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# Some results on $L\Sigma(\kappa)$-spaces

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*Abstract.* We present several results related to $L\Sigma(\kappa)$-spaces where $\kappa$ is a finite cardinal or $\omega$; we consider products and some constructions that lead from spaces of these classes to other spaces of similar classes.

*Keywords:* upper semicontinuous mappings, products, Lindelöf $\Sigma$-spaces

*Classification:* 54D20, 54C60, 54B10

All spaces in this article are assumed to be Tychonoff (= completely regular Hausdorff). We use terminology and notation as in [Eng2]. For multivalued mappings we do not require that images of points all be nonempty; if $p\colon X \to Y$ is a multivalued mapping and $A \subset X$, then $p(A)$ is defined as $\bigcup\{\, p(x) : x \in A \,\}$. The composition of two multivalued mappings $p\colon X \to Y$ and $q\colon Y \to Z$ is defined by the rule $(q \circ p)(x) = q(p(x))$. A multivalued mapping $p\colon X \to Y$ is *upper semicontinuous* if for every open set $V$ in $Y$ the set $\{\, x \in X : p(x) \subset V \,\}$ is open in $X$, or, equivalently, if for every point $x$ in $X$ and every neighborhood $V$ of $p(x)$ in $Y$ there is a neighborhood $U$ of $x$ in $X$ such that $p(U) \subset V$.

It is well-known that the composition of compact-valued upper semicontinuous mappings is compact-valued upper semicontinuous. In fact, it is easy to prove that a mapping is compact-valued upper semicontinuous iff it is the composition of a continuous single-valued function, the inverse of a perfect mapping and the inverse of a closed embedding (see, e.g., [KOS]).

The symbol $\mathfrak{c}$ denotes the cardinality of the continuum. If $\kappa$ is an infinite cardinal, $A(\kappa)$ denotes the one-point compactification of a discrete space of cardinality $\kappa$. The symbol $I$ stands for the closed interval $[0,1]$.

Let $\mathcal{K}$ be a cover of a space $X$. A family $\mathcal{N}$ of subsets of $X$ is called a *network modulo* $\mathcal{K}$ if for every element $K$ of $\mathcal{K}$ and a neighborhood $U$ of $K$, there is an element $N$ of $\mathcal{N}$ such that $K \subset N \subset U$ [Nag].

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Given a cardinal $\kappa$, finite or infinite, a space $X$ is called an $L\Sigma(<\kappa)$-*space* [KOS] if it satisfies one of the following equivalent conditions:

*There is a second-countable space $M$ and a compact-valued upper semicontinuous mapping $p\colon M \to X$ such that $p(M) = X$ and $w(p(z)) < \kappa$ for each $z \in M$;*

or

*There are a compact cover $\mathcal{K}$ of $X$ such that $w(K) < \kappa$ for every $K \in \mathcal{K}$ and a countable network modulo $\mathcal{K}$ in $X$.*

$X$ is an $L\Sigma(\leq \kappa)$-*space* if it is an $L\Sigma(< \kappa^+)$-space. $X$ is an $L\Sigma(\kappa)$-*space* if it is an $L\Sigma(\leq \kappa)$-space and is not an $L\Sigma(< \kappa)$-space; this concept is especially important in the case of finite cardinals $\kappa$. Of course, for finite $\kappa$, the weights of images of points and of the elements of the compact covers in the above characterizations can be replaced by the cardinalities.

The classes of $L\Sigma(<\kappa)$-spaces are invariant with respect to closed subspaces, continuous images and countable unions. Obviously, all $L\Sigma(\kappa)$-spaces are Lindelöf $\Sigma$-spaces in the sense of [Nag]; it is easy to see that $L\Sigma(\leq 1)$-spaces are exactly the spaces of countable network weight. The class of $L\Sigma(2)$-spaces includes the Double Arrow space, one-point compactifications of uncountable discrete spaces of cardinality less or equal to the continuum, and the one-point compactifications of $\Psi$-like spaces (that is, the spaces of the form $\Psi(\mathcal{A})$, where $\mathcal{A}$ is an almost disjoint family on $\omega$; see Section 2 for a detailed description). Assuming MA($\omega_1$), all scattered compact spaces of height 3 and cardinality $\omega_1$ are in $L\Sigma(\leq 3)$ [KOS].

If $\kappa \geq \mathfrak{c}$, then $L\Sigma(\leq \kappa)$-spaces are exactly Lindelöf $\Sigma$-spaces of network weight $\leq \kappa$.

## 1. Products of $L\Sigma(n)$-spaces

It is easy to see, using the fact that the product of compact-valued upper semicontinuous mappings is upper semicontinuous, that the product of an $L\Sigma(\kappa)$-space with an $L\Sigma(\lambda)$-space is an $L\Sigma(\leq \lambda\cdot\kappa)$-space. However, if $\lambda$ and $\kappa$ are finite, it turns out that the product may be of the "type" lower than $\lambda \cdot \kappa$. For example, the one-point compactification $A(\omega_1)$ of the discrete space of cardinality $\omega_1$ is an $L\Sigma(2)$-space; as shown in [KOS], for every $n \in \omega$, $A(\omega_1)^n$ is $L\Sigma(n+1)$. On the other hand, if $\omega_2 \leq \mathfrak{c}$, then the space $A(\omega_2)$ is also an $L\Sigma(2)$-space, but its square is in $L\Sigma(4)$. Thus, it may be interesting to find the exact $L\Sigma$-classes for products of some $L\Sigma(n)$-spaces. Several problems of this type were posed in [KOS] and [Oku]; here we present solutions to some of these problems.

**Theorem 1.1.** *Suppose $m, n \in \omega$, $X$ is an $L\Sigma(m)$-space and $Y$ is an $L\Sigma(n)$-space. Then $X \times Y$ is an $L\Sigma(k)$-space, where $n + m - 1 \leq k \leq mn$.*

Proof: Let $p_1\colon M_1 \to X$ and $p_2\colon M_2 \to Y$ be upper semicontinuous mappings from second countable spaces $M_1$ and $M_2$ onto $X$ and $Y$ such that $p_1$ is at most $m$-valued and $p_2$ is at most $n$-valued. Then the mapping $p_1 \times p_2\colon M_1 \times M_2 \to X \times Y$

(defined by the rule $(p_1 \times p_2)(m_1, m_2) = p_1(m_1) \times p_2(m_2)$) is upper semicontinuous and onto $X \times Y$. This proves that $X \times Y$ is an $L\Sigma(\leq mn)$-space; therefore, $X \times Y$ is an $L\Sigma(k)$-space for some $k \leq mn$.

To prove the second part of the inequality, suppose for contradiction that $X \times Y \in L\Sigma(k)$ and $k \leq n + m - 2$. Fix a second countable space $M$ and an at most $k$-valued upper semicontinuous mapping $p\colon M \to X \times Y$ such that $p(M) = X \times Y$. Let $\pi_X$, $\pi_Y$ be the projections of the product $X \times Y$; put

$$A = \{\, z \in M : |\pi_X(p(z))| \leq m - 1 \,\}.$$

Since the composition $\pi_X \circ p$ is upper semicontinuous and $X \notin L\Sigma(\leq m-1)$, there is a point $x_0 \in X$ such that $x_0 \notin \pi_X(p(A))$, hence $(\{x_0\} \times Y) \cap p(A) = \emptyset$.

Let $B = M \setminus A$ and $q\colon B \to Y$ be the multivalued mapping defined by the rule:

$$q(z) = \pi_Y\big(p(z) \cap \big(\{x_0\} \times Y\big)\big).$$

Since $\{x_0\} \times Y$ is closed in $X \times Y$, the mapping $q$ is upper semicontinuous, and from $p(M) = X \times Y$ and $(\{x_0\} \times Y) \cap p(A) = \emptyset$ it follows that $q(B) = Y$. For every $z \in B$, $p(z)$ has at most $n + m - 2$ points, and at least $m - 1$ of these points have their projections on $X$ different from $x_0$. Hence, $q(z)$ contains at most $n - 1$ points. Thus, $q$ is an upper semicontinuous, at most $(n-1)$-valued mapping from the second countable space $B$ onto the space $Y$, a contradiction with the assumption that $Y$ is an $L\Sigma(n)$-space. □

**Corollary 1.2.** *If $X$ is an $L\Sigma(n)$-space for some $n \in \omega$, then $X^m$ is an $L\Sigma(k)$-space for some $k \geq mn - m + 1$.*

In particular,

**Corollary 1.3.** *If there is an $n \in \omega$ such that $X^m$ is an $L\Sigma(\leq n)$-space for every $m \in \omega$, then $X$ has a countable network.*

It was shown in [KOS] that if $X^\omega$ is an $L\Sigma(< \omega)$-space, then there is an $n \in \omega$ such that $X^m$ is an $L\Sigma(\leq n)$-space for every $m \in \omega$; it was also shown that, consistently, this implies that $X$ has a countable network. Corollary 1.3 now allows to prove this in ZFC (thus giving an answer to Question 7.4 in [KOS]):

**Corollary 1.4.** *If $X^\omega$ is an $L\Sigma(< \omega)$-space, then $X$ has a countable network (and hence $X^\omega$ is in fact an $L\Sigma(\leq 1)$-space).*

Another interesting corollary of Theorem 1.1 is

**Corollary 1.5.** *If $X$ is an $L\Sigma(m)$-space for some $m \in \omega$, and $Y$ is an $L\Sigma(n)$-space for some $n \in \omega$, $n \geq 2$, then $X \times Y$ is not homeomorphic to $X$.*

*In particular, if $X$ is an $L\Sigma(m)$-space for some $m \in \omega$, $m \geq 2$, then all finite powers of $X$ are pairwise non-homeomorphic.*

Since the classes of $L\Sigma(\leq n)$-spaces are invariant with respect to closed subspaces and continuous images, we may further strengthen Corollary 1.5.:

**Corollary 1.6.** *If $X$ is an $L\Sigma(m)$-space for some $m \in \omega$, and $Y$ is an $L\Sigma(n)$-space for some $n \in \omega$, $n \geq 2$, then $X \times Y$ is not homeomorphic to a continuous image of any closed subspace of $X$.*

**Corollary 1.7.** *If $X$ is an $L\Sigma(\leq n)$-space for some $n \in \omega$, and there are natural $k$ and $m$ such that $k < m$ and $X^m$ is a continuous image of a closed subspace of $X^k$, then $X$ has a countable network.*

For example,

**Corollary 1.8.** *Let $X$ be the Double Arrow space. If $m, n \in \omega$ and $n > m$, then $X^n$ cannot be embedded into a continuous image of $X^m$.*

For some individual spaces, in particular, for products of given spaces, finding the exact $L\Sigma(k)$-class where they belong appears a non-trivial task. For example, it is still not clear whether the square of the Double Arrow space is in $L\Sigma(3)$ or $L\Sigma(4)$ (Problem 1(132) in [Oku]).

The next theorem solves Problem 3(134) in [Oku].

Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of $\omega$. Recall that the space $\Psi(\mathcal{A})$ is defined as the union $\omega \cup \mathcal{A}$ with the topology in which the points of $\omega$ are isolated, and basic neighborhoods of the points $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus F$ where $F \subset A$ is finite. Clearly, $\Psi(\mathcal{A})$ is a Hausdorff zero-dimensional (hence Tychonoff) locally compact space. Let $\alpha\Psi(\mathcal{A})$ be its one-point compactification. Then $\alpha\Psi(\mathcal{A})$ is an $L\Sigma(2)$-space, because it is a countable union of singletons (points of $\omega$) and the subspace homeomorphic to $A(|\mathcal{A}|)$; the latter space is in $L\Sigma(2)$, and the class $L\Sigma(2)$ is invariant with respect to countable unions (see [KOS]). Problem 3(134) in [Oku] was whether the square of a space $\alpha\Psi(\mathcal{A})$ can be an $L\Sigma(3)$-space and whether it can be an $L\Sigma(4)$-space.

**Theorem 1.9.** *Let $\mathcal{A}$, $\mathcal{B}$ be uncountable almost disjoint families of infinite subsets of $\omega$, and let $X = \alpha\Psi(\mathcal{A}) \times \alpha\Psi(\mathcal{B})$. Then*

*$X$ is an $L\Sigma(3)$-space iff both $\mathcal{A}$ and $\mathcal{B}$ have cardinality $\omega_1$;*

*$X$ is an $L\Sigma(4)$-space iff one of the families $\mathcal{A}$, $\mathcal{B}$ has cardinality greater than $\omega_1$.*

Proof: Since both $\alpha\Psi(\mathcal{A})$ and $\alpha\Psi(\mathcal{B})$ are $L\Sigma(2)$-spaces, their product is an $L\Sigma(\leq 4)$-space. By Theorem 1.1, $X$ is not $L\Sigma(2)$, so it is either $L\Sigma(3)$ or $L\Sigma(4)$.

If one of the families $\mathcal{A}$, $\mathcal{B}$ has cardinality greater or equal to $\omega_2$, then the one-point compactification of the corresponding $\Psi$-space contains a closed copy of $A(\omega_2)$ while the other contains a closed copy of $A(\omega_1)$. Hence, the product $X$ contains a closed subspace homeomorphic to $A(\omega_2) \times A(\omega_1)$, which is not an $L\Sigma(\leq 3)$-space by (the remark after the proof of) Theorem 4.7 in [KOS]. Since the class of $L\Sigma(\leq 3)$-spaces is hereditary with respect to closed subspaces, this proves that $X$ cannot be an $L\Sigma(3)$-space.

On the other hand, if both $\mathcal{A}$ and $\mathcal{B}$ have cardinality $\omega_1$, then each of them is the union of a countable space and the space $A(\omega_1)$. It follows that $X$ is the union of a countable set, countably many copies of $A(\omega_1)$, and a copy of $A(\omega_1) \times A(\omega_1)$. Since each of these spaces is in $L\Sigma(\leq 3)$, the space $X$ is in $L\Sigma(\leq 3)$. □

**Corollary 1.10.** *If $\mathfrak{c} = \omega_1$, then for any uncountable almost disjoint families $\mathcal{A}$, $\mathcal{B}$ on $\omega$, the product $\alpha\Psi(\mathcal{A}) \times \alpha\Psi(\mathcal{B})$ is an $L\Sigma(3)$-space.*

## 2. One-point compactifications

In [KOS], the consistently positive answer to Question 7.4 was obtained by showing that a counterexample would have to be a strong $S$-space and an $L\Sigma(n)$-space for some $n \in \omega$. It appears natural to ask if this kind of spaces can exist. In this section we present a construction that shows, in particular, that the answer is "yes".

**Theorem 2.1.** *Let $X$ be a locally compact space. Suppose that for some $n, m \in \omega$ there exist an $L\Sigma(\leq n)$-space $Y$ and a continuous mapping $j\colon X \to Y$ such that $j(X) = Y$ and $|j^{-1}(y)| \leq m$ for all $y \in Y$. Then the one-point compactification $\alpha X$ of $X$ is an $L\Sigma(\leq nm + 1)$-space.*

Proof: If $X$ is compact, then the mapping $j$ is perfect, so its inverse is upper semicontinuous and at most $m$-valued. If $p\colon M \to Y$ is an upper semicontinuous at most $n$-valued mapping from a second countable space $M$ onto $Y$, then the composition $j^{-1} \circ p$ is upper semicontinuous, onto $X$, and at most $nm$-valued, so $\alpha X = X$ is an $L\Sigma(\leq nm)$-space.

Thus, we may assume that $X$ is not compact. Let $\infty$ be the point such that $\{\infty\} = \alpha X \setminus X$.

Let $p\colon M \to Y$ be an upper semicontinuous mapping from a second-countable space $M$ onto $Y$ such that $|p(z)| \leq n$ for every $z \in M$. Define a multivalued mapping $q\colon M \to X$ by putting

$$q(z) = j^{-1}(p(z)) \cup \{\infty\}.$$

Obviously, the mapping $q$ is onto $\alpha X$ and is at most $(nm + 1)$-valued, so to complete the proof, it remains to verify that $q$ is upper semicontinuous.

Let $z_0$ be a point of $M$ and $U$ an open neighborhood of $q(z_0)$ in $\alpha X$; we need to find a neighborhood $V$ of $z_0$ in $M$ so that $q(V) \subset U$.

Since $\infty \in U$, the set $K = X \setminus U$ is compact. Put $W = Y \setminus j(K)$. The set $W$ is open in $Y$ and contains $p(z_0)$, so by the upper semicontinuity of $p$, there is a neighborhood $V$ of $z_0$ in $M$ such that $p(V) \subset W$. Then $q(V) = \{\infty\} \cup j^{-1}(p(V)) \subset \{\infty\} \cup j^{-1}(W) \subset U$, and the proof is complete. □

**Corollary 2.2.** *If $X$ is a locally compact space, and $X$ admits a continuous bijection onto a second-countable space, then $\alpha X$ is an $L\Sigma(2)$-space.*

The Kunen Line and the Todorčević line [Todor] are locally compact, admit weaker second-countable topologies, and are strong $S$-spaces. Since the Todorčević line is constructed assuming $\mathfrak{b} = \omega_1$, we arrive at the following.

**Corollary 2.3.** *Assume $\mathfrak{b} = \omega_1$. Then there exists a strong $S$-space which is an $L\Sigma(2)$-space.*

Arguments similar to that of the proof of Theorem 2.1 lead to the following versions:

**Theorem 2.4.** *Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(<\omega)$-space $Y$ and a continuous finite-to-one mapping $j\colon X \to Y$ such that $j(X) = Y$. Then the one-point compactification $\alpha X$ of $X$ is an $L\Sigma(<\omega)$-space.*

**Theorem 2.5.** *Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(<\omega)$-space $Y$ and a continuous mapping $j\colon X \to Y$ such that $j(X) = Y$ and $j^{-1}(y)$ is compact and metrizable for every $y \in Y$. Then the one-point compactification $\alpha X$ of $X$ is an $L\Sigma(\leq \omega)$-space.*

Recall that a mapping $j\colon X \to Y$ is called *compact-covering* if for every compact set $K$ in $Y$ there is a compact set $F$ in $X$ such that $j(F) = K$.

**Theorem 2.6.** *Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space $Y$ and a continuous compact-covering bijection $j\colon X \to Y$. Then the one-point compactification $\alpha X$ of $X$ is an $L\Sigma(\leq \omega)$-space.*

In all three latter theorems the mapping $q$ is defined in the same way as in the proof of Theorem 2.1, and the upper semicontinuity of $q$ is verified by the same argument. In Theorem 2.4, $q$ is trivially finite-valued, and in Theorem 2.5, $q$ has compact metrizable images of points because finite unions of metrizable compacta are metrizable compacta. In Theorem 2.6, the compactness and metrizability of images of points under $q$ are verified as follows: there is a compact subset $C$ of $X$ such that $p(z) \subset j(C)$; since $j$ is a continuous bijection, the restriction of $j$ to $C$ is a homeomorphism. Thus, $q(z)$ is the union of the set $j^{-1}(p(z))$, homeomorphic to $p(z)$, and a singleton, hence compact metrizable.

It is not clear if it is possible to omit the requirement that $j$ be compact-covering in Theorem 2.6. Hence,

**Problem 2.7.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space $Y$ and a continuous bijection $j\colon X \to Y$. Must the one-point compactification $\alpha X$ of $X$ be an $L\Sigma(\leq \omega)$-space?

It is also not clear whether Theorem 2.6 remains true if we require that $j$ be finite-to-one instead of being a bijection. The reason of course is that the preimage of a compact metrizable space under a perfect finite-to-one mapping need not be metrizable, so the argument as above does not work. Hence,

**Problem 2.8.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space $Y$ and a continuous finite-to-one compact-covering mapping $j\colon X \to Y$. Must the one-point compactification $\alpha X$ of $X$ be an $L\Sigma(\leq \omega)$-space?

**Problem 2.9.** Let $X$ be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space $Y$ and a continuous finite-to-one mapping $j\colon X \to Y$ such that $j(X) = Y$. Must the one-point compactification $\alpha X$ of $X$ be an $L\Sigma(\leq \omega)$-space?

## 3. The Alexandroff duplicates

One of intriguing questions in the theory of $L\Sigma(\leq \omega)$-spaces is the following (Question 7.5 in [KOS]; also Problem 13(144) in [Oku]): *Let $X$ be an $L\Sigma(\leq \omega)$-space and let $p\colon X \to Y$ be a finite-valued upper semicontinuous mapping such that $p(X) = Y$. Must $Y$ be an $L\Sigma(\leq \omega)$-space?*

Below we prove that the answer is positive for a particular case of the Alexandroff duplicate of an $L\Sigma(\leq \omega)$-space; this gives a positive answer to Problem 15(146) in [Oku].

Recall that the *Alexandroff duplicate* $AD(X)$ of a space $X$ is $X \times 2$ with the topology defined as follows: the points of $X \times \{1\}$ are isolated, and basic neighborhoods of the points $(x, 0)$ are of the form $(U \times 2) \setminus \{(x, 1)\}$ where $U$ is a neighborhood of $x$ in $X$ (see [Eng1] for a discussion of this construction). It is easy to see that the mapping $\pi\colon AD(X) \to X$ defined by the rule $\pi((x, i)) = x$ is two-to-one and perfect, so its inverse is 2-valued upper semicontinuous.

**Theorem 3.1.** *If $X$ is an $L\Sigma(\leq \omega)$-space, then so is $AD(X)$.*

Proof: Fix a second-countable space $M$ and an upper semicontinuous compact-valued mapping $p\colon M \to X$ so that $p(M) = X$ and $w(p(z)) \leq \omega$ for every $z \in M$. Since the cardinalities of $M$ and of $p(z)$, $z \in M$, are at most $\mathfrak{c}$, we have $|X| \leq \mathfrak{c}$, and we may fix a one-to-one function (not necessarily continuous) $j\colon X \to I = [0, 1]$. Define a multivalued mapping $q\colon M \times I \to AD(X)$ by the rule:

$$q(z, t) = \big(p(z) \times \{0\}\big) \cup \big((p(z) \cap j^{-1}(t)) \times \{1\}\big).$$

Since for every $(z, t) \in M \times I$ the set $j^{-1}(t)$ contains at most one point, the images of points under $q$ are compact and metrizable. Let us verify that $q$ is upper semicontinuous.

Let $(z_0, t_0) \in M \times I$, and let $U$ be a neighborhood of $q(z_0, t_0)$; we need to find a neighborhood $V$ of $(z_0, t_0)$ so that $q(V) \subset U$. Since $p(z_0)$ is compact, there is a neighborhood $W$ of $p(z_0)$ in $X$ and a finite set $F \subset X$ such that $F \cap j^{-1}(t_0) = \emptyset$ and $U \supset (W \times 2) \setminus (F \times \{1\})$. Indeed, for every point $x \in p(z_0)$ we can fix a standard open neighborhood $(W_x \times 2) \setminus \{(x, 1)\}$ of $(x, 0)$ contained in $U$; choose a finite subfamily $W_{x_1}, \dots, W_{x_n}$ of the family $\{\, W_x : x \in p(z_0) \,\}$ so that $p(z_0) \subset \bigcup_{i=1}^n W_{x_i}$, and put $W = \bigcup_{i=1}^n W_{x_i}$ and $F = \{x_1, \dots, x_n\} \setminus j^{-1}(t_0)$.
Let $S = j(F)$; then $S$ is finite and $t_0 \notin S$. By the upper semicontinuity of $p$, there is an open neighborhood $G$ of $z_0$ in $M$ such that $p(G) \subset W$. Put $V = G \times (I \setminus S)$. Now if $(z, t) \in V$, then $p(z) \subset W$ and $p(z) \cap j^{-1}(t) \subset W \setminus F$, so $q(z, t) \subset (W \times 2) \setminus (F \times \{1\}) \subset U$, and $V$ is as required.

Let us now verify that $q$ is onto $AD(X)$. If $x \in X$, then there is $z_0 \in M$ such that $x \in p(z_0)$. Put $t_0 = j(x)$. Then both $(x, 0)$ and $(x, 1)$ are in $q(z_0, t_0)$.

Thus, there is an upper semicontinuous compact-valued mapping with metrizable images of points from a second-countable space $M \times I$ onto $AD(X)$, and the proof is complete. □

Theorem 3.1 gives the positive answer to Problem 15(146) in [Oku].

A space $X$ is called a $KL\Sigma(\leq \omega)$-*space* if there is a *compact* second-countable space $M$ and a compact-valued upper semicontinuous mapping $p \colon M \to X$ such that $p(M) = X$ and $w(p(z)) \leq \omega$ for all $z \in M$ [KOS]. It is observed in [KOS] that a compact $L\Sigma(\leq \omega)$-space need not be a $KL\Sigma(\leq \omega)$-space. The same argument as in the proof of Theorem 3.1 can be used to prove the following:

**Theorem 3.2.** *If $X$ is a $KL\Sigma(\leq \omega)$-space, then so is $AD(X)$.*

Of course, the same argument works for the next statement:

**Theorem 3.3.** *Let $\kappa$ be an infinite cardinal. If $|X| \leq \mathfrak{c}$ and $X$ is an $L\Sigma(\leq \kappa)$-space ($KL\Sigma(\leq \kappa)$-space), then so is $AD(X)$.*

The condition "$|X| \leq \mathfrak{c}$" in Theorem 3.3 cannot be omitted unless $2^\kappa \leq \mathfrak{c}$. Indeed, if $2^\kappa > \mathfrak{c}$, let $X = 2^\kappa$ (with the product topology). Trivially, $X \in KL\Sigma(\leq \kappa)$. On the other hand, every $L\Sigma(\leq \kappa)$-space is a union of at most $\mathfrak{c}$ subspaces of weight at most $\kappa$, so its network weight is at most $\kappa \cdot \mathfrak{c}$. The network weight of $AD(2^\kappa)$ is $2^\kappa$, so it cannot be an $L\Sigma(\leq \kappa)$-space.

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