is a compact abelian group, by [5, Theorems 3, 4], the space $L(G)$ is dyadic if and only if the weight $w(G) \leq \aleph_1$. A natural question arises to characterize discrete groups $G$ for which $\mathcal{L}(G)$ is a dyadic space.

**Theorem 2.** *For a discrete abelian group $G$, the space $\mathcal{L}(G)$ is dyadic if and only if $|G| \leq \aleph_1$.*

Proof: By [2, Theorem 3], for every compact abelian group $G$ the space $L(G)$ is homeomorphic to $\mathcal{L}(\hat{G})$, where $\hat{G}$ is the dual group to $G$. Therefore, by Pontryagin's duality and by above-mentioned [5, Theorems 3, 4], the space $\mathcal{L}(G)$ is dyadic if and only if $|G| \leq \aleph_1$. $\square$

Our last theorem shows that, in contrast to Theorem 1, there are non-abelian groups $G$ with arbitrary large cellularity $c(\mathcal{L}(G))$.

**Theorem 3.** *For every infinite cardinal $\tau$ there exists a discrete group $G$ such that $c(\mathcal{L}(G)) = |G| = \tau$.*

Proof: Let $F$ be a free group generated by the set $\{x_\alpha, y_\alpha : \alpha < \tau\}$ of free generators. We shall define a group $G$ as the quotient $G = F/N$, where $N$ is a certain normal subgroup of the free group $F$ containing all the words $x_\alpha^2$, $y_\alpha^2$, $x_\beta x_\alpha x_\beta y_\alpha$, $\alpha < \beta < \tau$.

We fix $\alpha < \tau$ and put

$$A = \langle a_\alpha\rangle \times \langle b_\alpha\rangle,\ B = \otimes_{\alpha<\beta\tau}\langle c_\beta\rangle,$$

where $\langle a_\alpha\rangle$, $\langle b_\alpha\rangle$, $\langle c_\beta\rangle$ are the cyclic subgroups of order 2. Then we define a group $G_\alpha$ as the semidirect product $G_\alpha = A \leftthreetimes B$ with $c_\beta a_\alpha c_\beta = b_\alpha$ for all $\alpha < \beta < \tau$. Let $f_\alpha : F \to G_\alpha$ be a homomorphism defined by the rule

$$f_\alpha(x_\lambda) = f_\alpha(y_\lambda) = 1 \ \text{ for all } \ \lambda < \alpha,$$
$$f_\alpha(x_\alpha) = a_\alpha, \quad f_\alpha(y_\alpha) = b_\alpha,$$
$$f_\alpha(x_\beta) = f_\alpha(y_\beta) = c_\beta \ \text{ for all } \ \alpha < \beta < \tau.$$

Finally, we define

$$N = \bigcap\{\mathrm{Ker}\ f_\alpha : \alpha < \tau\},\ \textit{and}\ G = F/N.$$

Clearly, $f_\alpha(x_\lambda^2) = f_\alpha(y_\lambda^2) = 1$ for every $\lambda < \tau$, and it easily could be seen that $f_\alpha(x_\beta x_\lambda x_\beta) = f_\alpha(y_\lambda)$ for all $\lambda < \beta < \tau$. Hence, $N$ is a normal subgroup of the free group $F$ containing all the words $x_\alpha^2$, $y_\alpha^2$, $x_\beta x_\alpha x_\beta y_\alpha$, $\alpha < \beta < \tau$, but $x_\alpha, y_\alpha, x_\alpha y_\alpha^{-1} \notin N$. It follows that $x_\alpha N \neq 1$, $y_\alpha N \neq 1$, $x_\alpha N \neq y_\alpha N$ in the quotient group $G = F/N$. In the sequel we denote the cosets $wN$, i.e. the elements of the group $G$, simply by $w$.

In order to finish the proof of Theorem 3 we show that $c(\mathcal{L}(G)) = |G| = \tau$. Let us consider the family $\{[\{x_\alpha\}, G \setminus \{y_\alpha\}] : \alpha < \tau\}$ of non-empty open subsets of $\mathcal{L}(G)$, and show that this family is disjoint. We assume the contrary and choose $\alpha,\ \beta$ such that $\alpha < \beta < \tau$ and

$$[\{x_\alpha\}, G \setminus \{y_\alpha\}] \cap [\{x_\beta\}, G \setminus \{y_\beta\}] \neq \emptyset.$$

Then the subgroup $\langle x_\alpha,\ x_\beta\rangle$ generated by $x_\alpha$, $x_\beta$ is in the above intersection. On the other hand, $y_\alpha = x_\beta x_\alpha x_\beta$, so

$$y_\alpha \in \langle x_\alpha,\ x_\beta\rangle,\ \langle x_\alpha,\ x_\beta\rangle \in [\{x_\beta\}, G \setminus \{y_\alpha\}]$$

and we get a contradiction. $\square$

**Question.** *Does there exist a nilpotent group $G$ such that $c(\mathcal{L}(G)) > \aleph_0$ ?*

## References


[1] Engelking R., *General Topology*, PWN, Warszawa, 1985.
[2] Protasov I.V., *Dualisms of topological Abelian groups*, Ukrain. Mat. Zh. **31** (1979), 207–211; English translation: Ukrainian Math. J. **31** (1979), 164–166.
[3] Protasov I.V., *The Suslin number of a space of subgroups of a locally compact group*, Ukrain. Mat. Zh. **40** (1988), 654–658; English translation: Ukrainian Math. J. **40** (1988), 559–562.
[4] Shanin N.A., *On product of topological spaces*, Trudy Mat. Inst. Akad. Nauk SSSR **24** (1948), 1–112 (in Russian).
[5] Tsybenko Yu.V., *Dyadicity of a space of subgroups of a topological group*, Ukrain. Mat. Zh. **38** (1986), 635–639.
[6] Fuchs L., *Infinite Abelian Groups*, Vol. 1, Academic Press, New York and London, 1970.



Department of Mathematics, Ben Gurion University of the Negev, Beer Sheva, Israel

*E-mail*: arkady@math.bgu.ac.il

Taras Shevchenko Kyiv National University, Ukraine

*E-mail*: protasov@unicyb.kiev.ua


(*Received* September 20, 2007, *revised* March 21, 2008)