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# Abstract characterization of Orlicz-Kantorovich lattices associated with an $L_0$-valued measure

Botir Zakirov

*Abstract.* An abstract characterization of Orlicz-Kantorovich lattices constructed by a measure with values in the ring of measurable functions is presented.

*Keywords:* Orlicz-Kantorovich lattice, vector-valued measure, Orlicz function

*Classification:* 46B42, 46E30, 46G10

## 1. Introduction

The development of the theory of integration for measures with values in the algebra $L_0$ of all real measurable functions has inspired the study of Banach $L_0$-modules of measurable functions. The theory of $L_p$-spaces associated with a vector-valued measure is given in monographs [7], [10]. Precise description of Orlicz-Kantorovich spaces $L_M(\nabla, m)$ associated with a complete Boolean algebra $\nabla$, an $N$-function $M$ and an $L_0$-valued measure $m$ defined on $\nabla$ is given in [13], [14], [15]. Spaces $L_M(\nabla, m)$ are important examples of Banach-Kantorovich spaces (see, for example, [7], [8], [4] for definition and basic properties).

The abstract characterization of Banach lattices isomorphic to $L_p$-spaces is well known (see, for example, [9]). The same is done for Orlicz spaces in [2]. One can expect similar results for Banach $L_0$-modules $L_p(\nabla, m)$ and $L_M(\nabla, m)$. This problem was considered in [7] for $L_p(\nabla, m)$. Here we solve this problem for $L_M(\nabla, m)$.

We use terminology and notations from the theory of Boolean algebras from [11], the theory of vector latices from [12], [5], the theory of vector integration from [10], [8], the theory of lattice-normed spaces from [7], [8], and also terminology for Orlicz-Kantorovich lattices from [13], [14].

## 2. Preliminaries

Let $E$ be a vector lattice, $E_+$ be the set of all non-negative elements from $E$. Any element $x \in E$ can be uniquely decomposed as $x = x_+ - x_-$, where $x_+, x_- \in E_+$ and $x_+ \wedge x_- = 0$. The element $|x| = x_+ + x_-$ is called the absolute value of $x$, and elements $x_+$ and $x_-$ are called the positive and negative parts of $x$, respectively. Elements $x, y \in E$ are disjoint iff $|x| \wedge |y| = 0$.

Let $u \in E_+$. If no non-zero element is disjoint with $u$, then $u$ is called a weak order unit. Fix some weak order unit (if it exists) $\mathbf{I}$. An element $e \in E_+$ is called a *unitary element* if $e \wedge (\mathbf{I} - e) = 0$. The set $\nabla(E)$ of all unitary elements from $E$ is a Boolean algebra with respect to the order induced from $E$. A complement in $\nabla(E)$ is given as $\mathbf{I} - e$.

A vector lattice is called *complete* (*$\sigma$-complete*) if $\sup A$ and $\inf A$ exist for every (countable) bounded subset $A$.

Let $E$ be a $\sigma$-complete vector lattice with weak unit $\mathbf{I}$. For every $x \in E$, the element $e_x := \sup\{\mathbf{I} \wedge (n|x|) : n \in \mathbb{N}\}$ is unitary. It is called the *support* of $x$. Define $e_t^x := e_{(t\mathbf{I}-x)_+}$. The set $\{e_t^x\}_{t\in\mathbb{R}}$ is called a family of *spectral unitary elements* of $x$. If $x_n \in E$, $x = \inf x_n$, then $e_t^x = \sup_{n\geq 1} e_t^{x_n}$ for all $t \in \mathbb{R}$ (see [12, Lemma IV.10.2]).

Suppose that a $\sigma$-complete vector lattice $E$ is of countable type, i.e. every set of non-zero mutually disjoint elements from $E$ is at most countable. Then $E$ is order complete. Moreover, for every bounded set $A \subset E$, there exists a subset $\{x_n\}_{n=1}^{\infty} \subset A$, such that $\sup A = \sup_{n\geq 1} x_n$.

A Boolean algebra $\nabla$ is called *complete* (*$\sigma$-complete*) if $\sup A$ exists for every (countable) subset $A \subset \nabla$. Let $E$ be a complete ($\sigma$-complete) vector lattice with a weak unit. Then, the Boolean algebra $\nabla(E)$ (see above) is complete ($\sigma$-complete). Evidently, the operation sup is the same in $E$ and $\nabla(E)$. The decomposition of a unit in Boolean algebra is an arbitrary set $(e_\alpha)_{\alpha\in A}$ satisfying $\sup_{\alpha\in A} e_\alpha = \mathbf{I}$, $e_\alpha \neq 0$, $e_\alpha \wedge e_\beta = 0$, $\alpha \neq \beta$, $\alpha, \beta \in A$.

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measurable space. Let $L_0 = L_0(\Omega)$ be the algebra of all real measurable functions on $(\Omega, \Sigma, \mu)$ (functions equal a.e. are identified). $L_0$ is a complete vector lattice with respect to the natural order ($x \geq y$ if $x(\omega) \geq y(\omega)$ for almost all $\omega$). The weak order unit is $\mathbf{1}(\omega) \equiv 1$. The set $\nabla(\Omega)$ of all idempotents in $L_0$ is a complete Boolean algebra.

The support $e_x$ of an element $x \in L_0$ is also denoted by $s(x)$. It is clear that $s(x) = \chi_{\{|x|>0\}}$. Also, $xs(x) = x$. If $xy = 0$ then $s(x)y = 0$. In particular, $|x| \wedge |y| = 0$ if and only if $s(x)s(y) = 0$.

Let $e = \chi_A \in \nabla(\Omega)$. Set $e\Omega = (A, \Sigma_A, \mu)$, where $\Sigma_A = \{B \cap A : B \in \Sigma\}$. The rings $L_0(e\Omega)$ and $eL_0(\Omega)$ can be canonically identified. The Boolean algebras $\nabla(e\Omega)$ and $e\nabla(\Omega) = \{g \in \nabla(\Omega) : g \leq e\}$ can also be identified canonically. Define the map $\mu : \nabla(\Omega) \to [0, \infty]$ as $\mu(e) = \mu(A)$ if $e = \chi_A \in \nabla(\Omega)$. Obviously, $\mu$ is a strongly positive (i.e. $\mu(e) > 0$ for $e \neq 0$) countably additive $\sigma$-finite measure on $\nabla(\Omega)$.

A sequence $\{x_n\} \subset L_0$ converges locally with respect to a measure $\mu$ to the element $x \in L_0$ (notation: $x_n \xrightarrow{l.\mu} x$) if for any $A \in \Sigma$ with $\mu(A) < \infty$ the sequence $x_n\chi_A$ converges with respect to the measure to $x\chi_A$. If $\mu(\Omega) < \infty$, then local convergence with respect to the measure coincides with convergence with respect to the measure. There exists a countable set of non-zero disjoint idempotents

$\{e_n\} \subset \nabla(\Omega)$ such that $\sup_{n\geq 1} e_n = \mathbf{1}$ and $\mu(e_n) < \infty$. The algebra $L_0(\Omega)$ is canonically identified with the direct product $\prod_{n=1}^{\infty} L_0(e_n\Omega)$. Local convergence with respect to the measure is now identified with convergence of each coordinate with respect to the measure. $L_0(\Omega)$ with this topology is a complete metrizable topological vector lattice.

Now we define a Banach-Kantorovich space for an $L_0$-valued norm.

Let $E$ be a vector space over the field $\mathbb{R}$. A mapping $\|\cdot\| : E \to L_0$ is said to be *a vector* ($L_0$*-valued*) *norm* if it satisfies the following axioms:

1. $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$ $(x \in E)$;
2. $\|\lambda x\| = |\lambda|\|x\|$ $(\lambda \in \mathbb{R},\ x \in E)$;
3. $\|x + y\| \leq \|x\| + \|y\|$ $(x, y \in E)$.

A norm $\|\cdot\|$ is called *decomposable* or *Kantorovich* if the following property holds:

**Property 1.** If $e_1, e_2 \geq 0$ and $\|x\| = e_1 + e_2$, then there exist $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_k\| = e_k$ $(k = 1, 2)$.

If property 1 is valid only for disjoint elements $e_1, e_2 \in L_0$, the norm is called *disjointly decomposable* or, briefly, *d-decomposable*.

A pair $(E, \|\cdot\|)$ is called *a lattice-normed space* (shortly, LNS). If the norm $\|\cdot\|$ is decomposable ($d$-decomposable), then so is the space $(E, \|\cdot\|)$.

A sequence $\{x_n\} \subset E$ $(bo)$*-converges* to $x \in E$ if the sequence $\{\|x_n - x\|\}$ $(o)$-converges to 0 in $L_0$. A sequence $\{x_n\}$ is said to be a $(bo)$*-Cauchy sequence* if $\sup_{n,k\geq m} \|x_n - x_k\| \xrightarrow{(o)} 0$ as $m \to \infty$. An LNS is called $(bo)$*-complete* if any $(bo)$-Cauchy sequence $(bo)$-converges. *A Banach-Kantorovich space* (shortly, BKS) is a $d$-decomposable $(bo)$-complete LNS. It is well known that every BKS is a decomposable LNS.

Suppose that $(E, \|\cdot\|)$ is an LNS and a vector lattice simultaneously. The norm $\|\cdot\|$ is called *monotone* if $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$. BKS with a monotone norm is called *a Banach-Kantorovich lattice*.

Let $E$ be an $L_0$-module. It is called *a normal* $L_0$*-module* if

1. for any non-zero $e \in \nabla(\Omega)$, there exists $x \in E$ such that $ex \neq 0$;
2. for any decomposition of unit $\{e_n\}_{n=1}^{\infty} \subset \nabla(\Omega)$ and any $\{x_n\}_{n=1}^{\infty} \subset E$, there exists $x \in E$ such that $e_n x = e_n x_n$ for all $n$;
3. if $x \in E$ and $\{e_n\} \in \nabla(\Omega)$ is a disjoint sequence, then $e_n x = 0$ for all $n$ implies that $(\sup_{n\geq 1} e_n)x = 0$.

An ordered normal $L_0$-module $E$ is called *an* $L_0$*-vector lattice* if for any $x, y, z \in E$, $\lambda \in L_0$, $\lambda \geq 0$, the inequality $x \leq y$ implies $x + z \leq y + z$ and $\lambda x \leq \lambda y$. The simplest example of an $L_0$-vector lattice is $L_0$ itself considered as a module over $L_0$.

**Lemma 2.1.** *Let $E$ be an $L_0$-vector lattice, $x, y \in E$, $x \geq 0$, $y \geq 0$, $e, g \in \nabla(\Omega)$, $eg = 0$. Then the elements $ex$ and $gy$ are disjoint.*

Proof: Let $z = ex \wedge gy$. Since $ex \geq 0$, $gy \geq 0$, we have $z \geq 0$ and it follows that $0 \leq ez \leq egy = 0$, i.e. $ez = 0$. Further, $0 \leq (\mathbf{1} - e)z \leq (\mathbf{1} - e)ex = 0$, and therefore $(\mathbf{1} - e)z = 0$, i.e. $z = ez = 0$. $\square$

**Remark 2.2.** If $x, z \in E$, $e \in \nabla(\Omega)$ and $0 \leq z \leq ex$, then $z = ez$.

**Lemma 2.3.** *Let $E$ be an $L_0$-vector lattice with a weak order unit $\mathbf{I}$. Then*

(i) $\lambda\mathbf{I} \neq 0$ *for any non-zero* $\lambda \in L_0$*;*
(ii) $(\lambda\mathbf{I}) \vee 0 = \lambda_+\mathbf{I}$ *for any* $\lambda \in L_0$.

Proof: (1) Let $\lambda \in L_0$, $\lambda \geq 0, \lambda \neq 0$. Then $\lambda \geq \varepsilon e$ for some $e \in \nabla(\Omega)$, $e \neq 0$, $\varepsilon > 0$. Hence, $\lambda\mathbf{I} \geq \varepsilon e\mathbf{I}$. Let us show that $e\mathbf{I} \neq 0$. Select $x \in E$ such that $ex \neq 0$. Let $x = x_+ - x_-$. Either $ex_+ \neq 0$ or $ex_- \neq 0$. Let $ex_+ \neq 0$. Set $z = (ex_+) \wedge \mathbf{I} \neq 0$. If $e\mathbf{I} = 0$, then by Remark 2.2, $0 \leq ez = z \leq e\mathbf{I} = 0$, i.e. $z = 0$. Therefore, $e\mathbf{I} \neq 0$ and $\lambda\mathbf{I} \neq 0$. Let now $\lambda$ be an arbitrary element from $L_0$, and $\lambda = \lambda_+ - \lambda_-$, moreover $\lambda_- \neq 0$. Suppose $\lambda_+\mathbf{I} - \lambda_-\mathbf{I} = 0$. Then $\lambda_-\mathbf{I} = s(\lambda_-)\lambda_-\mathbf{I} = s(\lambda_-)\lambda_+\mathbf{I} = 0$, which is not the case.

(2) It is clear that $\lambda_+\mathbf{I} \geq 0$ and $\lambda_+\mathbf{I} - \lambda\mathbf{I} = \lambda_-\mathbf{I} \geq 0$, i.e. $\lambda_+\mathbf{I} \geq \lambda\mathbf{I} \vee 0$. On the other hand, if $a = \lambda\mathbf{I} \vee 0$, then

$$a \geq s(\lambda_+)a \geq s(\lambda_+)\lambda\mathbf{I} = \lambda_+\mathbf{I}.$$

Hence, $\lambda_+\mathbf{I} = (\lambda\mathbf{I}) \vee 0$. $\square$

Submodules and morphisms are defined in a usual way.

**Proposition 2.4.** *Let $E$ be an $L_0$-vector lattice and $\mathbf{I}$ be a weak order unit in $E$. Then $N = \{\lambda\mathbf{I} : \lambda \in L_0\}$ is a normal $L_0$-submodule in $E$ and a vector sublattice in $E$, canonically isomorphic to $L_0$. Moreover, $N(\Omega) = \{e\mathbf{I} : e \in \nabla(\Omega)\}$ is a $\sigma$-Boolean subalgebra in $\nabla(E)$.*

Proof: Only the second assertion needs to be proved. It follows from Lemma 2.1 that $N(\Omega)$ is a Boolean subalgebra of $\nabla$.

Let $\{e_n\} \subset \nabla(\Omega)$ and $e = \sup e_n$. If $g \in \nabla$ and $g \geq e_n\mathbf{I}$, then $\mathbf{I} - g \leq (\mathbf{1} - e_n)\mathbf{I}$, and therefore $e_n(\mathbf{I} - g) \leq e_n(\mathbf{1} - e_n)\mathbf{I} = 0$. Hence, $e_n(\mathbf{I} - g) = 0$. Then $e(\mathbf{I} - g) = 0$ because $E$ is normal. Hence, $e\mathbf{I} = \sup_{n\geq 1} e_n\mathbf{I}$. This means that $N(\Omega)$ is a $\sigma$-subalgebra in $\nabla(E)$. $\square$

**Proposition 2.5.** *Let $E$ be a $\sigma$-complete $L_0$-vector lattice, $\mathbf{I}$ a weak order unit in $E$ and let $\{\alpha_n\} \subset L_0$ be bounded from above (below). Then $\sup_{n\geq 1}(\alpha_n\mathbf{I}) = (\sup_{n\geq 1} \alpha_n)\mathbf{I}$ ($\inf_{n\geq 1}(\alpha_n\mathbf{I}) = (\inf_{n\geq 1} \alpha_n)\mathbf{I}$, respectively).*

Proof: First, let us show that the equality

$$e_{\alpha\mathbf{I}} := \sup_{n\geq 1}(\mathbf{I} \wedge n|\alpha|\mathbf{I}) = s(\alpha)\mathbf{I}$$

holds for any $\alpha \in L_0$. One can assume that $\alpha \geq 0$. Let $g_n = \{\alpha \geq \frac{1}{n}\}$ be a spectral idempotent for $\alpha$ in $L_0$. It is obvious that $g_n \uparrow s(\alpha)$ and by Proposition 2.4, $g_n\mathbf{I} \uparrow s(\alpha)\mathbf{I}$.

Let $f_n = s(\alpha) - g_n$ and $\beta_n = n\alpha f_n$, $n = 1, 2, \ldots$ . It is clear that $0 \leq \beta_n \leq f_n \leq \mathbf{1}$ and $\beta_n g_i = 0$ for all $i = 1, 2, \ldots, n$. Hence, $0 \leq \beta_n\mathbf{I} \leq f_n\mathbf{I} \leq f_i\mathbf{I} \leq \mathbf{I}$ as $n \geq i$. Let $a_n = \sup_{k\geq n} \beta_k\mathbf{I}$ and $a = \inf_{n\geq 1} a_n$. Since $a_n \leq f_n\mathbf{I}$, we have $a \leq f_n\mathbf{I}$ for all $n = 1, 2, \ldots$ . We thus have $0 \leq g_n a \leq g_n f_n\mathbf{I} = 0$, i.e. $g_n a = 0$, $n = 1, 2, \ldots$ . Hence, $s(\alpha)a = (\sup_{n\geq 1} g_n)a = 0$. On the other hand, $a \leq f_n\mathbf{I} \leq s(\alpha)\mathbf{I}$. By Remark 2.2 we obtain $a = s(\alpha)a$, and so $a = 0$. Thus, $\beta_n\mathbf{I} \xrightarrow{(o)} 0$. Since $\mathbf{1} \wedge n\alpha = g_n + \beta_n$, it follows that $\mathbf{I} \wedge (n\alpha)\mathbf{I} = (\mathbf{1} \wedge n\alpha)\mathbf{I} = g_n\mathbf{I} + \beta_n\mathbf{I}$. Hence, $e_{\alpha\mathbf{I}} = (o)\text{-}\lim(\mathbf{I} \wedge (n\alpha)\mathbf{I}) = (o)\text{-}\lim g_n\mathbf{I} + (o)\text{-}\lim \beta_n\mathbf{I} = s(\alpha)\mathbf{I}$. Now let us show that $\inf_{n\geq 1}(\alpha_n\mathbf{I}) = (\inf_{n\geq 1} \alpha_n)\mathbf{I}$ for any bounded from below sequence $(\alpha_n)$ in $L_0$. Let $\alpha = \inf_{n\geq 1} \alpha_n$, $x = \inf_{n\geq 1} \alpha_n\mathbf{I}$.

Consider in $E$ the families $\{e_t^x\}_{t\in\mathbb{R}}$ and $\{e_t^{\alpha_n\mathbf{I}}\}_{t\in\mathbb{R}}$ of spectral unitary elements for $x$ and $\alpha_n\mathbf{I}$, respectively. By Lemma 2.3(ii) we have

$$e_t^{\alpha_n\mathbf{I}} = e_{(t\mathbf{I}-\alpha_n\mathbf{I})_+} = e_{((t\mathbf{1}-\alpha_n)\mathbf{I})_+} = e_{(t\mathbf{1}-\alpha_n)_+\mathbf{I}} = s((t\mathbf{1} - \alpha_n)_+)\mathbf{I}.$$

This together with Proposition 2.4 and [11, Lemma IV.10.2] imply that

$$\begin{aligned} e_t^x &= \sup_{n\geq 1} e_t^{\alpha_n\mathbf{I}} = \sup_{n\geq 1} \left(s((t\mathbf{1} - \alpha_n)_+)\mathbf{I}\right) \\ &= \Big(\sup_{n\geq 1} s((t\mathbf{1} - \alpha_n)_+)\Big)\mathbf{I} = s((t\mathbf{1} - \alpha)_+)\mathbf{I} = g_t^\alpha, \end{aligned}$$

where $\{g_t^\alpha\}_{t\in\mathbb{R}}$ is the family of spectral idempotents for $\alpha$ in $L_0$. Similarly, for the family of spectral idempotents $\{e_t^{\alpha\mathbf{I}}\}_{t\in\mathbb{R}}$ we have

$$e_t^{\alpha\mathbf{I}} = e_{(t\mathbf{I}-\alpha\mathbf{I})_+} = s((t\mathbf{1} - \alpha)_+)\mathbf{I} = g_t^\alpha\mathbf{I}.$$

Hence, $e_t^x = e_t^{\alpha\mathbf{I}}$ for all $t \in \mathbb{R}$.

It follows from the spectral theorem for $\sigma$-complete vector lattices [11, Theorem IV.10.1] that $x = \alpha\mathbf{I}$, i.e. $\inf_{n\geq 1}(\alpha_n\mathbf{I}) = (\inf_{n\geq 1} \alpha_n)\mathbf{I}$. If $\{\alpha_n\}$ is a bounded from above sequence from $L_0$, then passing to the sequence $\{-\alpha_n\}$, we obtain $\sup_{n\geq 1}(\alpha_n\mathbf{I}) = (\sup_{n\geq 1} \alpha_n)\mathbf{I}$. □

**Remark 2.6.** Let $E$ be a $\sigma$-complete $L_0$-vector lattice with a weak order unit. Then $L_0$ can be identified with the normal $L_0$-submodule $N$ in $E$. In addition, operations sup and inf are identical in $L_0$ and $N$. The Boolean algebra $\nabla(\Omega)$ is a $\sigma$-subalgebra in $\nabla(E)$.

## 3. Banach $\mathbf{L_0}$-vector lattices

Let $E$ be a normal $L_0$-module. An $L_0$-valued norm $\|\cdot\| : E \to L_0$ is said to be *compatible with the structure of the $L_0$-module $E$* (shortly, *$L_0$-norm*) if $\|\lambda x\| = |\lambda|\|x\|$ for any $x \in E$ and $\lambda \in L_0$. Then, the pair $(E, \|\cdot\|)$ is called *a normed $L_0$-module*.

Let $E$ be a normed $L_0$-module. Let $t$ be the topology of local convergence with respect to the measure in $L_0$. A sequence $\{x_n\} \subset E$ *$t$-converges* to $x \in E$ if $\|x_n - x\| \xrightarrow{t} 0$. Cauchy sequences are defined as usual. A normed $L_0$-module $E$ is called *Banach* (*$t$-Banach*) if any $(bo)$-Cauchy ($t$-Cauchy, respectively) sequence in $E$ $(bo)$-converges ($t$-converges, respectively). $E$ is a Banach $L_0$-module if and only if it is a $t$-Banach $L_0$-module.

Let $E$ be a BKS over $L_0$. It is possible to define a structure of $L_0$-module on $E$. This structure makes $E$ a Banach $L_0$-module. Vice versa, any Banach $L_0$-module $E$ is a BKS over $L_0$.

If $E$ is a normed $L_0$-module and simultaneously an $L_0$-vector lattice with a monotone norm, then $E$ is called *a normed $L_0$-vector lattice*. Any norm complete $L_0$-vector lattice is called *a Banach $L_0$-vector lattice*. The class of Banach $L_0$-vector lattices coincides with the class of Banach-Kantorovich lattices over $L_0$.

Let us give examples of Banach $L_0$-vector lattices.

Suppose $\nabla$ is a complete Boolean algebra. Denote by $X(\nabla)$ the Stone compactification of $\nabla$. Let $L_0(\nabla)$ be the set of all continuous functions $x : X(\nabla) \to [-\infty, +\infty]$ such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subset of $X(\nabla)$ (see [10, V, §2]). Evidently, $L_0(\nabla)$ is a ring and an order complete vector lattice. The function $\mathbf{1}$, equal to 1 identically on $X(\nabla)$, is a weak order unit in $L_0(\nabla)$. The order ideal generated by the element $\mathbf{1}$ coincides with the space $C(X(\nabla))$ of all continuous real functions on $X(\nabla)$.

A mapping $m : \nabla \to L_0$ is called *an $L_0$-valued measure* on $\nabla$ if

1. $m(e) \geq 0$ for any $e \in \nabla$,
2. $m(e \vee g) = m(e) + m(g)$ if $e, g \in \nabla$ and $e \wedge g = 0$,
3. if $e_n \downarrow 0$, $e_n \in \nabla$, then $m(e_n) \downarrow 0$.

A measure $m$ is called *strongly positive* if $m(e) = 0$, $e \in \nabla$ implies $e = 0$. Using Lebesgue construction, one can obtain an integral $I_m : x \to \int x\,dm$ for every strongly positive $L_0$-valued measure $m$ (see [10], [8]). There exists the greatest order ideal $L := L_1(\nabla, m)$ in $L_0(\nabla)$ containing $\nabla$ with the following properties:

1. $I_m e = m(e)$ for any $e \in \nabla$,
2. $I_m(ax + by) = aI_m x + bI_m y$, $x, y \in L$, $a, b \in \mathbb{R}$,
3. if $x_n, x \in L$ and $x_n \uparrow x$ then $I_m x_n \xrightarrow{(o)} I_m x$.

The mapping $I_m$ satisfying the above properties is uniquely defined. The norm on $L_1(\nabla, m)$ is defined as $\|x\|_1 = \int |x|\,dm$. Now, $(L_1(\nabla, m), \|\cdot\|_1)$ is a $(bo)$-complete

LNS over $L_0$ (see [10]).

We suppose that $\nabla(\Omega)$ is a regular Boolean subalgebra in $\nabla$, i.e. $\sup A \in \nabla(\Omega)$ for every $A \subset \nabla(\Omega)$. We can always obtain this by considering the complete tensor product $\nabla \mathbin{\overline{\otimes}} \nabla(\Omega)$ of the Boolean algebras $\nabla$ and $\nabla(\Omega)$ (see [2, VII, §7.2]). One can canonically identify $L_0(\Omega)$ with a subalgebra in $L_0(\nabla)$. It is also a regular vector sublattice in $L_0(\nabla)$. Moreover, sup and inf operations in $L_0(\Omega)$ and $L_0(\nabla)$ coincide. Hence, $L_0(\nabla)$ becomes an $L_0$-vector lattice (multiplication of elements from $L_0(\nabla)$ by elements from $L_0$ coincides with the natural multiplication in $L_0(\nabla)$).

From now on, we require the measure $m : \nabla \to L_0$ to be compatible with the module structure, i.e. $m(ge) = gm(e)$ for all $e \in \nabla$, $g \in \nabla$. In this case, $L_1(\nabla, m)$ becomes a BKS over $L_0$. In addition, the following property holds:

Let $x \in L_1(\nabla, m)$ and $\alpha \in L_0$. Then, $\alpha x \in L_1(\nabla, m)$ and $\int \alpha x \, dm = \alpha \int x \, dm$. In particular, $L_0 \subset L_1(\nabla, m)$ and $\int \alpha \, dm = \alpha m(\mathbf{1})$ for all $\alpha \in L_0$ (see [6, 6.1.10]).

Let $p > 1$. Set

$$L_p(\nabla, m) := \{x \in L_0(\nabla) : |x|^p \in L_1(\nabla, m)\}.$$

Then $L_p(\nabla, m)$ is a normal $L_0$-module and a Banach $L_0$-vector lattice with respect to the norm $\|x\|_p := (\int |x|^p \, dm)^{1/p}$ (see [1, 4.2.2], or [2, VIII, §8.2]).

Now we give examples of $L_0$-valued measures compatible with the module structure.

**Example 1.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite complete measure space. Let $\mathcal{A} \subset \Sigma$ be a $\sigma$-subalgebra. Denote by $m(e) = E(e|\mathcal{A})$ the conditional expectation. It is clear that $m$ is a strongly positive $L_0(\Omega, \mathcal{A}, \mu)$-valued measure on $\nabla(\Omega, \Sigma, \mu)$ compatible with the module structure.

**Example 2.** Let $(\Omega, \Sigma, \mu)$ be the same space as in Example 1, $X$ be another complete Boolean algebra with a strongly positive scalar measure $\nu$. Step mappings $u : (\Omega, \Sigma, \mu) \to X$ are defined in the usual way. Let $\Gamma(X)$ be the set of all step mappings $u : (\Omega, \Sigma, \mu) \to X$. A mapping $u : (\Omega, \Sigma, \mu) \to X$ is said to be measurable if there exists a sequence $\{u_n\} \subset \Gamma(X)$ such that $\nu(u(\omega) \triangle u_n(\omega)) \to 0$ as $n \to \infty$ for a.e. $\omega \in \Omega$, Here, $e \triangle g = (e \wedge Cg) \vee (Ce \wedge g)$, $e, g \in X$. Let $\mathcal{L}_0(\Omega, X)$ be the set of all measurable maps from $(\Omega, \Sigma, \mu)$ into $X$. For arbitrary $u, v \in \mathcal{L}_0(\Omega, X)$ we set $u \le v$ if $u(\omega) \le v(\omega)$ for all $\omega \in \Omega$. Then, $\mathcal{L}_0(\Omega, X)$ becomes a Boolean algebra. Its unit is $\mathbf{1}(\omega) \equiv \mathbf{1}_X$. Its zero is $\mathbf{0}(\omega) = \mathbf{0}_X$. The complement is defined as $(Cu)(\omega) = C(u(\omega))$. Moreover $(u \vee v)(\omega) = u(\omega) \vee v(\omega)$, $(u \wedge v)(\omega) = u(\omega) \wedge v(\omega)$, $\omega \in \Omega$.

Consider the ideal $J = \{u \in \mathcal{L}_0(\Omega, X) : u(\omega) = 0 \text{ a.e.}\}$. Define $L_0(\Omega, X)$ as a Boolean factor-algebra $\mathcal{L}_0(\Omega, X)/J$. $L_0(\Omega, X)$ is a complete Boolean algebra (see [1]). $\nabla(\Omega) = \{u \in L_0(\Omega, X) : u = \chi_A,\ A \in \Sigma\}$ is a regular Boolean subalgebra in $L_0(\Omega, X)$. If $u \in \Gamma(X)$, then the scalar function $\nu \circ u \in L_0(\Omega)$. Hence, for any $v \in L_0(\Omega, X)$, the function $\nu(v(\omega)) = \lim_{n\to\infty} \nu(v_n(\omega)) \in L_0(\Omega)$. Here $v_n \in \Gamma(X)$,

$\nu(v(\omega)\triangle v_n(\omega)) \to 0$. So, we defined a mapping $\nu : L_0(\Omega, X) \to L_0(\Omega)$. It is an $L_0$-valued strongly positive measure on $L_0(\Omega, X)$ compatible with the module structure (see [1]).

Let $(E, \|\cdot\|$ be a normed $L_0$-vector lattice. A norm in $E$ is called *order continuous* if for any $\{x_n\} \subset E_+$, $x_n \downarrow 0$ implies $\|x_n\| \xrightarrow{t} 0$.

The following order and topological properties of normed $L_0$-vector lattices can be proved in the same way as in the case of normed lattices.

**Theorem 3.1.** *Let $(E, \|\cdot\|)$ be a normed $L_0$-vector lattice. Then*

1. *if $\{x_n\} \subset E$ is an increasing $t$-converging sequence, then*

$$\lim_{n\to\infty} x_n = \sup_n x_n.$$

2. (Amemiya theorem). *The following conditions are equivalent:*
   (a) *$E$ is a Banach $L_0$-vector lattice;*
   (b) *if $\{x_n\}$ is a $(bo)$-Cauchy increasing sequence from $E_+$, then $\{x_n\}$ $(bo)$-converges in $E$;*
   (c) *if $\{x_n\}$ is a $(bo)$-Cauchy increasing sequence from $E_+$, then there exists $x = (\sup_{n\geq 1} x_n) \in E$.*
3. *Let $(E, \|\cdot\|)$ be a $\sigma$-complete normed $L_0$-vector lattice with an order continuous norm. Then $E$ is of countable type. Therefore $E$ is an order complete vector lattice.*
4. *Let $E$ be a Banach $L_0$-vector space. The following conditions are equivalent:*
   (a) *$E$ is an order complete lattice and $\|\cdot\|$ is order continuous.*
   (b) *Any bounded sequence of positive mutually disjoint elements $t$-converges to zero.*

## 4. Orlicz-Kantorovich lattices associated with Orlicz $L_0$-modulators

Let us start with some definitions.

**Definition.** $\psi : [0,\infty) \to \mathbb{R}$ is called an *Orlicz function* if it is a convex non-negative function such that $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$. An additional requirement is the so called $(\delta_2, \Delta_2)$-condition, i.e. $\psi(2t) \leq c\psi(t)$ for all $t \geq 0$ and a constant $c > 0$.

Let $x \in L_0(\nabla)$. By definition, $G = \{t \in X(\nabla) : |x(t)| < \infty\} \subset X(\nabla)$ is an open and dense subset. Hence, we can define $y \in L_0(\nabla)$ as $y = \psi \circ |x| := \psi(|x|)$. Define

$$L_\psi := L_\psi(\nabla, m) := \{x \in L_0(\nabla) : \psi(|x|) \in L_1(\nabla, m)\}.$$

It is clear that $L_\psi$ is a normal $L_0$-submodule and a vector sublattice in $L_0(\nabla)$.

Let $\mathcal{P}(L_0) = \{\lambda \geq 0 \in L_0 : s(\lambda) = \mathbf{1}\}$. Obviously, for any $\lambda \in \mathcal{P}(L_0)$ there exists $\lambda^{-1} \in \mathcal{P}(L_0)$.

**Lemma 4.1.** *Let $x \in L_\psi$. There exists $\lambda \in \mathcal{P}(L_0)$ such that*

$$\int \psi(\lambda^{-1}|x|)\, dm \leq \mathbf{1}.$$

PROOF: Let $\lambda_0 = \int \psi(|x|)\, dm + \mathbf{1}$. It is clear that $\lambda_0 \in \mathcal{P}(L_0)$ and $0 \leq \lambda_0^{-1} \leq \mathbf{1}$. Since $\psi(st) \leq s\psi(t)$ for all $s \in [0,1]$, we are done. □

Hence, we can define an $L_0$-valued function

$$\|x\|_{(\psi)} = \inf \left\{ \lambda \in \mathcal{P}(L_0) : \int \psi(\lambda^{-1}|x|)\, dm \leq \mathbf{1} \right\}.$$

**Theorem 4.2.** *$(L_\psi, \|\cdot\|_{(\psi)})$ is a Banach $L_0$-vector lattice.*

We need some lemmas to prove Theorem 4.2.

**Lemma 4.3.** *Let $x_n, x \in L_0(\nabla)$, $0 \leq x_n \uparrow x$. Then $\psi(x_n) \uparrow \psi(x)$.*

The proof of this lemma is similar to that of Lemma 2.4 from [14].

**Lemma 4.4.** *$\|x\|_{(\psi)}$ is a monotone $L_0$-norm on $L_\psi$, i.e. $(L_\psi, \|\cdot\|_{(\psi)})$ is a normed $L_0$-vector lattice.*

PROOF: Obviously, $\|\cdot\|$ is monotone, convex and positive. Assume now that $\|x\| = 0$ for some $x \in Ł_\psi$. Consider $\lambda \in \mathcal{P}(L_0)$ such that $\int \psi(\lambda^{-1}|x|)\, dm \leq \mathbf{1}$. Then, $\lambda \wedge \mathbf{1} \in \mathcal{P}(L_0)$ and $0 \leq \lambda \wedge \mathbf{1} \leq \mathbf{1}$. Obviously, $(\lambda \wedge \mathbf{1})^{-1}|x| = \lambda^{-1}|x|\{\lambda < \mathbf{1}\} + |x|\{\lambda \geq \mathbf{1}\}$. Hence, $\psi((\lambda \wedge \mathbf{1})^{-1}|x|) = \psi(\lambda^{-1}|x|)\{\lambda < \mathbf{1}\} + \psi(|x|)\{\lambda \geq \mathbf{1}\}$. Therefore,

$$\begin{aligned}\int \psi((\lambda \wedge \mathbf{1})^{-1}|x|)\, dm &= \{\lambda < \mathbf{1}\} \int \psi(\lambda^{-1}|x|)\, dm + \{\lambda \geq \mathbf{1}\} \int \psi(|x|)\, dm \\ &\leq \{\lambda < \mathbf{1}\} + \int \psi(|x|)\, dm \leq \mathbf{1} + \int \psi(|x|)\, dm.\end{aligned}$$

However, $\psi((\lambda \wedge \mathbf{1})^{-1}|x|) \geq (\lambda \wedge \mathbf{1})^{-1}\psi(x)$. Therefore,

$$\int \psi(|x|)\, dm \leq (\lambda \wedge \mathbf{1}) \left( \mathbf{1} + \int \psi(|x|)\, dm \right).$$

Now, one can take infimum over all such $\lambda$ and obtain $\int \psi(|x|)\, dm = 0$. Hence, $x = 0$. □

**Lemma 4.5.** *Let $x \in L_\psi$, $e = \mathbf{1} - s(\|x\|_{(\psi)})$. Then*

$$\int \psi\left(\left(\|x\|_{(\psi)} + e\right)^{-1} |x|\right) dm \leq \mathbf{1}.$$

Proof: Obviously, $s(x) = s(\|x\|_{(\psi)})$. Hence $(\mathbf{1}-e)|x| = |x|$. Since $L_0$ has countable type, then there exists a sequence $\{\lambda_n\} \subset \mathcal{P}(L_0)$, such that $\int \psi(\lambda_n^{-1}|x|) \leq \mathbf{1}$ and $\lambda_n \downarrow= \|x\|_{(\psi)}$. Set $\alpha_n = \lambda_n(\mathbf{1}-e)+e$, $n = 1, 2, \dots$ . Then $\alpha_n \downarrow (\|x\|_{(\psi)} + e)$ and $\alpha_n^{-1} = (\lambda_n^{-1}(\mathbf{1}-e)+e) \uparrow (\|x\|_{(\psi)} + e)^{-1}$. Hence, $\psi(\alpha_n^{-1}|x|) \uparrow \psi((\|x\|_{(\psi)} + e)^{-1}|x|)$ (see Lemma 4.3). By the monotone convergence theorem (see [6, 6.1.5]), we have

$$\begin{aligned}\int \psi\left(\left(\|x\|_{(\psi)} + e\right)^{-1} |x|\right) dm &= \sup_{n\geq 1} \int \psi\left(\alpha_n^{-1}|x|\right) dm \\ &= \sup_{n\geq 1} \int \psi\left(\left(\lambda_n^{-1}(\mathbf{1}-e)+e\right)|x|\right) dm \\ &= \sup_{n\geq 1} \int \psi(\lambda_n^{-1}|x|)\, dm \leq \mathbf{1}.\end{aligned}$$

□

Proof of Theorem 4.2: Consider a $(bo)$-Cauchy increasing sequence $\{x_n\} \in (L_\psi)_+$. Obviously, the sequence $\|x_n\|_{(\psi)}$ is a $(o)$-Cauchy sequence in $L_0$. That is, $\|x_n\|_{(\psi)} \uparrow \alpha$. Set $e_n = \mathbf{1} - s(x_n)$ and $\alpha_n = \|x_n\|_{(\psi)} + e_n$. Then $0 \leq \alpha_n \leq \alpha + \mathbf{1}$. By Lemma 4.5, $\int \psi(\alpha_n^{-1}x_n) \leq \mathbf{1}$. Therefore, $\int \psi((\alpha+\mathbf{1})^{-1}x_n) \leq \mathbf{1}$. The sequence $\psi((\alpha+\mathbf{1})^{-1}x_n) \in L_1$ is monotone and $L_1$-bounded. Hence, $\psi((\alpha+1)^{-1}x_n) \uparrow y \in L_1$. Therefore, $x_n \uparrow (\alpha+\mathbf{1})\psi^{-1}(y) \in L_\psi$. □

A Banach $L_0$-vector lattice $(L_\psi, \|\cdot\|_{(\psi)})$ is called *the Orlicz-Kantorovich space*. See examples after Theorem 5.1.

Denote $\Phi(x) = \int \psi(|x|)\, dm$. It is easy to see that the mapping $\Phi : L_\psi \to L_0$ satisfies the following properties:

1. $\Phi(x) \geq 0$ and $\Phi(x) = 0 \Leftrightarrow x = 0$;
2. $\Phi(x) \leq \Phi(y)$ if $|x| \leq |y|$;
3. $\Phi(\alpha x + (\mathbf{1}-\alpha)y) \leq \alpha\Phi(x) + (\mathbf{1}-\alpha)\Phi(y)$, $\alpha \in L_0$, $0 \leq \alpha \leq \mathbf{1}$;
4. $\Phi(2x) \leq c\Phi(x)$ for some constant $c > 0$;
5. $\Phi(x+y) = \Phi(x) + \Phi(y)$ if $x \wedge y = 0$;
6. $\Phi(ex) = e\Phi(x)$ for all $e \in \nabla(\Omega)$;
7. $\Phi(t\mathbf{1}) = \varphi(t)\Phi(\mathbf{1})$ for all $t \geq 0$, where $\varphi : [0, \infty) \to [0, \infty)$ is a scalar function.

Now we define an Orlicz $L_0$-lattice. Let $E$ be an $L_0$-vector lattice with a weak order unit $\mathbf{I}$. A map $\Phi : E \to L_0$ is called *an Orlicz $L_0$-modulator* if $\Phi$ satisfies properties 1–7. Obviously, $\Phi(x) = \Phi(|x|)$ and $\Phi(\alpha x) \leq \alpha\Phi(x)$ for $\alpha \in L_0$, $0 \leq \alpha \leq \mathbf{1}$. The element $\Phi(\mathbf{I})$ is invertible in $L_0$. Indeed, let $e = s(\Phi(\mathbf{I}))$. Then $\Phi((\mathbf{1} - e)\mathbf{I}) = (\mathbf{1} - e)\Phi(\mathbf{I}) = 0$. Hence, $(\mathbf{1} - e)\mathbf{I} = 0$ and $e = \mathbf{1}$. Properties 1–7 imply that $\varphi$ is an Orlicz function satisfying the $(\delta_2, \Delta_2)$-condition.

Set $B(x) = \{\lambda \in \mathcal{P}(L_0) : \Phi(\lambda^{-1}x) \leq \mathbf{1}\}$. If $\lambda = \Phi(x) + \mathbf{1}$, then $\Phi(\lambda^{-1}x) \leq \lambda^{-1}\Phi(x) \leq \mathbf{1}$. Hence $B(x)$ is a non-empty set. For any $x \in E$, set $\|x\|_\Phi = \inf\{\lambda : \lambda \in B(x)\}$.

**Proposition 4.6.** *$(E, \|\cdot\|_\Phi)$ is a normed $L_0$-vector lattice.*

Proof: Obviously, $\|\cdot\|_\Phi$ is monotone, convex and positive. If $\|x\|_\Phi = 0$, then repeating the proof of Lemma 4.4 and using properties of the Orlicz $L_0$-modulator $\Phi$, we obtain $x = 0$. Let $x, y \in E$, $\lambda_1 \in B(x)$, $\lambda_2 \in B(y)$. Then

$$\begin{aligned}\Phi((\lambda_1 + \lambda_2)^{-1}(x + y)) &= \Phi(\lambda_1(\lambda_1 + \lambda_2)^{-1}\lambda_1^{-1}x + \lambda_2(\lambda_1 + \lambda_2)^{-1}\lambda_2^{-1}y)\\ &\leq \lambda_1(\lambda_1 + \lambda_2)^{-1}\Phi(\lambda_1^{-1}x) + \lambda_2(\lambda_1 + \lambda_2)^{-1}\Phi(\lambda_1^{-1}y) \leq \mathbf{1},\end{aligned}$$

i.e. $\lambda_1 + \lambda_2 \in B(x + y)$. This means that $B(x) + B(y) \subseteq B(x + y)$, and so

$$\|x + y\|_\Phi \leq \|x\|_\Phi + \|y\|_\Phi.$$

Let us now show that $\|ex\|_\Phi = e\|x\|_\Phi$ for any idempotent $e \in L_0$ and $x \in E$. Take $\lambda, \beta \in \mathcal{P}(L_0)$ such that $\Phi(\lambda^{-1}x) \leq \mathbf{1}$, $\Phi(\beta^{-1}xe) \leq \mathbf{1}$. Then $\gamma = \beta e + \lambda(\mathbf{1} - e) \in \mathcal{P}(L_0)$, in addition $\gamma^{-1} = \beta^{-1}e + \lambda^{-1}(\mathbf{1} - e)$ and

$$\begin{aligned}\Phi(\gamma^{-1}x) &= \Phi(\gamma^{-1}xe) + \Phi(\gamma^{-1}x(\mathbf{1} - e))\\ &= \Phi(\beta^{-1}xe) + \Phi(\lambda^{-1}x(\mathbf{1} - e))\\ &= e\Phi(\beta^{-1}xe) + (\mathbf{1} - e)\Phi(\lambda^{-1}x)\\ &\leq e + (\mathbf{1} - e) = \mathbf{1}.\end{aligned}$$

Hence, $\|x\|_\Phi \leq \gamma$ and therefore $e\|x\|_\Phi \leq \|ex\|_\Phi$.

Since $|ex| \leq |x|$, we have $\|ex\|_\Phi \leq \|x\|_\Phi$. That is why $e\|x\|_\Phi \leq e\|ex\|_\Phi \leq e\|x\|_\Phi$, i.e. $e\|x\|_\Phi = e\|ex\|_\Phi$.

Further, if $\lambda \in \mathcal{P}(L_0)$ and $\Phi(\lambda^{-1}ex) \leq \mathbf{1}$, then $\Phi(\beta^{-1}ex) = \Phi(\lambda^{-1}ex) \leq \mathbf{1}$ for $\beta = \lambda e + \varepsilon(\mathbf{1} - e)$. Hence $\|ex\|_\Phi(\mathbf{1} - e) = 0$ and $\|ex\|_\Phi = e\|ex\|_\Phi = e\|x\|_\Phi$.

Let now $\alpha$ be an invertible element from $L_0$. Then

$$\begin{aligned}\|\alpha x\|_\Phi &= \inf\left\{\lambda \in \mathcal{P}(L_0) : \Phi(\lambda^{-1}\alpha x) \leq \mathbf{1}\right\}\\ &= \inf\left\{|\alpha|\gamma : \Phi(\gamma^{-1}x) \leq \mathbf{1}, \gamma = \lambda|\alpha|^{-1} \in \mathcal{P}(L_0)\right\} = |\alpha|\|x\|_\Phi.\end{aligned}$$

If $\alpha$ is an arbitrary non-zero element from $L_0$, $e = \mathbf{1} - s(\alpha)$, then $\alpha + e$ is invertible in $L_0$, and therefore

$$\begin{aligned}\|\alpha x\|_\Phi &= \|(\alpha+e)(\mathbf{1}-e)x\|_\Phi = (|\alpha|+e)\|(\mathbf{1}-e)x\|_\Phi \\ &= (|\alpha|+e)(\mathbf{1}-e)\|x\|_\Phi = |\alpha|\|x\|_\Phi.\end{aligned}$$

Thus, $(E, \|\cdot\|_\Phi)$ is a normed $L_0$-vector lattice. □

**Definition.** A norm-complete $L_0$-vector lattice $(E, \|\cdot\|_\Phi)$ is called an *Orlicz $L_0$-lattice*.

The Orlicz-Kantorovich space $(L_\psi, \|\cdot\|_{(\psi)})$ is a good example of Orlicz $L_0$-lattices.

**Theorem 4.7.** *The Orlicz $L_0$-lattice $(E, \|\cdot\|_\Phi)$ is an order complete lattice, and the $L_0$-norm $\|\cdot\|_\Phi$ is order continuous.*

Proof: Consider a disjoint bounded sequence $\{x_n\} \subset E_+$. Since $x_n \leq x \in E_+$, we have $\sum_{i=1}^n x_i \leq x$. Using property 5, we obtain $\sum_{i=1}^n \Phi(x_i) \leq \Phi(x)$. Hence, $\Phi(x_n) \xrightarrow{(o)} 0$. For any fixed $i = 1, 2, \ldots$, $\Phi(2^i x_n) \xrightarrow{(o)} 0$. The element $\lambda = \Phi(x) + \mathbf{1} \in B(x)$. Hence, $\|x\|_\Phi \leq \Phi(x) + \mathbf{1}$. Therefore, $\|x_n\|_\Phi \leq 2^{-i}\Phi(2^i x_n) + 2^{-i}\mathbf{1}$. Thus, $(o)$-$\overline{\lim}\|x_n\|_\Phi \leq 2^{-i}\mathbf{1}$ for any $i$. Hence, $(o)$-$\overline{\lim}\|x_n\|_\Phi = 0$. By Theorem 3.1.4, we are done. □

**Lemma 4.8.** *Let $\|x\|_\Phi \leq \mathbf{1}$ and $\{\|x\|_\Phi = \mathbf{1}\} = 0$. Then $\Phi(x) \leq \|x\|_\Phi$.*

Proof: As in Proposition 2.7 from [13], one can choose $\lambda_n \in B(x)$ such that $\lambda_n \downarrow \|x\|_\Phi$. Let $\lambda \in L_0$, $\lambda \geq 0$, $\|x\|_\Phi \leq \lambda \leq \mathbf{1}$ and $\{\lambda = \|x\|_\Phi\} = 0$. Then, $\lambda$ is invertible. Set $f_n = \{\lambda < \lambda_n\}$. Obviously, $f_n \downarrow 0$. We have

$$\begin{aligned}\Phi\left(\lambda^{-1}x\right) &= \Phi\left(\left(\lambda_n^{-1}\lambda_n\lambda^{-1}\right)x\right) \\ &= f_n\Phi\left(\lambda_n^{-1}x\lambda_n\lambda^{-1}\right) + (\mathbf{1}-f_n)\Phi\left(\lambda_n^{-1}x\left(\lambda_n\lambda^{-1}\left(\mathbf{1}-f_n\right)\right)\right) \\ &\leq f_n\Phi\left(\lambda_n^{-1}x\lambda_n\lambda^{-1}\right) + (\mathbf{1}-f_n)\Phi(\lambda_n^{-1}x) \\ &\leq f_n\Phi(\lambda_n^{-1}x\lambda_n\lambda^{-1}) + (\mathbf{1}-f_n).\end{aligned}$$

Since $f_n \downarrow 0$, $f_n\Phi(\lambda_n^{-1}x\lambda_n\lambda^{-1}) \xrightarrow{(o)} 0$. After switching to $(o)$-limit, we obtain $\Phi(\lambda^{-1}x) \leq \mathbf{1}$. Since $\lambda \leq \mathbf{1}$, we have $\lambda^{-1}\Phi(x) \leq \Phi(\lambda^{-1}x) \leq \mathbf{1}$.

Let $\alpha_n = \|x\|_\Phi + n^{-1}(\mathbf{1} - \|x\|_\Phi)$. Then $\|x\|_\Phi \leq \alpha_n \leq \mathbf{1}$ and $\{\|x\|_\Phi = \alpha_n\} = 0$. Hence $\Phi(x) \leq \alpha_n$, $n = 1, 2, \ldots$ and $\Phi(x) \leq \|x\|_\Phi$. □

**Proposition 4.9.** *Let $(E, \|\cdot\|_\Phi)$ be an Orlicz $L_0$-lattice, $y_n \in E$. Then $\|y_n\|_\Phi \xrightarrow{(o)} 0$ if and only if $\Phi(y_n) \xrightarrow{(o)} 0$.*

Proof: Let $\Phi(y_n) \xrightarrow{(o)} 0$. Then, $\|y_n\|_\Phi \xrightarrow{(o)} 0$ (see the proof of Theorem 4.7).

Set $g_n = \{\|y_n\|_\Phi < \mathbf{1}\}$. Since $\|y_n\|_\Phi \xrightarrow{(o)} 0$, we have $g_n \xrightarrow{(o)} \mathbf{1}$. Obviously, $\|g_n y_n\|_\Phi = g_n\|y_n\|_\Phi \leq \mathbf{1}$ and $\{g_n\|y_n\|_\Phi = \mathbf{1}\} = 0$. By Lemma 4.8, $\Phi(g_n y_n) \leq \|g_n y_n\|_\Phi = g_n\|y_n\|_\Phi \xrightarrow{(o)} 0$. Since $(\mathbf{1} - g_n) \xrightarrow{(o)} 0$, we have $(\mathbf{1} - g_n)\Phi(y_n) \xrightarrow{(o)} 0$. Hence, $\Phi(y_n) = \Phi(g_n y_n) + \Phi((\mathbf{1} - g_n)y_n) \xrightarrow{(o)} 0$. □

**Proposition 4.10.** *Let $x_n \uparrow x$. Then $\Phi(x_n) \uparrow \Phi(x)$.*

Proof: Obviously, $\sup_{n\geq 1} \Phi(x_n) \leq \Phi(x)$. Further, for any number $a \in (0, 1]$, we have $x = (1 - a)x_n + a(x_n + a^{-1}(x - x_n))$. Using properties of $\Phi$, we obtain

$$\begin{aligned}\Phi(x) &\leq (1-a)\Phi(x_n) + a\Phi\left(x_n + a^{-1}(x - x_n)\right)\\ &\leq \Phi(x_n) + 2^{-1}ac\left(\Phi(x_n) + \Phi\left(a^{-1}(x - x_n)\right)\right).\end{aligned}$$

By Theorem 4.7, $\|a^{-1}(x - x_n)\|_\Phi \xrightarrow{(o)} 0$. By Proposition 4.9, $\Phi(a^{-1}(x - x_n)) \downarrow 0$. Hence,

$$\begin{aligned}\Phi(x) &\leq (o)\text{-}\limsup_{n\to\infty}\left(\Phi(x_n) + 2^{-1}ac\left(\Phi(x_n) + \Phi\left(a^{-1}(x - x_n)\right)\right)\right)\\ &= (1 + \frac{1}{2}ac)\sup_{n\geq 1}\Phi(x_n).\end{aligned}$$

Since $a$ is arbitrary, we obtain $\Phi(x) \leq \sup_{n\geq 1} \Phi(x_n)$. □

## 5. Abstract characterization of Orlicz-Kantorovich $\mathbf{L_0}$-spaces

**Definition** (compare with [2]). An Orlicz $L_0$-lattice $(E, \|\cdot\|_\Phi)$ is called *component-invariant* if

$$\Phi(te) = \Phi(e)\Phi^{-1}(\mathbf{I})\Phi(t\mathbf{I})$$

for all $t \geq 0$, $e \in \nabla$.

The Orlicz-Kantorovich space $(L_\psi(\nabla, m), \|\cdot\|_{(\psi)})$ is a component-invariant Orlicz $L_0$-lattice. The reverse assertion is proved in Theorem 5.1. This can be considered as an abstract characterization of Orlicz-Kantorovich spaces in the class of Banach $L_0$-vector lattices.

**Theorem 5.1.** *Let $(E, \|\cdot\|_\Phi)$ be a component-invariant Orlicz $L_0$-lattice. There exists a strongly positive measure $m$ on $\nabla$, with values in $L_0$, such that $(E, \|\cdot\|_\Phi)$ is isometrically isomorphic to the Orlicz-Kantorovich space $(L_\psi(\nabla, m), \|\cdot\|_{(\psi)})$. Here $\psi(t)\cdot\mathbf{1} = \Phi(t\mathbf{I})\Phi^{-1}(\mathbf{I})$.*

Proof: $E$ can be identified (see [12]) with a normal vector sublattice in $L_0(\nabla) = C_\infty(X(\nabla))$ so that $\mathbf{I}$ coincides with the $f \equiv \mathbf{1}$. Moreover, $e \in \nabla$ if and only if $e$ is a characteristic function of an open-closed set from $X(\nabla)$. For any $e \in \nabla$, set $m(e) = \Phi(e)$. Obviously, $m(e) \in L_0$, $m(e) \geq 0$. If $e \wedge g = 0$, $e, g \in \nabla$, then $m(e \vee g) = m(e) + m(g)$. Clearly, $m(e) = 0$ if and only if $e = 0$. Let $\{e_n\} \subset \nabla$ and $e_n \downarrow 0$. By Theorem 4.7, we have $\|e_n\|_\Phi \downarrow 0$. Proposition 4.9 implies $\Phi(e_n) \downarrow 0$. This means that $m$ is a strongly positive measure on $\nabla$ with values in $L_0$. Obviously, $m(eg) = em(g)$. Hence, $m$ is compatible with the module structure.

Let $x$ be a positive simple element from $L_0(\nabla)$, i.e. $x = \sum_{i=1}^n \lambda_i g_i$. Here, $\lambda_i \geq 0$ and $g_i \in \nabla$ are mutually disjoint. $\sup g_i = \mathbf{I}$. Obviously, $x \in E$ and $x \in L_\psi(\nabla, m)$.

Using the component invariance of $(E, \|\cdot\|_\Phi)$, we obtain

$$\Phi(x) = \sum_{i=1}^n \Phi(\lambda_i g_i) = \sum_{i=1}^n \Phi(g_i)\Phi^{-1}(\mathbf{I})\Phi(\lambda_i \mathbf{I}) = \sum_{i=1}^n \psi(\lambda_i) m(g_i)$$
$$= \int \sum_{i=1}^n \psi(\lambda_i) g_i \, dm = \int \psi\left(\sum_{i=1}^n \lambda_i g_i\right) dm = \int \psi(x)\, dm.$$

Thus, $\|x\|_\Phi = \inf\{\lambda \in \mathcal{P}(L_0) : \Phi(\lambda^{-1}x) \leq \mathbf{I}\} = \inf\{\lambda \in \mathcal{P}(L_0) : \int \psi(\lambda^{-1}x)\, dm \leq \mathbf{I}\} = \|x\|_{(\psi)}$ for any positive simple element $x$ from $L_0(\nabla)$.

However, simple elements are dense in $E$ as well as in $L_\psi$. □

We now use Theorem 5.1 to construct examples of Orlicz-Kantorovich spaces.

Let $(\Omega, \Sigma, \mu), (X, \nu)$ be as in Example 2. Let $L_\psi(X, \nu)$ be an Orlicz space associated with $(X, \nu)$ and with the Orlicz function $\psi$ satisfying the $(\delta_2, \Delta_2)$-condition. We denote by $\Gamma(L_\psi(X, \nu))$ the set of all step mappings $u : (\Omega, \Sigma, \mu) \to L_\psi(X, \nu)$ having the form $u = \sum_{i=1}^n x_i \chi_{A_i}$ where $x_i \in L_\psi(X, \nu)$, $A_i \in \Sigma$, $A_i \cap A_j = \emptyset$, $i \neq j$, $i, j = 1, \ldots, n$, $n \in \mathbb{N}$.

A mapping $u : (\Omega, \Sigma, \mu) \to L_\psi(X, \nu)$, is called measurable if there exists a sequence $\{u_k\} \subset \Gamma(L_\psi(X, \nu))$ such that $\|u(\omega) - u_n(\omega)\|_{L_\psi(X,\nu)} \to 0$ as $n \to \infty$ for a.e. $\omega \in \Omega$. Let $\mathcal{L}_0(\Omega, L_\psi(X, \nu))$ be the set of all measurable mappings from $(\Omega, \Sigma, \mu)$ into $L_\psi(X, \nu)$. Obviously, $\mathcal{L}_0(\Omega, L_\psi(X, \nu))$ is an $\mathcal{L}_0(\Omega)$-module, in addition $\|u(\omega)\|_{L_\psi(X,\nu)}$ is a measurable function on $(\Omega, \Sigma, \mu)$ for all $u \in \mathcal{L}_0(\Omega, L_\psi(X, \nu))$. Consider an $\mathcal{L}_0(\Omega)$-submodule $J = \{u \in \mathcal{L}_0(\Omega, L_\psi(X, \nu)) : u(\omega) = 0 \text{ a.e.}\}$ and denote by $L_0(\Omega, L_\psi(X, \nu))$ the factor-module $\mathcal{L}_0(\Omega, L_\psi(X, \nu))/J$. Then $(L_0(\Omega, L_\psi(X, \nu)), \|\cdot\|)$ is a Banach $L_0$-vector lattice [3], where $\|\widetilde{u}\| = [\|u(\omega)\|_{L_\psi(X,\nu)}]^\sim$.

The norm in $L_\psi(X,\nu)$ is order continuous, and therefore $g_n, g \in X$, $\nu(g_n \triangle g) \to 0$ implies that $\|g_n - g\|_{L_\psi(X,\nu)} \to 0$. Hence, the complete Boolean algebra $L_0(\Omega, X)$ from Example 2 is a subset of $L_0(\Omega, L_\psi(X,\nu))$. Moreover, the Boolean algebra of unitary elements from $L_0(\Omega, L_\psi(X,\nu))$ with respect to the weak unit $\mathbf{1}(\omega) = \mathbf{1}_X$, $\omega \in \Omega$ coincides with $L_0(\Omega, X)$. It is clear that $m(\widetilde{e}) = [\nu(e(\omega))]^\sim$ is a strongly positive $L_0$-valued measure on $L_0(\Omega, X)$ and $m$ is compatible with the module structure (see Example 2).

**Theorem 5.2.** *The Banach $L_0$-vector lattices $L_0(\Omega, L_\psi(X,\nu))$ and $L_\psi(L_0(\Omega, X), m)$ are order and isometrically isomorphic.*

Proof: Without loss of generality, one can assume that $\psi(1) = 1$. For any $u = \sum_{i=1}^n x_i \chi_{A_i} \in \Gamma(L_\psi(X,\nu))$, set

$$\Phi_0(u)(\omega) = \int \psi(|u(\omega)|)\, d\nu = \sum_{i=1}^n \left( \int \psi(|x_i|)\, d\nu \right) \chi_{A_i}(\omega), \omega \in \Omega.$$

Let $v \in L_0(\Omega, L_\psi(X,\nu))$ and $\{u_n\}$ be a sequence from $\Gamma(L_\psi(X,\nu))$ such that $\|v(\omega) - u_n(\omega)\|_{L_\psi(X,\nu)} \to 0$ as $n \to \infty$ for a.e. $\omega \in \Omega$. Fix $\omega \in \Omega$ for which $\|v(\omega) - u_n(\omega)\|_{L_\psi(X,\nu)} \to 0$. Let us show that $\lim_{n\to\infty} \Phi_0(u_n)(\omega) = \int \psi(|u(\omega)|)\, d\nu$. If not, then there exist $\varepsilon > 0$ and a sequence $\{u_{n_k}(\omega)\}$ such that

$$\left| \int \psi(|u(\omega)|)\, d\nu - \int \psi(|u_{n_k}(\omega)|)\, d\nu \right| > \varepsilon, \quad k = 1, 2, \ldots \tag{1}$$

Choose a subsequence $a_s = u_{n_{k_s}}(\omega)$ which is $(o)$-converging to $u(\omega)$ in $L_\psi(X,\nu)$ [11, VII, §2]. Then the sequence $\{\psi(a_s)\}$ $(o)$-converges to $\psi(u(\omega))$ in $L_1(X,\nu)$, which contradicts (1).

Thus there exists a limit

$$\Phi_0(v)(\omega) := \int \psi(|v(\omega)|)\, d\nu = \lim_{n\to\infty} \int \psi(|u_n(\omega)|)\, d\nu = \lim_{n\to\infty} \Phi_0(u_n)(\omega),$$

for a.e. $\omega \in \Omega$, in particular, $\Phi_0(v) \in \mathcal{L}_0(\Omega)$. Let $\Phi(\widetilde{u}) = [\Phi_0(u)]^\sim$. Clearly, $\Phi$ is a component-invariant $L_0$-modulator on $L_0(\Omega, L_\psi(X,\nu))$, in addition $\Phi(t\mathbf{1}) = \psi(t)\Phi(\mathbf{1})$, $t \geq 0$.

If $\widetilde{u} \in L_0(\Omega, L_\psi(X,\nu)), \lambda \in \mathcal{P}(L_0)$, then

$$\Phi(\lambda^{-1}\widetilde{u}) \leq \mathbf{1} \Leftrightarrow \int \psi(\lambda^{-1}(\omega)|u(\omega)|)\, d\nu \leq 1 \text{ a.e.} \quad \Leftrightarrow$$
$$\|u(\omega)\|_{L_\psi(X,\nu)} \leq \lambda(\omega) \text{ a.e.} \quad \Leftrightarrow \|\widetilde{u}\| \leq \lambda.$$

Hence,

$$\|\widetilde{u}\| = \inf\{\lambda \in \mathcal{P}(L_0) : \|\widetilde{u}\| \leq \lambda\} = \inf\{\lambda \in \mathcal{P}(L_0) : \Phi(\lambda^{-1}\widetilde{u}) \leq \mathbf{1}\} = \|\widetilde{u}\|_\Phi.$$

Thus, $(L_0(\Omega, L_\psi(X,\nu)), \|\ \|)$ is a component-invariant Orlicz $L_0$-lattice. In addition, $(L_0(\Omega, L_\psi(X,\nu)), \|\ \|)$ and $L_\psi(L_0(\Omega, X), m)$ are isometrically isomorphic (Theorem 5.1). □

**Remark 5.3.** If $\psi(t) = t^p$, $p \geq 1$, then $L_0(\Omega, L_p(X, \nu))$ is isometrically isomorphic to $L_p(L_0(\Omega, X), m)$.

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Institute of Mathematics and Information Technologies, Uzbekistan Academy of Sciences, 29, F. Khodjaev str., Tashkent 100125, Uzbekistan

*E-mail*: botirzakirov@list.ru

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