The Baire property in remainders of topological groups and other results

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Abstract. It is established that a remainder of a non-locally compact topological group G has the Baire property if and only if the space G is not Čech-complete. We also show that if G is a non-locally compact topological group of countable tightness, then either G is submetrizable, or G is the Čech-Stone remainder of an arbitrary remainder Y of G. It follows that if G and H are non-submetrizable topological groups of countable tightness such that some remainders of G and H are homeomorphic, then the spaces G and H are homeomorphic. Some other corollaries and related results are presented.

Keywords: Baire property, σ-compact, Čech-complete space, compactification, Čech-Stone compactification, Rajkov complete, paracompact p-space

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By a space we understand a Tychonoff topological space. A compactification of a space X is a Hausdorff compactification of X. A remainder of a space X is the subspace $bX \setminus X$ of a compactification bX of X. For the definition and properties of p-spaces see [1], [6], or [7]. We only recall that Lindelöf p-spaces can be characterized as preimages of separable metrizable spaces under perfect mappings ([1], [6]). A space X has the Baire property if the intersection of an arbitrary countable family of dense open subsets of X is dense in X.

In terminology and notation, we mostly follow [6], [7], and [10]. To these books a reader may also refer in the case of folklore type references.

1. The Baire property in remainders of topological groups

The Dichotomy Theorem in [5] can be reformulated as follows: *If G is a topological group, and some remainder of G is not pseudocompact, then every remainder of G is Lindelöf.*

A natural question arises: what if we strengthen the assumption and assume that some remainder of G does not have the Baire property? This question leads to the Second Dichotomy Theorem for remainders of topological groups:

Theorem 1.1. *Suppose that G is a non-locally compact topological group. Then either every remainder of G has the Baire property, or every remainder of G is σ-compact.*

To prove this statement, we need the next result of independent interest:

Theorem 1.2. *If Y is a Čech-complete subspace of a topological group G, then either Y is nowhere dense in G, or the space G is Čech-complete as well.*

Proof: Assume that Y is not nowhere dense in G. Then some non-empty open subset V of G is contained in the closure of Y in G. Clearly, $P_V = Y \cap V$ is a dense Čech-complete subspace of V.

Claim 1: For every non-empty open subset W of G, there exist a non-empty open subset U contained in W and a Čech-complete subspace Z of U such that Z is dense in U.

Indeed, since G is a topological group, we can use translations in G, in an obvious way, to establish Claim 1.

By Zorn's Lemma, we can take a maximal disjoint family γ of non-empty open subsets of G such that each element of γ contains a dense Čech-complete subspace. Put $M = \bigcup \gamma$. It follows from Claim 1 that M is dense in G. For each $U \in \gamma$, fix a Čech-complete subspace Z_U of U dense in U, and put $Z = \bigcup\{Z_U : U \in \gamma\}$. Obviously, Z is dense in G.

Let us show that the subspace Z is also Čech-complete. Fix a compactification B of G, and for each open subset U of G fix an open subset bU of B such that $U = G \cap bU$. Observe that U is dense in bU, since G is dense in B. Now take any $U \in \gamma$. Then Z_U is dense in bU, and since Z_U is Čech-complete, we can fix a countable family $\eta_U = \{W_n(U) : n \in \omega\}$ of open subsets of bU such that $Z_U = \bigcap \eta_U$. For what follows, it is essential to notice that the family $b\gamma = \{bU : U \in \gamma\}$ is disjoint. This is so, since γ is a disjoint family of open subsets of G and G is dense in B. Thus, $b\gamma$ is a disjoint family of open subsets of B. It also follows that the family $\xi_n = \{W_n(U) : U \in \gamma\}$ is disjoint, for each $n \in \omega$.

Put $W_n = \bigcup \xi_n = \bigcup\{W_n(U) : U \in \gamma\}$ for $n \in \omega$. Clearly, $Z \subset W_n$, for each $n \in \omega$. Hence, $Z \subset Z_1$, where $Z_1 = \bigcap\{W_n : n \in \omega\}$.

Claim 2: $Z_1 = Z$, and hence, Z is Čech-complete. Indeed, $Z_1 = \bigcap\{W_n : n \in \omega\} = \bigcup\{\bigcap\{W_n(U) : n \in \omega\} : U \in \gamma\} = \bigcup\{Z_U : U \in \gamma\} = Z$, since each family ξ_U is disjoint.

Claim 3: The topological group G is Rajkov complete.

Assume the contrary, and take the Rajkov completion H of G. Then $H \setminus G$ is non-empty. Recall that H is a topological group containing the group G as a dense subgroup. Fix $a \in H \setminus G$, and consider the subspaces aG and aZ of H. Clearly, aG and G are disjoint, since G is a subgroup of H and a is not in G. Observe that aG is dense in H, since G is dense in H. It follows that Z and aZ are disjoint Čech-complete subspaces of H dense in H. However, this is impossible. Indeed, the intersection of any two dense Čech-complete subspaces of any Tychonoff space is dense in this space, by the Baire property of compact Hausdorff spaces. Thus, G is Rajkov complete.

The existence of a dense Čech-complete subspace in G also implies that G contains a non-empty compact subspace with a countable base of open neighbourhoods. Hence, G is a paracompact p-space, since G is a topological group (see [7], Theorem 4.3.20 and Corollary 4.3.21).

However, M.M. Choban has shown that if a Rajkov complete topological group is a paracompact p-space, then this space is Čech-complete ([8]). Hence, G is Čech-complete. □

Proof of Theorem 1.1: Suppose that bG is a compactification of G such that the remainder $Y = bG \setminus G$ does not have the Baire property.

Claim: Y is σ-compact.

In other words, we have to show that G is Čech-complete. Since Y does not have the Baire property, we can find a countable family η of open dense subsets of Y such that $\bigcap \eta$ is not dense in Y. Note that G is nowhere locally compact, and therefore, Y is dense in bG. It follows that there exist a countable family ξ of open dense subsets of bG and a non-empty open subset U of bG such that $(\bigcap \xi) \cap (U \cap Y) = \emptyset$. Then the subspace $M = (\bigcap \xi) \cap (U \cap G) = (\bigcap \xi) \cap U$ is Čech-complete and dense in the open subset $U \cap G$ of G. This is so, since U is locally compact and hence has the Baire property. Therefore, M is not nowhere dense in G, and Theorem 1.2 implies that G is Čech-complete. □

Remark. Observe that a remainder Y of a non-locally compact topological group G cannot have the Baire property and be σ-compact at the same time. Indeed, otherwise the interior of Y in bG is not empty and clearly Y must be dense in the compactification bG. Therefore, Y has to intersect its complement G, since G is also dense in bG, a contradiction.

Corollary 1.3. *Every remainder (some remainder) of an arbitrary non-locally compact topological group G has the Baire property if and only if G is not Čech-complete (that is, if and only if the remainder of it is not σ-compact).*

Proof: This statement follows from Theorem 1.2. □

The last result shows that topological groups can be used to produce non-trivial topological spaces with the Baire property.

Corollary 1.4. *For an arbitrary topological group G with countable Souslin number, either G is Lindelöf and each remainder of G is a σ-compact p-space, or every remainder of G has the Baire property.*

Proof: Assume that the second alternative does not hold. Then G cannot be locally compact and, by Theorem 1.1, G is Čech-complete. Hence, G is paracompact ([7, Corollary 4.3.21]). Since the Souslin number of G is countable, it follows that G is Lindelöf. Thus, G is a Lindelöf p-space. Now a theorem in [4] implies that every remainder of G is a Lindelöf p-space. Observe that each remainder of G is σ-compact, since G is Čech-complete. □

In connection with the last result, we present the next theorem.

Theorem 1.5. *Suppose that G is an arbitrary topological group with countable Souslin number, and let Y be a remainder of G of countable pseudocharacter. Then the space Y is first countable.*

PROOF: Indeed, Y is either pseudocompact or Lindelöf, by a theorem in [5]. If Y is Lindelöf, then G is a paracompact p-space ([4]). Thus, Y is either pseudocompact or a p-space. Since each point in Y is a G_δ-point, in both cases it follows that Y is first countable. □

2. Remainders of topological groups and the Čech-Stone compactification

In this section, we will establish a curious property of remainders of topological groups: under certain general assumptions, the Čech-Stone remainder of any such space turns out to be homeomorphic to the group itself!

Let us start with the following general question: when a topological space has a remainder homeomorphic to a topological group? One, probably, would guess that this occurs rather rarely. We even may conjecture that a homogeneous remainder of a topological space is a rare specimen.

Recall that a space X is *Moscow* if the closure of an arbitrary open subset in X is the union of some family of G_δ-subsets of X ([3]; see also [7, Section 6.1, p. 346]).

A space X is said to be *submetrizable* if its topology contains a metrizable topology.

Theorem 2.1. *Suppose that G is a Moscow topological group, and that Y is a remainder of G in some compactification bG of G. Then at least one of the following three conditions is satisfied:*

1. *the space G contains a topological copy of D^{ω_1};*
2. *the space G is submetrizable;*
3. *the compactum bG is the Čech-Stone compactification of the space Y, and hence, G is the Čech-Stone remainder of Y.*

PROOF: Every locally compact non-metrizable topological group contains a copy of non-metrizable compact group, and therefore, contains a topological copy of D^{ω_1} ([7, Section 6.1, p. 226]).

Thus, we may assume that G is not locally compact. Then, of course, G is nowhere locally compact, since G is a topological group. It follows that Y is dense in bG, that is, bG in this case is indeed a compactification of the space Y.

Assume also that condition (3) is not satisfied. Then we can find closed sets A and B in Y and a real-valued continuous function f on Y such that $f(A) = \{0\}$ and $f(B) = \{1\}$, while some point $z \in G$ belongs to the intersection of the closures of A and B in bG. Using the continuity of f, we can find open subsets U and V of Y containing A and B, respectively, such that the closures of U and V in Y are disjoint. Fix now open subsets U_1 and V_1 of bG such that $U = U_1 \cap Y$ and $V = V_1 \cap Y$. Let F be the intersection of the closures of U_1 and V_1 in bG.

Note that F is compact. Clearly, U is dense in U_1, and V is dense in V_1, since Y is dense in bG. Therefore, no point of Y belongs to F, that is, $F \subset G$. Put $U' = U_1 \cap G$ and $V' = V_1 \cap G$. Then, by the construction, F is the intersection of the closures of U' and V' in G. Since G is a regular Moscow space, it follows that F is the union of closed G_δ-subsets of G. Since F is compact, we conclude that G contains a non-empty compact G_δ-subset P. We are going to consider two cases.

Case 1: P is metrizable. Then every point of P is a G_δ-point in G. Since G is a topological group, it follows that the space G is submetrizable ([7, Theorem 3.3.16]).

Case 2: P is not metrizable. By a fundamental theorem of M.M. Choban in [9], the space P is a dyadic compactum. Since P is non-metrizable, it follows that P contains a topological copy of D^{ω_1} ([10, 3.12.12]). □

Corollary 2.2. *Suppose that G is a non-submetrizable topological group of countable tightness, and that Y is a remainder of G in some compactification bG of G. Then the compactum bG is the Čech-Stone compactification of the space Y, and hence, G is the Čech-Stone remainder of Y.*

Proof: Observe that G is not locally compact, since otherwise G would be metrizable ([7, Theorem 3.3.12], [2]).

Since the tightness of G is countable, and G is a topological group, the space G is Moscow ([3], [7, Section 6.4]). The space G does not contain a topological copy of D^{ω_1}, since the tightness of G is countable ([10, 3.12.12]). By the assumption, G is not submetrizable. Now it follows from Theorem 2.1 that the conclusion in Corollary 2.2 holds. □

Corollary 2.3. *Suppose that G is a topological group algebraically generated by a non-metrizable compact subspace B of countable tightness, and let Y be a remainder of G. Then G is the Čech-Stone remainder of Y.*

Proof: It easily follows from the assumptions on G that G is covered by a countable family of compacta of countable tightness. Each of these compacta is a continuous image of a finite power of the compactum B (recall that the tightness of B^n is countable, for each $n \in \omega$, and that the tightness is not increased by perfect mappings, see [2]). It is known that the tightness of an arbitrary compactum covered by a countable family of compacta of countable tightness is also countable (D.V. Ranchin [11]). Therefore, the tightness of every compact subspace of G is countable. Therefore, G does not contain a topological copy of D^{ω_1}. Observe that the space G is not submetrizable, since B is a non-metrizable compactum. The space G is Moscow, since G is a σ-compact topological group ([3], [7, Section 6.4]).

Now it follows from Theorem 2.1 that the conclusion in Corollary 2.3 holds. □

For the definition and properties of free topological groups see [2] and [7, Chapter 7].

Corollary 2.4. *Suppose that $F(X)$ is the free topological group of a non-metrizable compact space X of countable tightness, and let Y be a remainder of $F(X)$. Then $F(X)$ is the Čech-Stone remainder of Y.*

Corollary 2.5. *Suppose that G and H are non-submetrizable topological groups of countable tightness. Then the spaces G and H are homeomorphic if and only if some remainders of G and H are homeomorphic.*

Proof: The necessity is clear. The sufficiency follows from Corollary 2.2, since both G and H turn out to be homeomorphic to the Čech-Stone remainder of the same space Y. $\square$

To demonstrate that the assumptions in Theorem 2.1 are not too excessive, we consider the next simple example. Let Q be the topological group of rational numbers, with the usual topology and operation. Clearly, Q has a compactification bQ homeomorphic to the circumference S^1 and such that the remainder $Y = bQ \setminus Q$ is homeomorphic to the space of irrational numbers. The space of irrational numbers is also homeomorphic to a topological group. Since bQ is metrizable, bQ is not the Čech-Stone compactification of Y. However, Q in this example is metrizable.

In fact, we have a general statement which complements Theorem 2.1 and generalizes the above situation.

Theorem 2.6. *Suppose that G is an arbitrary separable metrizable topological group, and that bG is any compactification of G. Then bG is not the Čech-Stone compactification of the space $Y = bG \setminus G$.*

Proof: If G is locally compact, then bG is not a compactification of Y, since Y is not dense in bG.

So we may assume that the space G is not locally compact. Then Y is dense in bG, and, clearly, Y is not compact. Observe that Y is a Lindelöf p-space, since G is a Lindelöf p-space ([4]). Therefore, Y is normal and Y is not countably compact, since Y is not compact. Hence, we can fix an infinite countable discrete closed subspace A in Y. Put $Z = \overline{A} \setminus A$, where $\overline{A}$ is the closure of A in bG.

Assume now that bG is the Čech-Stone compactification of Y. Then $\overline{A}$ is the Čech-Stone compactification of A, since Y is normal and A is closed in Y. Therefore, the space Z is not metrizable, since the space A is infinite and discrete. On the other hand, Z is metrizable, since Z is a subspace of G and G is metrizable. $\square$

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