Comment.Math.Univ.Carolin. 50,3 (2009) 325–347

Weyl quantization for the semidirect product of a compact Lie group and a vector space

Benjamin Cahen

Abstract. Let G be the semidirect product $V \rtimes K$ where K is a semisimple compact connected Lie group acting linearly on a finite-dimensional real vector space V. Let $\mathcal{O}$ be a coadjoint orbit of G associated by the Kirillov-Kostant method of orbits with a unitary irreducible representation π of G. We consider the case when the corresponding little group H is the centralizer of a torus of K. By dequantizing a suitable realization of π on a Hilbert space of functions on $\mathbb{C}^n$ where $n = \dim(K/H)$, we construct a symplectomorphism between a dense open subset of $\mathcal{O}$ and the symplectic product $\mathbb{C}^{2n} \times \mathcal{O}'$ where $\mathcal{O}'$ is a coadjoint orbit of H. This allows us to obtain a Weyl correspondence on $\mathcal{O}$ which is adapted to the representation π in the sense of [B. Cahen, *Quantification d'une orbite massive d'un groupe de Poincaré généralisé*, C.R. Acad. Sci. Paris t. 325, série I (1997), 803–806].

Keywords: Weyl quantization, Berezin quantization, semidirect product, coadjoint orbits, unitary representations

Classification: 81S10, 22E46, 22E99, 32M10

1. Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let π be a unitary irreducible representation of G on a Hilbert space H. Assume that the representation π is associated to a coadjoint orbit $\mathcal{O}$ of G by the Kirillov-Kostant method of orbits [19], [20], [21]. In [5] and [6] we introduced the notion of adapted Weyl correspondence on $\mathcal{O}$ in order to generalize the usual quantization rules directly [1], [15].

Definition 1.1. An *adapted Weyl correspondence* is an isomorphism W from a vector space $\mathcal{A}$ of complex-valued (or real-valued) smooth functions on the orbit $\mathcal{O}$ (called symbols) to a vector space $\mathcal{B}$ of (not necessarily bounded) linear operators on H satisfying the following properties:

(1) the elements of $\mathcal{B}$ preserve a fixed dense domain D of H;
(2) the constant function 1 belongs to $\mathcal{A}$, the identity operator I belongs to $\mathcal{B}$ and $W(1) = I$;
(3) $A \in \mathcal{B}$ and $B \in \mathcal{B}$ implies $AB \in \mathcal{B}$;
(4) for each f in $\mathcal{A}$ the complex conjugate $\bar{f}$ of f belongs to $\mathcal{A}$ and the adjoint of $W(f)$ is an extension of $W(\bar{f})$ (in the real case: for each f in $\mathcal{A}$ the operator $W(f)$ is symmetric);

(5) the elements of D are C^∞-vectors for the representation π, the functions $\tilde{X}$ ($X \in \mathfrak{g}$) defined on $\mathcal{O}$ by $\tilde{X}(\xi) = \langle \xi, X \rangle$ are in $\mathcal{A}$ and $W(i\tilde{X})v = d\pi(X)v$ for each $X \in \mathfrak{g}$ and each $v \in D$.

For example, if G is a connected simply-connected nilpotent Lie group then each coadjoint orbit $\mathcal{O}$ of G is diffeomorphic to $\mathbb{R}^{2n}$ where $n = 1/2 \dim \mathcal{O}$, the unitary irreducible representation of G associated with $\mathcal{O}$ can be realized in the Hilbert space $L^2(\mathbb{R}^n)$ and the usual Weyl correspondence gives an adapted symbol calculus on $\mathcal{O}$ [2], [28]. It is also known that the Berezin calculus on an integral coadjoint orbit $\mathcal{O}$ of a semisimple compact connected Lie group G provides an adapted symbol calculus on $\mathcal{O}$ [5] (see also [12] and, for a similar result for the discrete series representations of a semisimple noncompact Lie group, [11]). By combining the usual Weyl correspondence and the Berezin calculus, we have obtained an adapted Weyl correspondence on the principal series coadjoint orbits of a connected semisimple noncompact Lie group [5], [10] and on the integral coadjoint orbits of the semidirect product $V \rtimes K$ where K is a connected semisimple noncompact Lie group acting linearly on a finite-dimensional real vector space V, under the condition that the little group is a maximal compact subgroup of K [9].

In fact, an adapted Weyl correspondence provides a prequantization map in the sense of [16, Definition 1]. In [9], we briefly described the relationship between adapted Weyl correspondences and the notion of quantization introduced by Mark Gotay (see [16]). Our original motivation for constructing adapted Weyl correspondences was to obtain covariant star-products on coadjoint orbits [5]. More recently, it has been established that adapted Weyl correspondences are useful to study contractions of Lie group representations in the setting of the Kirillov-Kostant method of orbits [14], [7], [8].

In the present paper, we continue the study of the adapted Weyl correspondences for semidirect products started in [9]. We consider here the case of the semidirect product $G = V \rtimes K$ where K is a semisimple compact connected Lie group acting linearly on a real vector space V. Let $\mathcal{O}$ be an integral coadjoint orbit of G whose little group H is the centralizer of a torus of K and let π be a unitary irreducible representation of G associated with $\mathcal{O}$. The representation π is usually realized on a space of square integrable sections of a Hermitian G-homogeneous vector bundle over K/H or, equivalently, on a space of square integrable functions on K/H with values in the space of the corresponding little group representation. Here we use a parametrization of a dense open subset of the generalized flag manifold K/H in order to obtain a realization of π in a space of square integrable functions on $\mathbb{C}^n$ where $n = \dim K/H$ (Section 3). We calculate the corresponding derived representation $d\pi$ (Section 4) and we dequantize $d\pi$ by using the usual Weyl correspondence on $\mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$ and the Berezin calculus on the little group coadjoint orbit $\mathcal{O}'$ associated with $\mathcal{O}$ (Section 5). Then we obtain a symplectomorphism from the symplectic product $\mathbb{C}^{2n} \times \mathcal{O}'$ onto a dense open subset of $\mathcal{O}$ (Section 6). This allows us to construct an adapted Weyl correspondence on the orbit $\mathcal{O}$ (Section 6). In particular, these results can be applied to the case when V is the Lie algebra of K and the action of K on V is the adjoint action (Section 7).

2. Preliminaries

The coadjoint orbits of a semidirect product were described by J.H. Rawnsley in [23] (see also [3] for a detailed analysis of the geometrical structure of these orbits).

Let K be a semisimple compact connected Lie group with Lie algebra $\mathfrak{k}$. Let σ be a representation of K on a finite-dimensional real vector space V. For k in K and v in V we write $k.v$ instead of $\sigma(k)v$. We denote also by $(k,p) \to k.p$ the representation of K on V^* which is contragredient to σ and by $(A,v) \to A.v$ and $(A,p) \to A.p$ the corresponding derived representations of $\mathfrak{k}$ on V and V^*, respectively. For v in V and p in V^* we define $v \wedge p \in \mathfrak{k}^*$ by $(v \wedge p)(A) = p(A.v) = -(A.p)(v)$ for $A \in \mathfrak{k}$. Note that $\mathrm{Ad}^*(k)(v \wedge p) = k.p \wedge k.v$ for $k \in K$, $v \in V$ and $p \in V^*$.

We consider the semidirect product $G = V \rtimes K$. The group law of G is

$$(v,k).(v',k') = (v + k.v', kk')$$

for v, v' in V and k, k' in K. The Lie algebra $\mathfrak{g}$ of G is the space $V \times \mathfrak{k}$ with the Lie bracket

$$[(a,A),\,(a',A')] = (A.a' - A'.a,\,[A,A'])$$

for a, a' in V and A, A' in $\mathfrak{k}$. We identify the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ to $V^* \times \mathfrak{k}^*$. The coadjoint action of G on $\mathfrak{g}^*$ is then given by

$$(v,k).(p,f) = (k.p, \mathrm{Ad}^*(k)f + v \wedge k.p)$$

for $(v,k) \in G$ and $(p,f) \in \mathfrak{g}^*$. We identify K-equivariantly $\mathfrak{k}$ to its dual $\mathfrak{k}^*$ by using the Killing form of $\mathfrak{k}$ defined by $\langle A, B\rangle = \mathrm{Tr}(\mathrm{ad}\,A\,\mathrm{ad}\,B)$ for A and B in $\mathfrak{k}$. Then $\mathfrak{g}^*$ can be identified to $V^* \times \mathfrak{k}$.

Now we consider the orbit $\mathcal{O}(\xi_0)$ of the element $\xi_0 = (p_0, f_0)$ of $\mathfrak{g}^* \simeq V^* \times \mathfrak{k}$ under the coadjoint action of G on $\mathfrak{g}^*$. Henceforth we assume that the little group $H := \{k \in K \,:\, k.p_0 = p_0\}$ is the centralizer of a torus T_1 of K. Let $\mathfrak{h}$ denote the Lie algebra of H. Let $Z(p_0)$ be the orbit of p_0 under the action of K on V^*. Then $Z(p_0)$ is diffeomorphic to the generalized flag manifold K/H.

Let us describe how to endow $Z(p_0) \simeq K/H$ with a complex structure. Let T be a maximal torus of K containing T_1. Clearly $T \subset H$. Let $\mathfrak{t}$ be the Lie algebra of T. Let Δ be the root system of K relative to T and let Δ_1 be the root system of H relative to T. We can simultaneously choose a Weyl chamber P of T relative to K and a Weyl chamber P_1 of T relative to H so that if Δ^+ and Δ_1^+ are, respectively, the positive roots of Δ and Δ_1 relative to P and P_1 then

(1) $\Delta^+ \cap \Delta_1 = \Delta_1^+$ and

(2) if $\alpha \in \Delta^+ \setminus \Delta_1^+$, $\beta \in \Delta_1$ and $\alpha+\beta \in \Delta$ then $\alpha+\beta \in \Delta^+ \setminus \Delta_1^+$.

Moreover, if Δ^s is the set of simple roots of Δ relative to P and if Δ_1^s is the set of simple roots of Δ_1 relative to P_1, then $\Delta_1^s \subset \Delta^s$ (see [27, 6.2.8]).

Let $\mathfrak{k}^c$, $\mathfrak{h}^c$ and $\mathfrak{t}^c$ be the complexifications of $\mathfrak{k}$, $\mathfrak{h}$ and $\mathfrak{t}$, respectively. Let K^c, H^c and T^c be the connected complex Lie groups whose Lie algebras are $\mathfrak{k}^c$, $\mathfrak{h}^c$ and $\mathfrak{t}^c$, respectively. Let $\mathfrak{k}^c = \mathfrak{t}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{k}_\alpha$ be the root space decomposition of $\mathfrak{k}^c$. We set $\mathfrak{n}^+ = \sum_{\alpha\in\Delta^+\setminus\Delta_1^+} \mathfrak{k}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha\in\Delta^+\setminus\Delta_1^+} \mathfrak{k}_{-\alpha}$. Then, by [27, 6.2.1], $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie algebras satisfying $[\mathfrak{h}^c, \mathfrak{n}^\pm] \subset \mathfrak{n}^\pm$. We also have

$$\mathfrak{k}^c = \mathfrak{h}^c \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \qquad \mathfrak{h}^c = \mathfrak{t}^c \oplus \sum_{\alpha\in\Delta_1^+} \mathfrak{k}_\alpha \oplus \sum_{\alpha\in\Delta_1^+} \mathfrak{k}_{-\alpha}. \tag{2.1}$$

We denote by N^+ and N^- the analytic subgroups of K^c with Lie algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively. A complex structure on K/H is then defined by the diffeomorphism $K/H \simeq K^c/H^cN^-$ [27, 6.2.11]. This complex structure depends on the choice of P and P_1.

The natural projection $K^c \to K^c/H^cN^-$ induces a projection $\tau : K^c \to Z(p_0)$. The natural action of K^c on K^c/H^cN^- induces an action of K^c on $Z(p_0)$; we denote by kp the action of $k \in K^c$ on $p \in Z(p_0)$. Of course, if $k \in K$ then kp is the natural action $k.p$ of $k \in K$ on $p \in V^*$.

Now we introduce a parametrization of a dense open subset of $Z(p_0) \simeq K/H$. Recall that (1) each k in a dense open subset of K^c has a unique Gauss decomposition $k = n^+hn^-$ where $n^+ \in N^+$, $h \in H^c$ and $n^- \in N^-$ and (2) the map $\gamma : Z \to \tau(\exp Z)$ is a holomorphic diffeomorphism from $\mathfrak{n}^+$ onto a dense open subset of $Z(p_0)$ (see [17, Chapter VIII]). Then the action of K^c on $Z(p_0)$ induces an action (defined almost everywhere) of K^c on $\mathfrak{n}^+$. We denote by $k\cdot Z$ the action of $k \in K^c$ on $Z \in \mathfrak{n}^+$. Using the diffeomorphism $K/H \simeq K^c/H^cN^-$ again, we see that for each $Z \in \mathfrak{n}^+$ there exists an element $k_Z \in K$ for which $\tau(k_Z) = \tau(\exp Z)$ or, equivalently, $k_Z \cdot 0 = Z$.

Following [22], we introduce the projections $\kappa : N^+H^cN^- \to H^c$ and $\zeta : N^+H^cN^- \to N^+$. Then, for $k \in K^c$ and $Z \in \mathfrak{n}^+$ we have $k \cdot Z = \log \zeta(k \exp Z)$. We set $(X+iY)^* = -X+iY$ for $X, Y \in \mathfrak{k}$ and we denote by $k \to k^*$ the involutive anti-automorphism of K^c which is obtained by exponentiating $X+iY \to (X+iY)^*$ to K^c. Also, let θ be the conjugation of $\mathfrak{k}^c$ with respect to $\mathfrak{k}$ and let $\tilde{\theta}$ be the automorphism of K^c for which $d\tilde{\theta} = \theta$. Then we have $\theta(X) = -X^*$ for $X \in \mathfrak{k}^c$ and $\tilde{\theta}(k) = (k^*)^{-1}$ for $k \in K^c$.

In the rest of the paper, we fix a Cartan-Weyl basis for $\mathfrak{k}^c$, $(E_\alpha)_{\alpha\in\Delta} \cup (H_\alpha)_{\alpha\in\Delta_s}$, as in [20, Chapter 5]. In particular, $\mathfrak{k}$ is spanned by the elements $E_\alpha - E_{-\alpha}$, $i(E_\alpha + E_{-\alpha})$ for $\alpha \in \Delta^+$ and iH_α for $\alpha \in \Delta_s$ and we have the property $E_\alpha^* = E_{-\alpha}$ for $\alpha \in \Delta$.

Now we describe the K-invariant measure on $Z(p_0)$. Let $d\mu_L(Z)$ be the Lebesgue measure on $\mathfrak{n}^+$ defined as follows. Let $(\alpha_k)_{1\leq k\leq n}$ be an enumeration of $\Delta^+\setminus\Delta_1^+$. Then $(E_{\alpha_k})_{1\leq k\leq n}$ is a basis for $\mathfrak{n}^+$ and we denote by $z_1=x_1+iy_1, z_2=x_2+iy_2,\dots,z_n=x_n+iy_n$ the coordinates of $Z\in\mathfrak{n}^+$ in this basis. Then we set $d\mu_L(Z)=dx_1dy_1dx_2dy_2\cdots dx_ndy_n$. Now a K-invariant measure on $\mathfrak{n}^+$ is given by $d\mu(Z)=\chi_\Lambda(\kappa(\exp Z^*\exp Z))\,d\mu_L(Z)$ where $\chi_\Lambda(h)=\mathrm{Det}_{\mathfrak{n}^+}\mathrm{Ad}(h)$ is the character of H^c corresponding to the weight $\Lambda=\sum_{\alpha\in\Delta^+\setminus\Delta_1^+}\alpha$, that is, $\Lambda=d\chi_\Lambda|_{\mathfrak{t}^c}$ (see for instance [22] or [12]). Hence, a K-invariant measure on $Z(p_0)$ is $d\tilde{\mu}=\gamma^*(d\mu)$.

The two next lemmas will be needed later. First, we reformulate [23, Lemma 1] as follows.

Lemma 2.1. *The space $\mathcal{V}:=\{v\wedge p_0\,:\,v\in V\}\subset\mathfrak{k}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$. We also have $\mathcal{V}=\{Y+\theta(Y)\,:\,Y\in\mathfrak{n}^-\}$.*

Proof: The first assertion of the lemma follows from the equality $(v\wedge p_0)(A)=p_0(A.v)=-(A.p_0)(v)$ for $A\in\mathfrak{k}$ and $v\in V$. To prove the second assertion, we note that $\mathfrak{h}$ is spanned by the elements $E_\alpha-E_{-\alpha}$, $i(E_\alpha+E_{-\alpha})$ for $\alpha\in\Delta_1^+$ and iH_α for $\alpha\in\Delta_s$. On the other hand, the space $\{Y+\theta(Y)\,:\,Y\in\mathfrak{n}^-\}$ is spanned by the elements $E_\alpha+\theta(E_\alpha)=E_\alpha-E_{-\alpha}$ and $iE_\alpha+\theta(iE_\alpha)=i(E_\alpha+E_{-\alpha})$ for $\alpha\in\Delta^+\setminus\Delta_1^+$. Recalling that $\langle E_\alpha,E_\beta\rangle=\delta_{\alpha,-\beta}$ and $\langle E_\alpha,H_\beta\rangle=0$, the result then follows. □

Observe that, for $v\in V$, one has $(v,e).(p_0,f_0)=(p_0,f_0+v\wedge p_0)$ where e denotes the identity element of K. Then, by Lemma 2.1, we may assume without loss of generality that $\xi_0=(p_0,\varphi_0)$ with $\varphi_0\in\mathfrak{h}$. We shall denote by $\mathcal{O}(\varphi_0)\subset\mathfrak{h}$ the orbit of $\varphi_0\in\mathfrak{h}$ under the adjoint action of H.

Lemma 2.2. (1) *For $k\in N^+H^cN^-$, we have*

$$\kappa(\zeta(k)^*\zeta(k))=(\kappa(k)^*)^{-1}\kappa(k^*k)\kappa(k)^{-1}$$

(2) *For $Z\in\mathfrak{n}^+$, we have $\kappa(\exp Z^*\exp Z)=\kappa(k_Z^{-1}\exp Z)^*\kappa(k_Z^{-1}\exp Z)$.*

Proof: (1) Write $k=zhy$ where $z\in N^+$, $h\in H^c$ and $y\in N^-$. Then $k^*k=y^*h^*z^*zhy$. Hence $\kappa(k^*k)=h^*\kappa(z^*z)h$. This gives the desired result.

(2) Applying (1) to $k=k_Z=\exp Zhy$ where $h\in H^c$ and $y\in N^-$, we get $\kappa(\exp Z^*\exp Z)=(h^*)^{-1}h^{-1}$. Now $k_Z^{-1}\exp Z=y^{-1}h^{-1}=h^{-1}(hy^{-1}h^{-1})$ gives $\kappa(k_Z^{-1}\exp Z)=h^{-1}$ and the result follows. □

3. Representations

In the rest of the paper, we assume that the orbit $\mathcal{O}(\varphi_0)$ is associated with a unitary irreducible representation (ρ,E) of H as in [29, Section 4]. This correspondence can be described as follows. Let λ be the highest weight of (ρ,E). Let $\varphi_0\in\mathfrak{t}$ such that $\lambda(A)=i\langle\varphi_0,A\rangle$ for each $A\in\mathfrak{t}$. Then orbit of φ_0 under the adjoint action of H is said to be associated with the representation (ρ,E).

Since $\mathcal{O}(\varphi_0)$ is integral, the orbit $\mathcal{O}(\xi_0)$ is also integral [23]. In fact, $\mathcal{O}(\xi_0)$ is associated with the unitarily induced representation

$$\tilde{\pi} = \mathrm{Ind}_{V\times H}^{G}\,(e^{ip_0}\otimes\rho).$$

By a result of G. Mackey, π is irreducible because ρ is irreducible [25]. We denote by π_0 the usual realization of $\tilde{\pi}$ defined on a Hermitian vector bundle as follows [21], [24]. We introduce the Hilbert G-bundle $L := G\times_{e^{ip_0}\otimes\rho} E$ over $Z(p_0)\simeq K/H$. Recall that an element of L is an equivalence class

$$[g,u] = \{(g.(v,h), e^{-i\langle p_0,v\rangle}\rho(h)^{-1}u)\;:\; v\in V,\; h\in H\}$$

where $g\in G$, $u\in E$ and that G acts on L by left translations: $g\,[g',u] := [g.g',u]$. The action of G on $Z(p_0)\simeq K/H$ being given by $(v,k).p = k.p$, the projection map $[(v,k),u]\to k.p_0$ is G-equivariant. The G-invariant Hermitian structure on L is given by

$$\langle [g,u],[g,u']\rangle = \langle u,u'\rangle_E$$

where $g\in G$ and u, $u'\in E$. Let $\mathcal{H}_0$ be the space of sections s of L which are square-integrable with respect to the measure $d\mu(p)$, that is,

$$\|s\|^2_{\mathcal{H}_0} = \int_{Z(p_0)} \langle s(p),\, s(p)\rangle\, d\mu(p) < +\infty.$$

Then π_0 is the action of G on $\mathcal{H}_0$ defined by

$$(\pi_0(g)\,s)(p) = g\,s(g^{-1}.p).$$

Now, following [24], we introduce an alternative realization of $\tilde{\pi}$ on a space of functions. We associate with any $s\in\mathcal{H}_0$ the function $f_s:\mathfrak{n}^+\to E$ defined by $s(\gamma(Z)) = [(0,k_Z), f_s(Z)]$. For s and s' in $\mathcal{H}_0$, we have

$$\langle s(\gamma(Z)), s'(\gamma(Z))\rangle = \langle f_s(Z),\, f_{s'}(Z)\rangle_E.$$

This implies that

$$\langle s,\, s'\rangle_{\mathcal{H}_0} = \int_{\mathfrak{n}^+} \langle f_s(Z),\, f_{s'}(Z)\rangle_E\, \delta(Z)\, d\mu_L(Z).$$

where $\delta(Z) = \chi_\Lambda(\kappa(\exp Z^*\exp Z))$ (see Section 2). This leads us to introduce the Hilbert space $\mathcal{H}^0$ of functions $f:\mathfrak{n}^+\to E$ which are square-integrable with respect to the measure $\delta(Z)\,d\mu_L(Z)$. The norm on $\mathcal{H}^0$ is defined by

$$\|f\|^2_{\mathcal{H}^0} = \int_{\mathfrak{n}^+} \langle f(Z),\, f(Z)\rangle_E\, \delta(Z)\, d\mu_L(Z).$$

Moreover, for $s \in \mathcal{H}_0$, $g = (v, k) \in G$ and $Z \in \mathfrak{n}^+$, we have

$$
\begin{aligned}
(\pi_0(g)\, s)(\gamma(Z)) &= g\, s(g^{-1}.\gamma(Z)) = g\,[(0, k_{k^{-1}.Z}), f_s(k^{-1}\cdot Z)] \\
&= [(v, k k_{k^{-1}.Z}), f_s(k^{-1}\cdot Z)] = [(0, k_Z).(k_Z^{-1}.v, k_Z^{-1} k k_{k^{-1}.Z}), f_s(k^{-1}\cdot Z)] \\
&= [(0, k_Z), e^{i\langle p_0, k_Z^{-1}.v\rangle} \rho(k_Z^{-1} k k_{k^{-1}.Z}) f_s(k^{-1}\cdot Z)].
\end{aligned}
$$

Hence we conclude that the equality

$$
\pi^0(v,k) f\,(Z) = e^{i\langle k_Z . p_0, v\rangle}\, \rho(k_Z^{-1} k k_{k^{-1}.Z}) f(k^{-1}\cdot Z) \tag{3.1}
$$

defines a unitary representation π^0 on $\mathcal{H}^0$ which is unitarily equivalent to π_0.

Now we deduce from π^0 another realization of $\tilde{\pi}$ which is more convenient for explicit computations and for the Weyl calculus. First, we extend ρ to a representation $\tilde{\rho}$ of $H^c N^-$ on E which is trivial on N^- and we note that

$$
\begin{aligned}
\rho(k_Z^{-1} k k_{k^{-1}.Z}) = \tilde{\rho}(k_Z^{-1} \exp Z) \tilde{\rho}(\exp(-Z) k \exp(k^{-1}\cdot Z)) \\
\tilde{\rho}((\exp(k^{-1}\cdot Z))^{-1} k_{k^{-1}.Z}).
\end{aligned} \tag{3.2}
$$

On the other hand, by (2) of Lemma 2.2, we have

$$
\begin{aligned}
&\langle \tilde{\rho}(k_Z^{-1} \exp Z) u,\, \tilde{\rho}(k_Z^{-1}\exp Z) u'\rangle_E \\
&\quad= \langle \tilde{\rho}(\kappa(k_Z^{-1}\exp Z)) u, \tilde{\rho}(\kappa(k_Z^{-1}\exp Z)) u'\rangle_E \\
&\quad= \langle \tilde{\rho}(\kappa(k_Z^{-1}\exp Z))^* \tilde{\rho}(\kappa(k_Z^{-1}\exp Z)) u,\, u'\rangle_E \\
&\quad= \langle \tilde{\rho}(\kappa(\exp Z^* \exp Z)) u,\, u'\rangle_E.
\end{aligned} \tag{3.3}
$$

Let us denote by $R(v) = v^{1/2}$ the square root of a positive self-adjoint operator on E. In order to simplify the notation, we set $h(Z) := \kappa(\exp Z^* \exp Z)$ and $q(Z) = R(\tilde{\rho}(h(Z)))$. Then by (3.3) we have

$$
\langle \tilde{\rho}(k_Z^{-1}\exp Z) u,\, \tilde{\rho}(k_Z^{-1}\exp Z) u'\rangle_E = \langle q(Z) u,\, q(Z) u'\rangle_E. \tag{3.4}
$$

Let us introduce the Hilbert space $\mathcal{H}$ of functions $\phi : \mathfrak{n}^+ \to E$ which are square-integrable with respect to the measure $d\mu_L(Z)$. From equations (3.1), (3.2) and (3.4) we deduce immediately that π^0 is unitarily equivalent to the representation π of G on $\mathcal{H}$ defined by

$$
\begin{aligned}
\pi(v,k)\phi\,(Z) = e^{i\langle \exp Z\, p_0, v\rangle}\, \delta(Z)^{1/2} \delta(k^{-1}\cdot Z)^{-1/2} q(Z) \\
\tilde{\rho}(\exp(-Z) k \exp(k^{-1}\cdot Z)) q(k^{-1}\cdot Z)^{-1} \phi(k^{-1}\cdot Z)
\end{aligned} \tag{3.5}
$$

the intertwining operator $f \in \mathcal{H}^0 \mapsto \phi \in \mathcal{H}$ being given by

$$
\phi(Z) = \delta(Z)^{1/2} q(Z) \tilde{\rho}(k_Z^{-1}\exp Z)^{-1} f(Z).
$$

4. Derived representation

In this section, we compute the differential $d\pi$ of the representation π of G. For $(w,A)\in\mathfrak{g}$, we can write

$$
\begin{aligned}
(d\pi(w,A)\,\phi)(Z) &= i\langle \exp Z p_0,\, w\rangle\,\phi(Z) \\
&+ \delta(Z)^{1/2}\frac{d}{dt}\delta(k(t)^{-1}\cdot Z)^{-1/2}\big|_{t=0}\phi(Z) \\
&+ q(Z)\,\frac{d}{dt}q(k(t)^{-1}\cdot Z)^{-1}\big|_{t=0}\phi(Z) \\
&+ q(Z)d\tilde\rho\left(\frac{d}{dt}\exp(-Z)k(t)\exp(k(t)^{-1}\cdot Z)\big|_{t=0}\right)\,q(Z)^{-1}\phi(Z) \\
&+ \frac{d}{dt}\phi(k(t)^{-1}\cdot Z)\big|_{t=0}
\end{aligned}
\tag{4.1}
$$

where $k(t):=\exp(tA)$. Recall that we have set $h(Z)=\kappa(\exp Z^*\exp Z)$ and $q(Z)=R(\tilde\rho(h(Z)))$ where R denotes square root. The following lemma can be easily deduced from results of [12]. We denote by $p_{\mathfrak{h}^c}$, $p_{\mathfrak{n}^+}$ and $p_{\mathfrak{n}^-}$ the projections of $\mathfrak{k}^c$ on $\mathfrak{h}^c$, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ associated with the direct decomposition $\mathfrak{k}^c=\mathfrak{h}^c\oplus\mathfrak{n}^+\oplus\mathfrak{n}^-$.

Lemma 4.1. *Let $A\in\mathfrak{k}$ and $k(t)=\exp(tA)$. Then we have*

$$
\begin{aligned}
\frac{d}{dt}\tilde\rho\big(\exp(-Z)k(t)\exp(k(t)^{-1}\cdot Z)\big)\big|_{t=0} &= \frac{d}{dt}\tilde\rho\big(\kappa(k(t)^{-1}\exp Z)\big)^{-1}\big|_{t=0} \\
&= d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\big)
\end{aligned}
\tag{4.2}
$$

and

$$
\frac{d}{dt}\,k(t)^{-1}\cdot Z\big|_{t=0} = -\frac{\mathrm{ad}\,Z}{1-e^{-\,\mathrm{ad}\,Z}}\,p_{\mathfrak{n}^+}(e^{-\,\mathrm{ad}\,Z}\,A).
\tag{4.3}
$$

Proof: Immediate consequence of [12], Proposition 4.1 and Proposition 5.1. □

Lemma 4.2. *Let $A\in\mathfrak{k}$ and $k(t)=\exp(tA)$. Then we have*

$$
\begin{aligned}
&\frac{d}{dt}q(Z)\,q(k(t)^{-1}\cdot Z)^{-1}\big|_{t=0} = -\,\mathrm{Ad}\,\tilde\rho(h(Z))^{1/2}(\mathrm{id}+\mathrm{Ad}\,\tilde\rho(h(Z))^{-1/2})^{-1} \\
&\big(d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\big)+\mathrm{Ad}\,\tilde\rho(h(Z))^{-1}d\tilde\rho\big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*\big)\big)
\end{aligned}
\tag{4.4}
$$

and similarly

$$
\begin{aligned}
&\frac{d}{dt}\delta(Z)^{1/2}\delta(k(t)^{-1}\cdot Z)^{-1/2}\big|_{t=0} \\
&= -\frac{1}{2}\big(\Lambda(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A))+\Lambda(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*)\big).
\end{aligned}
\tag{4.5}
$$

Proof: First, note that $\exp(k(t)^{-1} \cdot Z) = \zeta(k(t)^{-1} \exp Z)$. Then, applying Lemma 2.2, we have

$$h(k(t)^{-1} \cdot Z) = \kappa(k(t)^{-1} \exp Z)^{*-1} h(Z) \kappa(k(t)^{-1} \exp Z)^{-1}.$$

Hence

$$q(k(t)^{-1} \cdot Z)^{-1} = R\left(\tilde{\rho}\big(\kappa(k(t)^{-1} \exp Z) h(Z)^{-1} \kappa(k(t)^{-1} \exp Z)^{*}\big)\right)$$

and

$$\begin{aligned}
&\frac{d}{dt} q(k(t)^{-1} \cdot Z)^{-1}\Big|_{t=0} \\
&= dR(\tilde{\rho}(h(Z)^{-1})) d\tilde{\rho}(h(Z)^{-1}) \left(\frac{d}{dt} \kappa(k(t)^{-1} \exp Z) h(Z)^{-1} \kappa(k(t)^{-1} \exp Z)^{*}\Big|_{t=0}\right) \\
&= dR(\tilde{\rho}(h(Z)^{-1})) \left(d\tilde{\rho}(U) \tilde{\rho}(h(Z))^{-1}\right)
\end{aligned}$$

where

$$U := \frac{d}{dt}\, \kappa(k(t)^{-1} \exp Z) h(Z)^{-1} \kappa(k(t)^{-1} \exp Z)^{*} h(Z)\Big|_{t=0}.$$

Applying Lemma 4.1, we find

$$\begin{aligned}
U =& \frac{d}{dt} \kappa(k(t)^{-1} \exp Z)\Big|_{t=0} + \operatorname{Ad}(h(Z)^{-1}) \frac{d}{dt} \kappa(k(t)^{-1} \exp Z)^{*}\Big|_{t=0} \\
=& - p_{\mathfrak{h}^c}(\operatorname{Ad} \exp(-Z)\, A) - \operatorname{Ad}(h(Z)^{-1}) p_{\mathfrak{h}^c}(\operatorname{Ad} \exp(-Z)\, A)^{*}.
\end{aligned} \tag{4.6}$$

On the other hand, using the equality

$$dR(u)v = (\mathrm{id} + \operatorname{Ad} u^{1/2})^{-1} (v u^{-1/2})$$

for any positive definite self-adjoint operator u on E, we get

$$\begin{aligned}
q(Z)\, &\frac{d}{dt} q(k(t)^{-1} \cdot Z)^{-1}\Big|_{t=0} \\
&= \tilde{\rho}(h(Z))^{1/2} (\mathrm{id} + \operatorname{Ad} \tilde{\rho}(h(Z))^{-1/2})^{-1} \left(d\tilde{\rho}(U) \tilde{\rho}(h(Z))^{-1/2}\right) \\
&= \operatorname{Ad} \tilde{\rho}(h(Z))^{1/2} (\mathrm{id} + \operatorname{Ad} \tilde{\rho}(h(Z))^{-1/2})^{-1} (d\tilde{\rho}(U)).
\end{aligned}$$

Taking equation (4.6) into account, we then obtain (4.4). Moreover, writing (4.6) for $\tilde{\rho} = \chi_\Lambda$, we also obtain (4.5). □

Proposition 4.1. *For $(w, A) \in \mathfrak{g}$ and $\phi \in C_0^\infty(\mathfrak{n}^+, E)$ we have*

$$\begin{aligned}
d\pi(w, A)\,\phi(Z) &= \frac{d}{dt}\,(\pi(tw, \exp(tA))\phi)(Z)\big|_{t=0}\\
&= i\langle \exp Z p_0,\, w\rangle\,\phi(Z)\\
&\quad - \frac{1}{2}\big(\Lambda(p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)) + \Lambda(p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)^*)\big)\phi(Z)\\
&\quad + (\mathrm{id} + \operatorname{Ad}\tilde\rho(h(Z))^{-1/2})^{-1}\big(d\tilde\rho\big(p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)\big)\\
&\quad - \operatorname{Ad}\tilde\rho(h(Z))^{-1/2}d\tilde\rho\big(p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)^*\big)\big)\phi(Z)\\
&\quad - \partial_Z\phi(Z, Z^*)\left(\frac{\operatorname{ad} Z}{1 - e^{-\operatorname{ad} Z}}\,p_{\mathfrak{n}^+}(e^{-\operatorname{ad} Z}\,A)\right)\\
&\quad - \partial_{Z^*}\phi(Z, Z^*)\left(\frac{\operatorname{ad} Z}{1 - e^{-\operatorname{ad} Z}}\,p_{\mathfrak{n}^+}(e^{-\operatorname{ad} Z}\,A)\right)^*.
\end{aligned}$$

Proof: Using Lemma 4.1 and Lemma 4.2 and writing

$$\begin{aligned}
&q(Z)d\tilde\rho\left(\frac{d}{dt}\exp(-Z)k(t)\exp(k(t)^{-1}\cdot Z)\big|_{t=0}\right)\,q(Z)^{-1}\\
&= \operatorname{Ad}\tilde\rho(h(Z))^{1/2}d\tilde\rho\big(p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)\big)\\
&= \operatorname{Ad}\tilde\rho(h(Z))^{1/2}(\mathrm{id} + \operatorname{Ad}\tilde\rho(h(Z))^{-1/2})^{-1}(\mathrm{id} + \operatorname{Ad}\tilde\rho(h(Z))^{-1/2})\\
&\quad d\tilde\rho\big(p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)\big)
\end{aligned}$$

we see that

$$\begin{aligned}
&q(Z)d\tilde\rho\left(\frac{d}{dt}\exp(-Z)k(t)\exp(k(t)^{-1}\cdot Z)\big|_{t=0}\right)\,q(Z)^{-1}\\
&\quad + q(Z)\,\frac{d}{dt}q(k(t)^{-1}\cdot Z)^{-1}\big|_{t=0}\\
&= (\mathrm{id} + \operatorname{Ad}\tilde\rho(h(Z))^{-1/2})^{-1}\Big(d\tilde\rho\big(p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)\big)\\
&\quad - \operatorname{Ad}\tilde\rho(h(Z))^{-1/2}d\tilde\rho\big(p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)^*\big)\Big).
\end{aligned}$$

The result then follows. □

5. Dequantization

We first introduce the Berezin calculus on the orbit $\mathcal{O}(\varphi_0)$. The Berezin calculus associates with each operator B on the finite-dimensional complex vector space E a complex-valued function $s(B)$ on the orbit $\mathcal{O}(\varphi_0)$ called the symbol of the operator B (see [4]). The following properties of the Berezin calculus can be found in [13], [5], [12].

Proposition 5.1. (1) *The map $B \to s(B)$ is injective.*

(2) *For each operator B on E, we have $s(B^*) = \overline{s(B)}$.*

(3) *For $\varphi \in \mathcal{O}(\varphi_0)$, $h \in H$ and for an operator B on E, we have*

$$s(B)(\mathrm{Ad}(h)\varphi) = s(\rho(h)^{-1}B\rho(h))(\varphi).$$

(4) *For $A \in \mathfrak{h}$ and $\varphi \in \mathcal{O}(\varphi_0)$, we have $s(d\rho(A))(\varphi) = i\langle \varphi, A\rangle$.*

Now we introduce the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$. We first recall the definition of the Berezin-Weyl calculus on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ (see [9]). We say that a smooth function $f : (T, S, \varphi) \to f(T, S, \varphi)$ is a symbol on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ if for each $(T, S) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ the function $\varphi \to f(T, S, \varphi)$ is the symbol in the Berezin calculus on $\mathcal{O}(\varphi_0)$ of an operator on E denoted by $\hat{f}(T, S)$. A symbol f on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ is called an $\mathcal{S}$-symbol if the function $\hat{f}$ belongs to the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with values in $\mathrm{End}(E)$. Now we consider the Weyl calculus for $\mathrm{End}(E)$-valued functions [18]. For any $\mathcal{S}$-symbol f on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ we define an operator $\mathcal{W}(f)$ on the Hilbert space $L^2(\mathbb{R}^{2n}, E)$ by

$$(5.1)\quad (\mathcal{W}(f)\phi)(T) = (2\pi)^{-2n} \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{i\langle S,S'\rangle} \hat{f}\big(T + \frac{1}{2}S, S'\big)\, \phi(T+S)\, dS\, dS'$$

for $\phi \in C_0^\infty(\mathbb{R}^{2n}, E)$.

The Weyl-Berezin calculus can be extended to much larger classes of symbols (see for instance [18]). Here we are only concerned with a class of polynomial symbols. We say that a symbol f on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ is a P-symbol if the function $\hat{f}(T, S)$ is polynomial in S. Let f be the P-symbol defined by $f(T, S, \varphi) = u(T)S^\alpha$ where $u \in C^\infty(\mathbb{R}^{2n}, E)$ and $S^\alpha := s_1^{\alpha_1} s_2^{\alpha_2} \dots s_{2n}^{\alpha_{2n}}$ for each multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{2n})$. Then we have (see [26]):

$$(5.2)\qquad (\mathcal{W}(f)\phi)(T) = (i\partial_S)^\alpha \big(u(T + \frac{1}{2}S)\, \phi(T+S)\big)\Big|_{S=0}.$$

In particular, if $f(T, S, \varphi) = u(T)$ then

$$(5.3)\qquad (\mathcal{W}(f)\phi)(T) = u(T)\, \phi(T)$$

and if $f(T, S, \varphi) = u(T)s_k$ then

$$(5.4)\qquad (\mathcal{W}(f)\phi)(T) = i\left(\frac{1}{2}(\partial_{t_k} u)(T)\, \phi(T) + u(T)(\partial_{t_k}\phi)(T)\right).$$

The correspondence $f \mapsto \mathcal{W}(f)$ is called the Berezin-Weyl calculus on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$. In order to obtain the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$, we just rewrite the Berezin-Weyl calculus on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ in complex coordinates.

Let $j : \mathbb{R}^{2n} \to \mathfrak{n}^+$ be the map defined by

$$j(t_1, t_2, \ldots, t_n, t'_1, t'_2, \ldots, t'_n) = \sum_{k=1}^{n} (t_k + it'_k) E_{\alpha_k}$$

and let $\tilde{j} : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0) \to \mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ be the map given by

$$\tilde{j}(T, S, \varphi) = (j(T), j(S), \varphi).$$

We say that a function $f : \mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0) \to \mathbb{C}$ is a symbol (resp. an S-symbol, a P-symbol) on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ if $f \circ \tilde{j}$ is a symbol (resp. an S-symbol, a P-symbol) on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ and we define the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ by

$$W(f)\phi \circ j = \mathcal{W}(f \circ \tilde{j})\phi$$

for each $\phi \in C_0^\infty(\mathfrak{n}^+, E)$. Let $Y = \sum_{k=1}^{n} y_k E_{\alpha_k}$ be the decomposition of $Y \in \mathfrak{n}^+$ in the basis (E_{α_k}). An easy computation shows that if $f(Z, Y, \varphi) = u(Z)$ then

$$(W(f)\phi)(Z) = u(Z)\phi(Z), \tag{5.5}$$

if $f(Z, Y, \varphi) = u(Z) y_k$ then

$$(W(f)\phi)(Z) = i(\partial_{\overline{z}_k} u)(Z)\phi(Z) + 2iu(Z)(\partial_{\overline{z}_k}\phi)(Z) \tag{5.6}$$

and if $f(Z, Y, \varphi) = u(Z)\overline{y}_k$ then

$$(W(f)\phi)(Z) = i(\partial_{z_k} u)(Z)\phi(Z) + 2iu(Z)(\partial_{z_k}\phi)(Z). \tag{5.7}$$

In order to dequantize the derived representation $d\pi$, that is, to calculate the Berezin-Weyl symbol of the operators $d\pi(X)$ ($X \in \mathfrak{g}$), we need the following lemma.

Lemma 5.1. *For $A \in \mathfrak{k}^c$ let $u_A : \mathfrak{n}^+ \to \mathfrak{n}^+$ be the holomorphic map defined by*

$$u_A(Z) = \frac{\operatorname{ad} Z}{1 - e^{-\operatorname{ad} Z}}\, p_{\mathfrak{n}^+}(e^{-\operatorname{ad} Z} A).$$

Then

$$\operatorname{Tr}_{\mathfrak{n}^+}\, du_A(Z) = \Lambda(p_{\mathfrak{h}^c}(e^{-\operatorname{ad} Z} A)).$$

Proof: Since $\mathfrak{n}^+$ is a nilpotent Lie algebra, we can write $u_A(Z) = s(\operatorname{ad} Z) p_{\mathfrak{n}^+}(e^{-\operatorname{ad} Z} A)$ where $s(z) = \sum_{k=0}^{N} a_k z^k$ is a polynomial. For $Y \in \mathfrak{n}^+$ and $Z \in \mathfrak{n}^+$, we have

$$\begin{aligned} du_A(Z)(Y) &= \frac{d}{dt} s(\operatorname{ad}(Z + tY))\Big|_{t=0} p_{\mathfrak{n}^+}(e^{-\operatorname{ad} Z} A) \\ &\quad + s(\operatorname{ad} Z) p_{\mathfrak{n}^+}\left(\frac{d}{dt} \operatorname{Ad}(\exp(-Z - tY))A\Big|_{t=0} \right). \end{aligned}$$

Now

$$\frac{d}{dt}s(\operatorname{ad}(Z+tY))\Big|_{t=0} = \sum_{k=0}^{N} a_k \frac{d}{dt}(\operatorname{ad} Z + t \operatorname{ad} Y)^k\Big|_{t=0}$$
$$= \sum_{k=0}^{N} a_k \Big(\sum_{r=0}^{k-1} (\operatorname{ad} Z)^r \operatorname{ad} Y (\operatorname{ad} Z)^{k-r-1} \Big).$$

Then, since for each $r = 0, 1, \ldots, k-1$ the endomorphism of $\mathfrak{n}^+$ defined by

$$Y \to (\operatorname{ad} Z)^r \operatorname{ad} Y (\operatorname{ad} Z)^{k-r-1} p_{\mathfrak{n}^+}(e^{-\operatorname{ad} Z} A)$$
$$= -(\operatorname{ad} Z)^r \operatorname{ad}\big((\operatorname{ad} Z)^{k-r-1} p_{\mathfrak{n}^+}(e^{-\operatorname{ad} Z} A)\big)(Y)$$

is nilpotent, the endomorphism of $\mathfrak{n}^+$ given by

$$Y \to \frac{d}{dt}s(\operatorname{ad}(Z+tY))\Big|_{t=0} p_{\mathfrak{n}^+}(e^{-\operatorname{ad} Z} A)$$

has trace zero. On the other hand we have

$$\frac{d}{dt}\operatorname{Ad}(\exp(-Z-tY))\,A\Big|_{t=0}$$
$$= \frac{d}{dt}\operatorname{Ad}\left(\exp(-Z)\exp(Z+tY)\right)^{-1}\operatorname{Ad}\exp(-Z)\,A\Big|_{t=0}$$
$$= -\operatorname{ad}\left(\frac{1-e^{-\operatorname{ad} Z}}{\operatorname{ad} Z}Y\right)\operatorname{Ad}\exp(-Z)\,A$$
$$= \operatorname{ad}\left(\operatorname{Ad}\exp(-Z)\,A\right)\left(\frac{1-e^{-\operatorname{ad} Z}}{\operatorname{ad} Z}\right)Y.$$

The trace of the endomorphism of $\mathfrak{n}^+$ defined by

$$Y \to s(\operatorname{ad} Z)p_{\mathfrak{n}^+}\left(\frac{d}{dt}\operatorname{Ad}(\exp(-Z-tY))\,A\Big|_{t=0}\right)$$

is then

$$\operatorname{Tr}_{\mathfrak{n}^+}\left(s(\operatorname{ad} Z)p_{\mathfrak{n}^+}\circ\operatorname{ad}(\operatorname{Ad}\exp(-Z)\,A)\,\frac{1-e^{-\operatorname{ad} Z}}{\operatorname{ad} Z}\right)$$
$$= \operatorname{Tr}_{\mathfrak{n}^+}\left(\frac{1-e^{-\operatorname{ad} Z}}{\operatorname{ad} Z}s(\operatorname{ad} Z)\,p_{\mathfrak{n}^+}\circ\operatorname{ad}(\operatorname{Ad}\exp(-Z)\,A)\right)$$
$$= \operatorname{Tr}_{\mathfrak{n}^+}\left(p_{\mathfrak{n}^+}\circ\operatorname{ad}(\operatorname{Ad}\exp(-Z)\,A)\right).$$

Consequently, the lemma will be proved if we show that, for each A in $\mathfrak{k}^c$, we have

$$\operatorname{Tr}_{\mathfrak{n}^+}(p_{\mathfrak{n}^+}\circ\operatorname{ad} A) = \Lambda(p_{\mathfrak{h}^c}(A)).$$

If $A \in \mathfrak{n}^+$ then $p_{\mathfrak{n}^+}\circ\operatorname{ad} A = \operatorname{ad} A$ is a nilpotent endomorphism of $\mathfrak{n}^+$. Thus $\operatorname{Tr}_{\mathfrak{n}^+}(p_{\mathfrak{n}^+}\circ\operatorname{ad} A) = 0$. If $A \in \mathfrak{n}^-$ then for each $k = 1, 2, \ldots, n$ we have $\operatorname{ad} A(E_{\alpha_k}) \in$

$\mathfrak{h}^c + \sum_{\alpha<\alpha_k} \mathfrak{k}_{\alpha_k}$ and we also find that $\mathrm{Tr}_{\mathfrak{n}^+}(p_{\mathfrak{n}^+} \circ \mathrm{ad}\, A) = 0$. Finally, if $A \in \mathfrak{h}^c$ then

$$\mathrm{Tr}_{\mathfrak{n}^+}(p_{\mathfrak{n}^+} \circ \mathrm{ad}\, A) = \mathrm{Tr}_{\mathfrak{n}^+}(\mathrm{ad}\, A) = \sum_{k=1}^{n} \alpha_k(A) = \Lambda(A).$$

This ends the proof of the lemma. □

We consider the Cartan decomposition $K^c = K \exp(i\mathfrak{k})$ [17, Chapter VI]. For $k \in K^c$ we can write $k = up$ where $u \in K$ and $p \in \exp(i\mathfrak{k})$. Since $u^*u = e$ and $p^* = p$ we have $k^*k = p^*u^*up = p^2$ and we can introduce the notation $p =: (k^*k)^{1/2}$.

Proposition 5.2. *For $X = (w, A) \in \mathfrak{g}$, the Berezin-Weyl symbol of the operator $-id\pi(X)$ is the P-symbol f_X on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ given by*

$$\begin{aligned} &f_X(Z, Y, \varphi) = \langle \exp Z p_0, w \rangle \\ &+ \langle \varphi,\, (\mathrm{id} + \mathrm{Ad}(h(Z))^{-1/2})^{-1} \big(p_{\mathfrak{h}^c}(\mathrm{Ad} \exp(-Z)\, A) \\ &- \mathrm{Ad}(h(Z))^{-1/2} p_{\mathfrak{h}^c}(\mathrm{Ad} \exp(-Z)\, A)^*\big)\rangle \\ &+ \mathrm{Re}\langle u_A(Z),\, Y^* \rangle \end{aligned}$$

where

$$u_A(Z) = \frac{\mathrm{ad}\, Z}{1 - e^{-\mathrm{ad}\, Z}}\, p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\, Z}\, A).$$

Proof: Write $u_A(Z) = \sum_{k=1}^{n} u_k(Z) E_{\alpha_k}$. Then, by using (5.5), (5.6) and (5.7), we see that the operator

$$\phi \mapsto i(\partial_Z \phi)(Z, Z^*)(u_A(Z)) = i \sum_{k=1}^{n} u_k(Z) \partial_{z_k} \phi$$

has symbol

$$\frac{1}{2} \sum_{k=1}^{n} u_k(Z) \overline{y}_k - \frac{1}{2} i \sum_{k=1}^{n} \partial_{z_k} u_k = \frac{1}{2} \langle u_A(Z), Y^* \rangle - \frac{1}{2} i \Lambda(p_{\mathfrak{h}^c}(e^{-\mathrm{ad}\, Z}\, A)).$$

Similarly, the operator

$$\phi \mapsto i(\partial_{Z^*} \phi)(Z, Z^*)(u_A(Z)^*) = i \sum_{k=1}^{n} \overline{u_k(Z)} \partial_{\overline{z}_k} \phi$$

has symbol

$$\frac{1}{2} \sum_{k=1}^{n} \overline{u_k(Z)} y_k - \frac{1}{2} i \sum_{k=1}^{n} \partial_{\overline{z}_k} \overline{u}_k = \frac{1}{2} \overline{\langle u_A(Z), Y^* \rangle} - \frac{1}{2} i \overline{\Lambda(p_{\mathfrak{h}^c}(e^{-\mathrm{ad}\, Z}\, A))}.$$

The result follows from Proposition 4.1 and Proposition 5.1(3). □

6. Adapted Weyl correspondence

In this section we show how the dequantization procedure used in Section 5 allows us to obtain an explicit symplectomorphism from $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ onto a dense open subset of $\mathcal{O}(\xi_0)$. Using this symplectomorphism we then construct an adapted Weyl correspondence on $\mathcal{O}(\xi_0)$. We retain the notation from the previous sections. Moreover, for $A \in \mathfrak{k}^c$, we set $\operatorname{Re}(A) = \frac{1}{2}(A + \theta(A))$.

Recall that f_X denotes the Berezin-Weyl symbol of the operator $-id\pi(X)$ for $X \in \mathfrak{g}$. Since the map $X \to f_X(Z, Y, \varphi)$ is linear there exists a map Ψ from $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ to $\mathfrak{g}^* \simeq V^* \oplus \mathfrak{k}$ such that

$$f_X(Z, Y, \varphi) = \langle \Psi(Z, Y, \varphi),\, X\rangle \tag{6.1}$$

for each $X \in \mathfrak{g}$ and each $(Z, Y, \varphi) \in \mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$. From Proposition 5.2 we deduce a precise expression for Ψ.

Proposition 6.1. *For $(Z, Y, \varphi) \in \mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$, we have*

$$\begin{aligned}\Psi(Z, Y, \varphi) = \Bigg(\exp Z p_0,\, \operatorname{Re}\operatorname{Ad}(\exp Z)\Bigg[p_{\mathfrak{n}^-}\bigg(\frac{\operatorname{ad} Z}{1 - e^{\operatorname{ad} Z}}\,\theta(Y)\bigg)\\ + 2(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi\Bigg]\Bigg).\end{aligned}$$

Proof: For $(w, A) \in \mathfrak{g}$, we transform the expression for $f_X(Z, Y, \varphi)$ given in Proposition 5.2 as follows. First we have

$$\begin{aligned}&\langle \varphi, (\mathrm{id} + \operatorname{Ad}(h(Z))^{-1/2})^{-1} p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)\rangle\\ &\quad= \langle (\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi,\, p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)\rangle\\ &\quad= \langle (\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi,\, \operatorname{Ad}\exp(-Z)\,A\rangle\\ &\quad= \langle \operatorname{Ad}(\exp Z)(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi,\, A\rangle.\end{aligned}$$

On the other hand, by using the properties $(\operatorname{Ad}(k^{-1})B)^* = \operatorname{Ad}(k^*)B^*$ for $k \in \mathfrak{k}^c$ and $B \in \mathfrak{k}^c$ and $\langle B_1^*, B_2^*\rangle = \overline{\langle B_1, B_2\rangle}$ for B_1 and B_2 in $\mathfrak{k}^c$, we have

$$\begin{aligned}&\langle \varphi, (\mathrm{id} + \operatorname{Ad}(h(Z))^{-1/2})^{-1}\operatorname{Ad}(h(Z))^{-1/2} p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)^*\rangle\\ &\quad= \langle (\mathrm{id} + \operatorname{Ad}(h(Z))^{-1/2})^{-1}\varphi,\, p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)^*\rangle\\ &\quad= -\overline{\langle (\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi,\, p_{\mathfrak{h}^c}(\operatorname{Ad}\exp(-Z)\,A)\rangle}\\ &\quad= -\overline{\langle (\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi,\, \operatorname{Ad}\exp(-Z)\,A\rangle}\\ &\quad= -\overline{\langle \operatorname{Ad}(\exp Z)\,(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi,\, A\rangle}.\end{aligned}$$

Then

$$\Big\langle\varphi,(\mathrm{id}+\mathrm{Ad}(h(Z))^{-1/2})^{-1}\Big(p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)\\ -\mathrm{Ad}(h(Z))^{-1/2}p_{\mathfrak{h}^c}(\mathrm{Ad}\exp(-Z)\,A)^*\Big)\Big\rangle\\ =\Big\langle 2\,\mathrm{Re}\Big(\,\mathrm{Ad}(\exp Z)\,(\mathrm{id}+\mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi\Big),\,A\Big\rangle.$$

Moreover we have

$$\begin{aligned}\langle u_A(Z),\,Y^*\rangle &= \Big\langle \frac{\mathrm{ad}\,Z}{1-e^{-\mathrm{ad}\,Z}}p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\,Z}A),\,Y^*\Big\rangle\\ &=\Big\langle p_{\mathfrak{n}^+}(e^{-\mathrm{ad}\,Z}A),\,-\frac{\mathrm{ad}\,Z}{1-e^{\mathrm{ad}\,Z}}Y^*\Big\rangle\\ &=\Big\langle e^{-\mathrm{ad}\,Z}A,\,p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\,Z}{1-e^{\mathrm{ad}\,Z}}\theta(Y)\Big)\Big\rangle\\ &=\Big\langle A,\,e^{\mathrm{ad}\,Z}p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\,Z}{1-e^{\mathrm{ad}\,Z}}\theta(Y)\Big)\Big\rangle.\end{aligned}$$

The result therefore follows. □

Let ω_0 and ω_1 be the Kirillov 2-forms on $\mathcal{O}(\xi_0)$ and $\mathcal{O}(\varphi_0)$, respectively. Denote by $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ the Poisson brackets associated with ω_0 and ω_1. We endow $\mathfrak{n}^+\times\mathfrak{n}^+$ with the symplectic form

$$\omega_2:=\frac{1}{2}\sum_{k=1}^{n}(dz_k\wedge d\overline{y}_k+d\overline{z}_k\wedge dy_k).$$

The corresponding Poisson bracket on $C^\infty(\mathfrak{n}^+\times\mathfrak{n}^+)$ is

$$\{f,\,g\}_2:=2\sum_{k=1}^{n}\left(\partial f_{z_k}\partial_{\overline{y}_k}g-\partial_{\overline{y}_k}f\partial_{z_k}g+\partial f_{\overline{z}_k}\partial_{y_k}g-\partial_{y_k}f\partial_{\overline{z}_k}g\right).$$

We endow the product $\mathfrak{n}^+\times\mathfrak{n}^+\times\mathcal{O}(\varphi_0)$ with the symplectic form $\omega:=\omega_2\otimes\omega_1$ and we denote by $\{\cdot,\cdot\}$ the corresponding Poisson bracket. Let $u,\,v\in C^\infty(\mathfrak{n}^+\times\mathfrak{n}^+)$ and $a,\,b\in C^\infty(\mathcal{O}(\varphi_0))$. Then, for $f(Z,Y,\varphi)=u(Z,Y)a(\varphi)$ and $g(Z,Y,\varphi)=v(Z,Y)b(\varphi)$ we have

$$\{f,\,g\}=u(Z,Y)v(Z,Y)\{a,\,b\}_1+a(\varphi)b(\varphi)\{u,\,v\}_2.$$

Lemma 6.1. *Suppose that f and g are two P-symbols on $\mathfrak{n}^+\times\mathfrak{n}^+\times\mathcal{O}(\varphi_0)$ of the form*

$$u(Z)+\langle v(Z),\varphi\rangle+\sum_{k=1}^{n}(w_k(Z)y_k+w'_k(Z)\overline{y}_k)$$

where $u \in C^\infty(\mathfrak{n}^+)$, $v \in C^\infty(\mathfrak{n}^+, \mathfrak{k}^c)$ *and* w_k, $w'_k \in C^\infty(\mathfrak{n}^+)$ *for* $k = 1, 2, \ldots, n$. *Then we have*

$$[W(f), W(g)] = -i\, W(\{f, g\}).$$

Proof: By using (5.5), (5.6) and (5.7), one can prove the result by a direct computation. One can also deduce it from Lemma 6.2 of [9] by using the fact that $\tilde{j}$ is a symplectomorphism from $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathcal{O}(\varphi_0)$ endowed with its natural symplectic structure onto $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$. □

Let $\tilde{\mathcal{O}}(\xi_0)$ be the dense open subset of $\mathcal{O}(\xi_0)$ defined by

$$\tilde{\mathcal{O}}(\xi_0) = \{(v, k).(p_0, \varphi_0) \,:\, v \in V,\, k \in K \cap N^+ H^c N^-\}.$$

Proposition 6.2. *The map* Ψ *is a symplectomorphism from* $(\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0), \omega)$ *onto* $(\tilde{\mathcal{O}}(\xi_0), \omega_0)$.

Proof: (1) First, we show that for any $\xi \in \tilde{\mathcal{O}}(\xi_0)$ there exists a unique element (Z, Y, φ) in $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ such that $\Psi(Y, Z, \varphi) = \xi$. Let $\xi \in \tilde{\mathcal{O}}(\xi_0)$. Write $\xi = (v, k).(p_0, \varphi_0)$ where $v \in V$ and $k \in K \cap N^+ H^c N^-$. If $\Psi(Y, Z, \varphi) = \xi$ then

$$(0, k)^{-1}.\Psi(Z, Y, \varphi) = (p_0, \varphi_0 + (k^{-1}.v) \wedge p_0). \tag{6.2}$$

This gives $k^{-1} \exp Z p_0 = p_0$ or, equivalently, $k^{-1} \exp Z \in H^c N^-$ and we can write $k^{-1} \exp Z = yh$ where $y \in N^-$ and $h \in H^c$. Thus, equation (6.2) implies

$$\begin{aligned} &2 \operatorname{Re} \operatorname{Ad}(yh)(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi + \operatorname{Re} \operatorname{Ad}(yh) p_{\mathfrak{n}^-}\Big(\frac{\operatorname{ad} Z}{1 - e^{\operatorname{ad} Z}}\theta(Y)\Big) \\ &\qquad = \varphi_0 + (k^{-1}.v) \wedge p_0. \end{aligned} \tag{6.3}$$

Hence, noting that the element $Y_{Z,\varphi}$ defined by

$$Y_{Z,\varphi} := \operatorname{Ad}(y)\operatorname{Ad}(h)(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi - \operatorname{Ad}(h)(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi$$

belongs to $\mathfrak{n}^-$ and applying Lemma 2.1, we see that equation (6.3) is equivalent to

$$\begin{cases} \text{(E1)} \qquad \operatorname{Re}\Big(Y_{Z,\varphi} + \operatorname{Ad}(yh) p_{\mathfrak{n}^-}\Big(\frac{\operatorname{ad} Z}{1 - e^{\operatorname{ad} Z}}\theta(Y)\Big)\Big) = (k^{-1}.v) \wedge p_0 \\ \text{(E2)} \qquad 2 \operatorname{Re}\big(\operatorname{Ad}(h)(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi\big) = \varphi_0. \end{cases}$$

But we have

$$\begin{aligned} &2 \operatorname{Re}\big(\operatorname{Ad}(h)(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi\big) \\ &\quad = \operatorname{Ad}(h)(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi + \operatorname{Ad}(\theta(h))(\mathrm{id} + \operatorname{Ad}(\theta(h(Z)))^{1/2})^{-1}\theta(\varphi) \\ &\quad = \operatorname{Ad}(h)(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi + \operatorname{Ad}(h^*)^{-1}(\mathrm{id} + \operatorname{Ad}(h(Z))^{-1/2})^{-1}\varphi \\ &\quad = \operatorname{Ad}(h)(\mathrm{id} + \operatorname{Ad}(h^* h)^{-1}\operatorname{Ad}(h(Z))^{1/2})(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi \end{aligned}$$

and, since $h^*h = h(Z)$, we can write

$$\begin{aligned}
&2\,\mathrm{Re}\big(\mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi\big)\\
&\quad = \mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{-1/2})(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi\\
&\quad = \mathrm{Ad}(h)\,\mathrm{Ad}(h(Z))^{-1/2}\varphi.
\end{aligned}$$

Finally, writing $h = up$, $u \in K$, $p = (h^*h)^{1/2} \in \exp(i\mathfrak{k})$ for the Cartan decomposition of h, we obtain

$$2\,\mathrm{Re}\big(\mathrm{Ad}(h)(\mathrm{id} + \mathrm{Ad}(h(Z))^{1/2})^{-1}\varphi\big) = \mathrm{Ad}(u)\varphi$$

where $u \in H^c \cap K = H$. Consequently, equation (E2) gives $\varphi = \mathrm{Ad}(u^{-1})\varphi_0$. Since $Z = \log\zeta(k)$, we have shown that Z and φ are unique. In order to verify that Y is also unique, we have just to use equation (E1) and the following facts: (1) the map $Y \to \mathrm{Re}(Y)$ from $\mathfrak{n}^+$ to the ortho-complement of $\mathfrak{h}$ in $\mathfrak{k}$ is injective and (2) the map

$$Y \to p_{\mathfrak{n}^-}\Big(\frac{\mathrm{ad}\,Z}{1 - e^{\mathrm{ad}\,Z}}\theta(Y)\Big)$$

is a bijection from $\mathfrak{n}^+$ onto $\mathfrak{n}^-$, the inverse bijection being

$$U \to \theta\left(p_{\mathfrak{n}^-}\Big(\frac{1 - e^{\mathrm{ad}\,Z}}{\mathrm{ad}\,Z}U\Big)\right).$$

It is also clear that the element (Y, Z, φ) obtained below satisfies the equation $\Psi(Y, Z, \varphi) = \xi$. Moreover, by similar considerations, we show that Ψ takes values in $\tilde{\mathcal{O}}(\xi_0)$ and we can conclude that Ψ is a bijection from $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ onto $\tilde{\mathcal{O}}(\xi_0)$.

(2) For $X \in \mathfrak{g}$, we denote by $\tilde{X}$ the function on $\tilde{\mathcal{O}}(\xi_0)$ defined by $\tilde{X}(\xi) = \langle \xi, X\rangle$. Observe that $f_X = \tilde{X} \circ \Psi$.

Let X and Y in $\mathfrak{g}$. Then by Proposition 5.2 and Lemma 6.1 we have

$$[W(f_X),\, W(f_Y)] = -iW(\{f_X,\, f_Y\}).$$

But we also have

$$[W(f_X),\, W(f_Y)] = [-id\pi(X), -id\pi(Y)] = -d\pi([X,\, Y]) = -iW(f_{[X,Y]}).$$

Hence $f_{[X,Y]} = \{f_X,\, f_Y\}$. Since $\widetilde{[X,Y]} = \{\tilde{X}, \tilde{Y}\}_0$, we obtain

$$\big\{\tilde{X}, \tilde{Y}\big\}_0 \circ \Psi = \big\{\tilde{X} \circ \Psi,\, \tilde{Y} \circ \Psi\big\}.$$

This implies that $\Psi^*(\omega_0) = \omega$. Since the 2-form ω is non-degenerate, we also have that the map Ψ is regular. Finally, Ψ is a symplectomorphism. $\square$

Remark 6.1. The map Ψ might define a symplectomorphism from $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$ onto $\tilde{\mathcal{O}}(\xi_0)$ even when the orbit $\mathcal{O}(\varphi_0)$ is not assumed to be integral.

Now, we are in position to construct an adapted Weyl transform on $\mathcal{O}(\xi_0)$ by transferring to $\mathcal{O}(\xi_0)$ the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$. We say that a smooth function f on $\mathcal{O}(\xi_0)$ is a symbol (resp. a P-symbol, an S-symbol) on $\mathcal{O}(\xi_0)$ if $f \circ \Psi$ is a symbol (resp. a P-symbol, an S-symbol) for the Berezin-Weyl calculus on $\mathfrak{n}^+ \times \mathfrak{n}^+ \times \mathcal{O}(\varphi_0)$.

Proposition 6.3. *Let $\mathcal{A}$ be the space of P-symbols on $\mathcal{O}(\xi_0)$ and let $\mathcal{B}$ be the space of differential operators on $\mathfrak{n}^+$ with coefficients in $C^\infty(\mathfrak{n}^+, E)$. Then the map $\tilde{W} : \mathcal{A} \to \mathcal{B}$ defined by the $\tilde{W}(f) = W(f \circ \Psi)$ is an adapted Weyl correspondence in the sense of Definition* 1.1.

Proof: The properties (1), (2) and (3) of Definition 1.1 are clearly satisfied with $D = C_0^\infty(\mathfrak{n}^+, E)$. The property (4) follows from the corresponding properties for the Berezin calculus (see Proposition 5.1) and for the usual Weyl calculus [18]. Finally, the property (5) is an immediate consequence of Proposition 5.2 and Proposition 6.1. ∎

7. Final remarks and examples

7.1. If ρ is a character of H then $\mathcal{O}(\varphi_0)$ reduces to the point φ_0 and Ψ is given by

$$\Psi(Z, Y, \varphi) = \Big(\exp Z p_0, \operatorname{Re} \operatorname{Ad}(\exp Z)\Big[\varphi_0 + p_{\mathfrak{n}^-}\Big(\frac{\operatorname{ad} Z}{1 - e^{\operatorname{ad} Z}}\,\theta(Y)\Big)\Big]\Big). \tag{7.1}$$

7.2. If $Z(p_0) \simeq G/H$ is a symmetric space then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are abelian and $[\mathfrak{n}^+, \mathfrak{n}^-] \subset \mathfrak{h}^c$ (see [17, Lemma VII 2.16]). Thus, for each Y and Z in $\mathfrak{n}^+$, we have

$$p_{\mathfrak{n}^-}\Big(\frac{\operatorname{ad} Z}{1 - e^{\operatorname{ad} Z}}\,\theta(Y)\Big) = \theta(Y).$$

Hence the expression for Ψ is

$$\Psi(Z, Y, \varphi) = \Big(\exp Z p_0, \operatorname{Re} \operatorname{Ad}(\exp Z)\,\Big[2(\mathrm{id} + \operatorname{Ad}(h(Z))^{1/2})^{-1}\varphi + \theta(Y)\Big]\Big)\,. \tag{7.2}$$

7.3. In this subsection, we consider the case when V is equal to the Lie algebra $\mathfrak{k}$ of K and σ is the adjoint action of K on $\mathfrak{k}$. We identify $V^* = \mathfrak{k}^*$ to $V = \mathfrak{k}$ by means of the Killing form. Then we have $v \wedge p = [v, p]$ for each $v \in V = \mathfrak{k}$ and each $p \in V^* \simeq \mathfrak{k}$. The coadjoint action of G on $\mathfrak{g}^*$ is thus given by

$$(v, k).(p, f) = (\operatorname{Ad}(k)p, \operatorname{Ad}(k)f + [v, \operatorname{Ad}(k)p]).$$

Moreover, if $\xi_0 = (p_0, \varphi_0)$ is an element of $\mathfrak{g}^*$ such that $p_0 \neq 0$ and $\mathcal{O}(\varphi_0)$ is integral then the stabilizer H of p_0 in K is the centralizer of the torus of K generated by $\exp p_0$ and one can apply to $\mathcal{O}(\xi_0)$ the results of the previous sections.

7.4. We illustrate here the situation described in the previous subsection by the following example. We take $K = SU(m+n)$ and p_0 to be the element of $\mathfrak{k}$ defined by

$$p_0 = i \begin{pmatrix} -nI_m & 0 \\ 0 & mI_n \end{pmatrix}.$$

The torus T_1 generated by $\exp p_0$ consists of the matrices

$$\begin{pmatrix} e^{ia}I_m & 0 \\ 0 & e^{ib}I_n \end{pmatrix} \qquad a,\, b \in \mathbb{R}, \qquad (e^{ia})^m(e^{ib})^n = 1.$$

The torus T_1 is contained in the maximal torus $T \subset K$ consisting of the matrices

$$\mathrm{Diag}(e^{ia_1}, e^{ia_2}, \ldots, e^{ia_{m+n}}), \qquad a_1, a_2, \ldots, a_{m+n} \in \mathbb{R}, \qquad \prod_{k=1}^{m+n} e^{ia_k} = 1.$$

Moreover, the subgroup $H = \{k \in K \,:\, k.p_0 = p_0\}$ is the centralizer of T_1 in K and consists of the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad A \in U(m),\, D \in U(n), \qquad \mathrm{Det}\, A.\, \mathrm{Det}\, D = 1,$$

that is, we have $H = S(U(m) \times U(n))$. The complexification T^c of T has Lie algebra

$$\mathfrak{t}^c = \left\{ X = \mathrm{Diag}(x_1, x_2, \ldots, x_{m+n}) \ :\ x_k \in \mathbb{C},\, \sum_{k=1}^{m+n} x_k = 0 \right\}.$$

The set of roots of $\mathfrak{t}^c$ on $\mathfrak{g}^c$ is $\lambda_i - \lambda_j$ for $1 \le i \ne j \le m+n$ where $\lambda_i(X) = x_i$ for $X \in \mathfrak{t}^c$ as above. The set of roots of $\mathfrak{t}^c$ on $\mathfrak{h}^c$ is $\lambda_i - \lambda_j$ for $1 \le i \ne j \le m$ and $m+1 \le i \ne j \le m+n$. We take the set of positive roots Δ^+ to be $\lambda_i - \lambda_j$ for $1 \le i < j \le m+n$ and the set of positive roots Δ_1^+ to be $\lambda_i - \lambda_j$ for $1 \le i < j \le m$ and $m+1 \le i < j \le m+n$. Then we have

$$N^+ = \left\{ \begin{pmatrix} I_m & Z \\ 0 & I_n \end{pmatrix} : Z \in M_{mn}(\mathbb{C}) \right\},\, N^- = \left\{ \begin{pmatrix} I_m & 0 \\ Y & I_n \end{pmatrix} : Y \in M_{nm}(\mathbb{C}) \right\}.$$

We identify $\mathfrak{n}^+$ to $M_{mn}(\mathbb{C})$ by means of the map

$$Z \mapsto \tilde{Z} = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}.$$

We also have

$$H^c = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} A \in M_m(\mathbb{C}), D \in M_n(\mathbb{C}),\, \mathrm{Det}\, A.\, \mathrm{Det}\, D = 1 \right\}.$$

We easily see that the $N^+H^cN^-$-decomposition of a matrix $k \in K^c$ is given by

$$k = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_m & BD^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_m & 0 \\ D^{-1}C & I_n \end{pmatrix}.$$

Observe that a matrix $k \in K^c$ have such a decomposition if and only if $\operatorname{Det} D \neq 0$. In particular we have $K \subset N^+H^cN^-$. Moreover, we deduce from the preceding decomposition that the action of K^c on $\mathfrak{n}^+$ is given by

$$k \cdot Z = (AZ + B)(CZ + D)^{-1}, \qquad k = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Note that, for $k \in K^c$, we have $k^* = \overline{k}^t$ (conjugate transpose of k) and $\tilde{\theta}(k) = (\overline{k}^t)^{-1}$. For $X \in \mathfrak{k}^c$, we have $X^* = \overline{X}^t$ and $\theta(X) = -\overline{X}^t$.

We are here in the situation of the subsection 7.2 and Ψ is then given by equation (7.2) with

$$h(\tilde{Z}) = \begin{pmatrix} (I_m + ZZ^\star)^{-1} & 0 \\ 0 & I_n + Z^\star Z \end{pmatrix}$$

and

$$\exp \tilde{Z}\, p_0 = i \begin{pmatrix} (I_m + ZZ^\star)^{-1}(mZZ^\star - nI_m) & (m+n)Z(I_n + Z^\star Z)^{-1} \\ (m+n)(I_n + Z^\star Z)^{-1}Z^\star & (mI_n - nZ^\star Z)(I_n + Z^\star Z)^{-1} \end{pmatrix}.$$

In particular, in the case when $m = n = 1$, we can take

$$\varphi_0 = \begin{pmatrix} -ia & 0 \\ 0 & ia \end{pmatrix}$$

where $a \in \mathbb{N} \setminus (0)$. We get

$$\exp \tilde{Z} p_0 = \frac{1}{\sqrt{1 + |z|^2}} i \begin{pmatrix} |z|^2 - 1 & 2z \\ 2\overline{z} & 1 - |z|^2 \end{pmatrix}$$

and

$$\Psi(\tilde{Z}, \tilde{Y}) = \left(\exp \tilde{Z} p_0, \frac{1}{2} \begin{pmatrix} -2ai + y\overline{z} - \overline{y}z & 2aiz + y + \overline{y}z^2 \\ 2ai\overline{z} - y - y\overline{z}^2 & 2ai - y\overline{z} + \overline{y}z \end{pmatrix} \right).$$

References

[1] Ali S.T., Englis M., *Quantization methods: a guide for physicists and analysts*, Rev. Math. Phys. **17** (2005), no. 4, 391–490.

[2] Arnal D., Cortet J.-C., *Nilpotent Fourier Transform and Applications*, Lett. Math. Phys. **9** (1985), 25–34.

[3] Baguis P., *Semidirect products and the Pukansky condition*, J. Geom. Phys. **25** (1998), 245–270.

[4] Berezin F.A., *Quantization*, Math. USSR Izv. 8, 5 (1974), 1109–1165.

[5] Cahen B., *Deformation program for principal series representations*, Lett. Math. Phys. **36** (1996), 65–75.

[6] Cahen B., *Quantification d'une orbite massive d'un groupe de Poincaré généralisé*, C.R. Acad. Sci. Paris Sér. I Math. **325** (1997), 803–806.

[7] Cahen B., *Quantification d'orbites coadjointes et théorie des contractions*, J. Lie Theory **11** (2001), 257–272.

[8] Cahen B., *Contractions of $SU(1,n)$ and $SU(n+1)$ via Berezin quantization*, J. Anal. Math. **97** (2005), 83–102.

[9] Cahen B., *Weyl quantization for semidirect products*, Differential Geom. Appl. **25** (2007), 177–190.

[10] Cahen B., *Weyl quantization for principal series*, Beiträge Algebra Geom. **48** (2007), no. 1, 175–190.

[11] Cahen B., *Berezin quantization for discrete series*, preprint Univ. Metz (2008), to appear in Beiträge Algebra Geom.

[12] Cahen B., *Berezin quantization on generalized flag manifolds*, preprint Univ. Metz (2008), to appear in Math. Scand.

[13] Cahen M., Gutt S., Rawnsley J., *Quantization on Kähler manifolds I, Geometric interpretation of Berezin quantization*, J. Geom. Phys. **7** (1990), 45–62.

[14] Cotton P., Dooley A.H., *Contraction of an adapted functional calculus*, J. Lie Theory **7** (1997), 147–164.

[15] Folland B., *Harmonic Analysis in Phase Space*, Princeton Univ. Press, Princeton, 1989.

[16] Gotay M., *Obstructions to quantization*, in Mechanics: From Theory to Computation (Essays in Honor of Juan-Carlos Simo), J. Nonlinear Science Editors, Springer, NewYork, 2000, pp. 271–316.

[17] Helgason S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Graduate Studies in Mathematics, Vol. 34, American Mathematical Society, Providence, Rhode Island, 2001.

[18] Hörmander L., *The Analysis of Linear Partial Differential Operators III. Pseudodifferential Operators*, Grundlehren der Mathematischen Wissenschaften, 274, Springer, Berlin-Heidelberg-NewYork, 1985.

[19] Kirillov A.A., *Elements of the Theory of Representations*, Grundlehren der Mathematischen Wissenschaften, 220, Springer, Berlin-Heidelberg-New York, 1976.

[20] Kirillov A.A., *Lectures on the Orbit Method*, Graduate Studies in Mathematics, Vol. 64, American Mathematical Society, Providence, Rhode Island, 2004.

[21] Kostant B., *Quantization and unitary representations*, in Modern Analysis and Applications, Lecture Notes in Mathematics 170, Springer, Berlin-Heidelberg-New York, 1970, pp. 87–207.

[22] Neeb K.-H., *Holomorphy and Convexity in Lie Theory*, de Gruyter Expositions in Mathematics, Vol. 28, Walter de Gruyter, Berlin, New York, 2000.

[23] Rawnsley J.H., *Representations of a semi direct product by quantization*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 345–350.

[24] Simms D.J., *Lie Groups and Quantum Mechanics*, Lecture Notes in Mathematics, 52, Springer, Berlin-Heidelberg-New York, 1968.

[25] Taylor M.E., *Noncommutative Harmonis Analysis*, Mathematical Surveys and Monographs, Vol. 22, American Mathematical Society, Providence, Rhode Island, 1986.

[26] Voros A., *An algebra of pseudo differential operators and the asymptotics of quantum mechanics*, J. Funct. Anal. **29** (1978), 104–132.

[27] Wallach N.R., *Harmonic Analysis on Homogeneous Spaces*, Pure and Applied Mathematics, Vol. 19, Marcel Dekker, New York, 1973.

[28] Wildberger N.J., *Convexity and unitary representations of a nilpotent Lie group*, Invent. Math. **89** (1989), 281–292.
[29] Wildberger N.J., *On the Fourier transform of a compact semisimple Lie group*, J. Austral. Math. Soc. A **56** (1994), 64–116.

Université de Metz, UFR-MIM, Département de mathématiques, LMMAS, ISGMP-Bât. A, Ile du Saulcy 57045, Metz cedex 01, France
Email: cahen@univ-metz.fr

(*Received* January 2, 2009)