Regular methods of summability in some locally convex spaces

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Abstract. Suppose that X is a Fréchet space, $\langle a_{ij}\rangle$ is a regular method of summability and (x_i) is a bounded sequence in X. We prove that there exists a subsequence (y_i) of (x_i) such that: either (a) all the subsequences of (y_i) are summable to a common limit with respect to $\langle a_{ij}\rangle$; or (b) no subsequence of (y_i) is summable with respect to $\langle a_{ij}\rangle$. This result generalizes the Erdös-Magidor theorem which refers to summability of bounded sequences in Banach spaces. We also show that two analogous results for some ω_1-locally convex spaces are consistent to ZFC.

Keywords: Fréchet space, regular method of summability, summable sequence, Galvin-Prikry theorem, Erdös-Magidor theorem

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1. Introduction, preliminaries

The results of the present paper are motivated by the Erdös-Magidor theorem [4], concerning summability of bounded sequences in Banach spaces. In Section 2, we generalize the Erdös-Magidor theorem for Fréchet spaces. This result is based on Galvin-Prikry theorem [5], as that of Erdös-Magidor. In Section 3 we show that two analogous results, for some ω_1-locally convex spaces, are consistent to ZFC. These are based on the work of B. Balcar, J. Pelant and P. Simon given in [1] and also on a theorem of S. Plewik [7], concerning unions of completely Ramsey sets.

Let X be a topological vector space; denote by τ the topology of X. A sequence (x_n) in X is called *τ-Cauchy* if for every neighborhood V of $0 \in X$ there exists $n_0 \in \mathbb{N}$ such that $x_n - x_m \in V$ whenever $n, m \geq n_0$. If d is an invariant metric on X which induces the topology τ, then obviously, (x_n) is τ-Cauchy if and only if (x_n) is d-Cauchy. The space X is said to be *sequentially complete* if every Cauchy sequence in X converges to a point of X. A family $\mathcal{P}$ of seminorms on X is called *separating* if for every $x \in X$ with $x \neq 0$, there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$. The *local topological weight* of X is defined to be the least cardinal number α such that there is a basis $\mathcal{B}$ of neighborhoods of $0 \in X$ with $\mathrm{card}(\mathcal{B}) = \alpha$. The topological vector space X is called a *Fréchet space* if it is locally convex and its topology is induced by a complete and invariant metric. A locally convex space is called an ω_1-locally convex space if its local weight is not greater than ω_1.

Suppose that X is an ω_1-locally convex space. Then we can find a basis $\mathcal{B}$ of neighborhoods of $0 \in X$, consisting of open, convex and balanced sets, with $\mathrm{card}(\mathcal{B}) \leq \omega_1$. For every $U \in \mathcal{B}$ we write p_U for the Minkowski functional corresponding to the set U, that is, the map $p_U : X \to \mathbb{R}$ given by $p_U(x) = \inf\{t > 0 \mid x \in tU\}$. Then p_U is a continuous seminorm on X and $U = \{x \in X \mid p_U(x) < 1\}$. The topology induced on X by the family $\mathcal{P} = \{p_U \mid U \in \mathcal{B}\}$ of seminorms, is the topology of X and $\mathrm{card}(\mathcal{P}) \leq \omega_1$. For the basic theory of locally convex spaces, we refer to [8].

If M is an infinite subset of $\mathbb{N}$, let $[M]^\omega$ denote the set of all infinite subsets of M. Let N be an infinite subset of $\mathbb{N}$ and α a finite subset of $\mathbb{N}$. We set $\alpha < N$ if $\alpha \neq \emptyset$ and $\max \alpha < \min N$, or $\alpha = \emptyset$. Moreover, for an infinite subset M of $\mathbb{N}$ and a finite subset α of $\mathbb{N}$, we set

$$[\alpha, M] = \{\alpha \cup L \mid L \in [M]^\omega \ \&\ \alpha < L\}\,.$$

A subset $\mathcal{A}$ of $[\mathbb{N}]^\omega$ is called *completely Ramsey* if for every $M \in [\mathbb{N}]^\omega$ and every finite subset α of $\mathbb{N}$ with $\alpha < M$, there is $N \in [M]^\omega$ such that: either $[\alpha, N] \subseteq \mathcal{A}$ or $[\alpha, N] \cap \mathcal{A} = \emptyset$. Considering on $[\mathbb{N}]^\omega$ the topology of pointwise convergence, the *Galvin-Prikry* theorem [5] (see also [9]) is the following.

Theorem 1.1. *Let $\mathcal{A}$ be a Borel subset of $[\mathbb{N}]^\omega$. Then $\mathcal{A}$ is completely Ramsey.*

The distributivity number of the quotient algebra $\mathcal{P}(\omega)/\,\mathrm{fin}$ is denoted by $\mathfrak{h}$. This notion was introduced and studied by Balcar, Pelant and Simon in [1]. They proved that $\mathfrak{h}$ is a regular cardinal with $\omega_1 \leq \mathfrak{h} \leq c$, and that the value of $\mathfrak{h}$ depends on the axioms of set theory. In particular, there are models of ZFC set theory in which $\mathfrak{h} = \omega_2$. Therefore, the assumption that $\mathfrak{h} = \omega_2$, is consistent to ZFC axioms.

A topological characterisation of the completely Ramsey sets was given by E. Ellentuck [3] (see also [6]). S. Plewik [7], using this characterisation, proved the following.

Theorem 1.2. *The union of less than $\mathfrak{h}$ completely Ramsey sets is completely Ramsey.*

It follows that the assumption that the union of ω_1 completely Ramsey sets is completely Ramsey, is consistent to ZFC axioms. We will use the next consequence of this and of Theorem 1.1.

Theorem 1.3. *Assume that $\mathfrak{h} = \omega_2$. Then the intersection of less or equal to ω_1 Borel subsets of $[\mathbb{N}]^\omega$ is completely Ramsey.*

An infinite matrix $\langle a_{ij}\rangle_{i,j\in\mathbb{N}}$ of real numbers is called a *regular method of summability* if, given a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of a sequentially complete locally convex space X converging to $x \in X$, the sequence $x'_i = \sum_{j=1}^{\infty} a_{ij}x_j$ is well-defined and also converges to x. A sequence (x_i) in a sequentially complete locally convex space is called *summable with respect to* $\langle a_{ij}\rangle$ if the sequence (x'_i), where $x'_i = \sum_{j=1}^{\infty} a_{ij} x_j$, is well-defined and converges. The following proposition characterizes the regular methods of summability.

Proposition 1.1. *Let $\langle a_{ij} \rangle$ be an infinite matrix of real numbers. The following assertions are equivalent.*

(1) *$\langle a_{ij} \rangle$ is a regular summability method.*
(2) *The following conditions hold:*
   (a) $\sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$;
   (b) $\lim_{i\to\infty} a_{ij} = 0$ *for every j and*
   (c) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = 1$.

Proof: The implication (1)⇒(2) is well-known (see [2, p. 75]). The converse implication for a sequentially complete locally convex space X is proved as in the case of a Banach space, by considering all the seminorms belonging to a family $\mathcal{P}$ of seminorms defining the topology of X. We give this proof for completeness. Suppose that (x_i) is a sequence in X converging to x and let $p \in \mathcal{P}$ and $\epsilon > 0$ be given. Then it is clear that the sequence (x'_i), with $x'_i = \sum_{j=1}^{\infty} a_{ij} x_j$, is well-defined. Condition (c) implies that there exists $i_1 \in \mathbb{N}$ such that for $i \geq i_1$,

$$p(x)\Big| \sum_{j=1}^{\infty} a_{ij} - 1 \Big| < \epsilon/3.$$

Since the sequence (x_j) converges, there is $K_1 < \infty$ such that $p(x_j - x) < K_1$ for all j. We set $K_2 = \sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$. Since $\lim_{j\to\infty} p(x_j - x) = 0$ there is $j_0 \in \mathbb{N}$ such that for $j \geq j_0$, $p(x_j - x) < \frac{\epsilon}{3K_2}$. Condition (b) implies that $\lim_{i\to\infty} \sum_{j=1}^{j_0} |a_{ij}| = 0$, hence there is $i_2 \in \mathbb{N}$ such that for $i \geq i_2$,

$$\sum_{j=1}^{j_0} |a_{ij}| < \frac{\epsilon}{3K_1}.$$

For $i \geq \max\{i_1, i_2\}$,

$$\begin{aligned} p(x'_i - x) &= p\Big(\sum_{j=1}^{\infty} a_{ij} x_j - x \Big) \\ &= p\Big(\sum_{j=1}^{\infty} a_{ij} x_j - \sum_{j=1}^{\infty} a_{ij} x + \sum_{j=1}^{\infty} a_{ij} x - x \Big) \\ &\leq p\Big(\sum_{j=1}^{\infty} a_{ij} \, (x_j - x) \Big) + p\,(x) \Big| \sum_{j=1}^{\infty} a_{ij} - 1 \Big| \\ &\leq p\Big(\sum_{j=1}^{j_0} a_{ij} \, (x_j - x) \Big) + p\Big(\sum_{j>j_0} a_{ij} \, (x_j - x) \Big) + \frac{\epsilon}{3} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{j_0} |a_{ij}| p\,(x_j - x) + \sum_{j>j_0} |a_{ij}| p\,(x_j - x) + \frac{\epsilon}{3} \\
&< K_1 \sum_{j=1}^{j_0} |a_{ij}| + \frac{\epsilon}{3K_2} \sum_{j>j_0} |a_{ij}| + \frac{\epsilon}{3} \\
&< K_1 \frac{\epsilon}{3K_1} + \frac{\epsilon}{3K_2} K_2 + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

□

2. Summability in Fréchet spaces

In this section we prove the following theorem.

Theorem 2.1. *Suppose that X is a Fréchet space, $\langle a_{ij} \rangle_{i,j \in \mathbb{N}}$ is a regular method of summability and $(x_i)_{i \in \mathbb{N}}$ is a bounded sequence in X. Then there exists a subsequence (y_i) of (x_i) such that: either*

(a) *all subsequences of (y_i) are summable, with respect to $\langle a_{ij} \rangle$; or*
(b) *no subsequence of (y_i) is summable, with respect to $\langle a_{ij} \rangle$.*

Moreover, in the first case we can find a subsequence (z_i) of (y_i) such that all its subsequences are summable to the same limit.

This theorem, in case X is a Banach space, is the Erdös-Magidor theorem [4]. In the following by a basis of neighborhoods of $0 \in X$ we shall mean a countable basis $\mathcal{B}$ of neighborhoods of $0 \in X$ consisting of open, convex and balanced sets. For the proof we need the following two lemmas.

Lemma 2.1. *Let (z_j) be a bounded sequence in the Fréchet space X. For every i, we define the function*

$$f_i : [\mathbb{N}]^\omega \to X \;\; by$$

$$A = \{k_1 < k_2 < \ldots\} \longmapsto f_i(A) = \sum_{j=1}^{\infty} a_{ij} z_{k_j}.$$

Then f_i is continuous.

Proof: Fix $A = \{k_1 < k_2 < \ldots\} \in [\mathbb{N}]^\omega$. Let $U = U_n \in \mathcal{B}$ be a basic neighborhood of $0 \in X$ and let $p = p_n$ be the corresponding Minkowski functional. Since (z_j) is bounded, the sequence $(p(z_j))$ is also bounded, so there exists $K < \infty$ such that $p(z_j) < K$ for every $j \in \mathbb{N}$. Furthermore, it follows from Proposition 1.1 that $\sum_{j=1}^{\infty} |a_{ij}| < \infty$, hence there exists ζ such that $\sum_{j>\zeta} |a_{ij}| < \frac{1}{2K}$. Put

$$\mathcal{C} = \{B = \{m_1 < m_2 < \ldots\} \in [\mathbb{N}]^\omega \mid m_j = k_j \;\; \text{for} \;\; j \leq \zeta\}.$$

Clearly, $\mathcal{C}$ is an open neighborhood of A in $[\mathbb{N}]^\omega$. We show that $f_i[\mathcal{C}] \subseteq f_i(A)+U$. Indeed, if $B \in \mathcal{C}$,

$$\begin{aligned} p\left(f_i(B)-f_i(A)\right) &= p\Big(\sum_{j=1}^{\infty} a_{ij} z_{m_j} - \sum_{j=1}^{\infty} a_{ij} z_{k_j}\Big) \\ &= p\Big(\sum_{j=1}^{\infty} a_{ij}\left(z_{m_j}-z_{k_j}\right)\Big) \\ &\leq \sum_{j>\zeta} \left|a_{ij}\right| p\big(z_{m_j}-z_{k_j}\big) \\ &\leq \sum_{j>\zeta} \left|a_{ij}\right| 2K \ < 1, \end{aligned}$$

and hence $f_i(B)-f_i(A) \in U$, that is $f_i(B) \in f_i(A)+U$. Thus f_i is continuous at A; since A is arbitrary, the proof is complete. □

Lemma 2.2. *Let (z_j) be a bounded sequence in the Fréchet space X which is summable to z with respect to $\langle a_{ij}\rangle$ and let $v_1,\dots,v_N \in X$. Then the sequence $(v_1,\dots,v_N,z_{N+1},\dots)$ is also summable to z with respect to $\langle a_{ij}\rangle$.*

Proof: For every i we set

$$w_i = \sum_{j=1}^{N} a_{ij} v_j + \sum_{j>N} a_{ij} z_j.$$

We need to prove that the sequence (w_i) converges to z. Indeed, let $\mathcal{P}$ be a family of seminorms on X, defining the topology of X, and let $p \in \mathcal{P}$. Then for every i we have:

$$\begin{aligned} p(w_i-z) &= p\Big(\sum_{j=1}^{N} a_{ij} v_j + \sum_{j>N} a_{ij} z_j \ - z\Big) \\ &= p\Big(\sum_{j=1}^{N} a_{ij} v_j - \sum_{j=1}^{N} a_{ij} z_j + \sum_{j=1}^{\infty} a_{ij} z_j \ - z\Big) \\ &\leq p\Big(\sum_{j=1}^{N} a_{ij}\big(v_j-z_j\big)\Big) + p\Big(\sum_{j=1}^{\infty} a_{ij} z_j \ - z\Big) \\ &\leq \sum_{j=1}^{N} |a_{ij}| p\big(v_j-z_j\big) + p\Big(\sum_{j=1}^{\infty} a_{ij} z_j \ - z\Big). \end{aligned}$$

Now $\lim_{i\to\infty} p(\sum_{j=1}^{\infty} a_{ij} z_j \ - z) = 0$ and it follows from condition (b) of Proposition 1.1 that $\lim_{i\to\infty} \sum_{j=1}^{N} |a_{ij}| p(v_j-z_j) = 0$. Thus $\lim_{i\to\infty} p(w_i-z)=0$ and the result follows since $p \in \mathcal{P}$ is arbitrary. □

Proof of Theorem 2.1: Let $\mathcal{B} = \{U_l \mid l \in \mathbb{N}\}$ be a basis of neighborhoods of $0 \in X$, let $\mathcal{P} = \{p_l \mid l \in \mathbb{N}\}$ be the corresponding family of Minkowski functional and let d be a complete and invariant metric on X which induces the topology τ of X. Consider the set:

$$\mathcal{A} = \left\{ A = \{k_1 < k_2 < \dots\} \in [\mathbb{N}]^\omega \mid \left(x_{k_i}\right) \text{ is summable with respect to } \langle a_{ij} \rangle \right\}.$$

Claim 1. The set $\mathcal{A}$ is a Borel subset of $[\mathbb{N}]^\omega$.
Indeed, observe that

$$\begin{aligned}
\{k_1 < k_2 < \dots\} \in \mathcal{A} \quad &\Leftrightarrow \quad \left(x_{k_i}\right) \text{ is summable with respect to } \langle a_{ij} \rangle \\
&\Leftrightarrow \quad x'_i = \sum_{j=1}^{\infty} a_{ij} x_{k_j} \text{ converges in } X \\
&\Leftrightarrow \quad (x'_i) \text{ converges with respect to the metric } d \\
&\Leftrightarrow \quad (x'_i) \text{ is } d\text{-Cauchy} \\
&\Leftrightarrow \quad (x'_i) \text{ is } \tau\text{-Cauchy} \\
&\Leftrightarrow \quad (\forall\, U_l \in \mathcal{B})\, (\exists\, s \in \mathbb{N})\, [(\forall\, n, m \geq s)\, ((x'_n - x'_m) \in U_l)]\,.
\end{aligned}$$

Therefore,

$$\mathcal{A} = \bigcap_{l \in \mathbb{N}} \bigcup_{s \in \mathbb{N}} \bigcap_{n,m \geq s} \mathcal{D}_{l,n,m},$$

where

$$\mathcal{D}_{l,n,m} = \left\{ \{k_1 < k_2 < \dots\} \in [\mathbb{N}]^\omega \mid \left(\sum_{j=1}^{\infty} a_{nj} x_{k_j} - \sum_{j=1}^{\infty} a_{mj} x_{k_j} \right) \in U_l \right\}.$$

By Lemma 2.1, the set $\mathcal{D}_{l,n,m}$ is open, being the inverse image of the open set U_l by the continuous function $f_n - f_m$ (here f_n and f_m are as in Lemma 2.1). Hence the set $\mathcal{A}$ is Borel.

By the Galvin-Prikry theorem, there is $M = \{k_1 < k_2 < \dots\} \in [\mathbb{N}]^\omega$ such that: either $[M]^\omega \subseteq \mathcal{A}$, or $[M]^\omega \cap \mathcal{A} = \emptyset$. Therefore, for the sequence $z_i = x_{k_i}$, either

(I) every subsequence of (z_i) is summable with respect to $\langle a_{ij} \rangle$; or

(II) no subsequence of (z_i) is summable with respect to $\langle a_{ij} \rangle$.

It remains to prove that in case (I) we can find a subsequence of (z_i) all of whose subsequences are summable to the same limit. Let $Z = \overline{\operatorname{span}}\{z_i \mid i \in \mathbb{N}\}$, be the closed linear span of (z_i). Then (Z, d) is a separable metric space. Choose a countable cover $\{B^1_n \mid n \in \mathbb{N}\}$ of Z consisting of open balls of radius 1. Consider the following subset of $[\mathbb{N}]^\omega$:

$$\mathcal{F} = \Big\{ A = \{k_1 < k_2 < \dots\} \in [\mathbb{N}]^\omega \mid \text{ the subsequence } (z_{k_i}) \text{ is summable to some point of the ball } B^1_1 \Big\}.$$

Claim 2. $\mathcal{F}$ is a Borel subset of $[\mathbb{N}]^\omega$.
Indeed, we have:

$$\begin{aligned} A = \{k_1 < k_2 < \ldots\} \in \mathcal{F} \quad &\Leftrightarrow \quad (z_{k_i}) \text{ is summable to some point of the ball } B_1^1 \\ &\Leftrightarrow \quad \text{the limit of } z_i' = \sum_{j=1}^{\infty} a_{ij} z_{k_j} \text{ belongs to the ball } B_1^1 \\ &\Leftrightarrow \quad (\exists\, k \in \mathbb{N}\, \exists\, l \in \mathbb{N})\, (\forall\, i \geq l)\, \big[d\,(z_i', z) < 1 - \frac{1}{k}\big], \end{aligned}$$

where z is the center of the ball B_1^1. Therefore,

$$\mathcal{F} = \bigcup_{k \in \mathbb{N}} \bigcup_{l \in \mathbb{N}} \bigcap_{i \geq l} \mathcal{G}_{k,i},$$

where

$$\mathcal{G}_{k,i} = \left\{ \{k_1 < k_2 < \ldots\} \in [\mathbb{N}]^\omega \mid d\left(\sum_{j=1}^{\infty} a_{ij} z_{k_j}, z \right) < 1 - \frac{1}{k} \right\}.$$

By Lemma 2.1, the set $\mathcal{G}_{k,i}$ is open, being the inverse image of some open set by a continuous function. Hence the set $\mathcal{F}$ is Borel.

The Galvin-Prikry theorem, now implies that there exists $M_1 \in [\mathbb{N}]^\omega$ such that: either $[M_1]^\omega \subseteq \mathcal{F}$, or $[M_1]^\omega \cap \mathcal{F} = \emptyset$, that is, either

- each subsequence of $(z_i)_{i \in M_1}$ is summable to a point in the ball B_1^1; or
- each subsequence of $(z_i)_{i \in M_1}$ is summable to a point outside the ball B_1^1.

Repeating the same argument we find a sequence, $\mathbb{N} \supseteq M_1 \supseteq M_2 \supseteq \ldots$, of infinite subsets of $\mathbb{N}$ such that for each k, either

(1) each subsequence of $(z_i)_{i \in M_k}$ is summable to a point of the ball B_k^1; or
(2) each subsequence of $(z_i)_{i \in M_k}$ is summable to a point outside the ball B_k^1.

If each M_k is given its natural order, we let $L_1 = \{l_1^1 < l_2^1 < \ldots\}$ be the diagonal sequence, where l_k^1 is the k-th term of M_k.

Claim 3. There is $k_1 \in \mathbb{N}$ such that condition (1) holds for M_{k_1}.
Indeed, let us suppose that for all k, every subsequence of $(z_i)_{i \in M_k}$ is summable to a point outside the ball B_k^1. The sequence $(z_{l_j^1})$, being a subsequence of (z_i), is summable, say to $z \in Z$. If $M_k = \{m_1 < m_2 < \ldots\}$, then, by Lemma 2.2, the sequence $(z_{m_1}, \ldots z_{m_{k-1}}, z_{l_k^1}, z_{l_{k+1}^1}, \ldots)$ is also summable to z. Since this is a subsequence of $(z_i)_{i \in M_k}$, we obtain $z \notin B_k^1$. Since this happens for all k, we have reached a contradiction.

Using Lemma 2.2 again, we find that every subsequence of $(z_i)_{i \in L_1}$ is summable to a point of the ball $B_{k_1}^1$.

Now consider a countable cover $\{B_n^2 \mid n \in \mathbb{N}\}$ of the ball $B_{k_1}^1$, consisting of open balls in $B_{k_1}^1$ of radius $1/2$. Repeat the previous procedure to the sequence $(z_i)_{i \in L_1}$

to obtain an infinite subset L_2 of L_1 and a $k_2 \in \mathbb{N}$ such that every subsequence of $(z_i)_{i\in L_2}$ is summable to a point of the ball $B^2_{k_2}$.

We inductively construct a sequence $\mathbb{N} \supseteq L_1 \supseteq L_2 \supseteq \ldots$, of infinite subsets of $\mathbb{N}$ and a sequence $B^1_{k_1} \supseteq B^2_{k_2} \supseteq \ldots$, of open balls in Z, such that for every n the following properties hold:

(i) $\operatorname{diam}(B^n_{k_n}) \leq \frac{2}{n}$

(ii) every subsequence of $(z_i)_{i\in L_n}$ is summable to a point of the ball $B^n_{k_n}$. Clearly, $\operatorname{diam}(\bigcap_{n=1}^{\infty} B^n_{k_n}) \leq \operatorname{diam}(B^n_{k_n}) \leq \frac{2}{n}$ for every n. Thus, $\operatorname{diam}(\bigcap_{n=1}^{\infty} B^n_{k_n}) = 0$, that is, the set $\bigcap_{n=1}^{\infty} B^n_{k_n}$ is at most a singleton.

If each L_n is given its natural order, we let $L = \{l_1 < l_2 < \ldots\}$ be the diagonal sequence, where l_n is the n-th term of L_n. Then every subsequence of $(z_i)_{i\in L}$ is summable to a point of $\bigcap_{n=1}^{\infty} B^n_{k_n}$ (by the construction and Lemma 2.2). Therefore the sequence $(z_i)_{i\in L}$ is the desired subsequence of (x_i). □

3. Summability in ω_1-locally convex spaces

In this section, assuming that $\mathfrak{h} = \omega_2$ we quote first the following theorem, analogous to Theorem 2.1.

Theorem 3.1. *Assume that $\mathfrak{h} = \omega_2$. Let X be a sequentially complete ω_1-locally convex space. Suppose that there exists a countable family of neighborhoods of $0 \in X$ consisting of open, convex and balanced sets such that the family of corresponding Minkowski functionals is separating. Let $\langle a_{ij}\rangle_{i,j\in\mathbb{N}}$ be a regular method of summability and (x_i) be a bounded sequence in X. Then there exists a subsequence (y_i) of (x_i) such that: either*

(a) *all subsequences of (y_i) are summable, with respect to $\langle a_{ij}\rangle$; or*
(b) *no subsequence of (y_i) is summable, with respect to $\langle a_{ij}\rangle$.*

Moreover, in the first case we can find a subsequence (z_i) of (y_i) such that all its subsequences are summable to the same limit.

Proof: There exists a basis $\mathcal{B}$ of neighborhoods of $0 \in X$, consisting of open, convex and balanced sets, such that $\operatorname{card}(\mathcal{B}) \leq \omega_1$. Moreover we can find a countable subfamily $\mathcal{B}'$ of $\mathcal{B}$ such that the family of the corresponding Minkowski functionals is separating. Consider the set:

$$\mathcal{A} = \{A = \{k_1 < k_2 < \ldots\} \in [\mathbb{N}]^\omega \mid (x_{k_i}) \text{ is summable with respect to } \langle a_{ij}\rangle\}.$$

Then,

$$\begin{aligned}\{k_1 < k_2 < \ldots\} \in \mathcal{A} \quad &\Leftrightarrow \quad \text{the sequence } (x'_i),\ x'_i = \sum_{j=1}^{\infty} \alpha_{ij} x_{k_j}, \quad \text{converges in } X \\ &\Leftrightarrow \quad (\forall\, U \in \mathcal{B})\, (\exists\, s \in \mathbb{N})\, [(\forall\, n, m \geq s)\, ((x'_n - x'_m) \in U)].\end{aligned}$$

Therefore,

$$\mathcal{A} = \bigcap_{U\in\mathcal{B}} \bigcup_{s\in\mathbb{N}} \bigcap_{n,m\geq s} \mathcal{D}_{U,n,m},$$

where

$$\mathcal{D}_{U,n,m} = \left\{ \{k_1 < k_2 < \dots\} \in [\mathbb{N}]^\omega \mid \left(\sum_{j=1}^{\infty} a_{nj} x_{k_j} - \sum_{j=1}^{\infty} a_{mj} x_{k_j} \right) \in U \right\}.$$

It is easy to verify that Lemma 2.1 holds if X is any sequentially complete locally convex space. So the set $\mathcal{D}_{U,n,m}$ is open, being the inverse image of the open set U, by the continuous function $f_n - f_m$. Thus, by Theorem 1.3, the set $\mathcal{A}$ is completely Ramsey being the intersection of ω_1 Borel sets. Therefore there is $M = \{k_1 < k_2 < \dots\} \in [\mathbb{N}]^\omega$ such that either $[M]^\omega \subseteq \mathcal{A}$ or $[M]^\omega \cap \mathcal{A} = \emptyset$. By setting $(y_i) = (x_{k_i})$, we have that either

(1) every subsequence of (y_i) is summable, with respect to $\langle a_{ij} \rangle$; or
(2) no subsequence of (y_i) is summable, with respect to $\langle a_{ij} \rangle$.

Finally, it is easy to see that in case (1) we can find a subsequence (z_i) of (y_i) such that all its subsequences are summable to the same limit. Indeed, denote by τ the topology of X and by τ' the topology on X induced by the family $\mathcal{B}'$. Since the family $\{p_U \mid U \in \mathcal{B}'\}$ is separating, the topology τ' is Hausdorff. Therefore (X, τ') is a locally convex space whose topology is induced by the countable family of seminorms $\{p_U \mid U \in \mathcal{B}'\}$. Hence, this topology is induced by an invariant metric. As $\tau' \subseteq \tau$, every subsequence of (y_i) is summable with respect to τ'. By repeating the second part of the proof of Theorem 2.1 we find $x \in X$ and a subsequence (z_i) of (y_i) such that each subsequence of (z_i) is summable to x with respect to τ'. But then every subsequence of (z_i) is summable to x with respect to τ. Thus, (z_i) is the desired subsequence. □

In the following theorem, as there is no completeness, the method of summability $\langle a_{ij} \rangle$ we consider is such that for every i, $a_{ij} \neq 0$ only for finitely many j. Such a method of summability is, for instance, the Cesàro method of summability.

Theorem 3.2. *Assume that $\mathfrak{h} = \omega_2$. Let X be a vector space and $\mathcal{T}$ be a family of locally convex topologies on X such that* $\operatorname{card}(\mathcal{T}) \leq \omega_1$ *and for each $\tau \in \mathcal{T}$ the local weight of (X, τ) is not greater than ω_1. We assume the existence of $\tau_0 \in \mathcal{T}$ such that the space (X, τ_0) is a Fréchet space. Let X be endowed with the locally convex topology induced by the family $\mathcal{T}$. Let $\langle a_{ij} \rangle$ be a method of summability such that for every i, $a_{ij} \neq 0$ only for finitely many j. Let (x_i) be a bounded sequence in X. Then there exists a subsequence (y_i) of (x_i) such that: either*

(a) *all subsequences of (y_i) are summable to a common limit, with respect to $\langle a_{ij} \rangle$; or*
(b) *no subsequence of (y_i) is summable, with respect to $\langle a_{ij} \rangle$.*

Proof: Since the space (X, τ_0) is a Fréchet space, from Theorem 2.1 we conclude that there exists a subsequence (z_i) of (x_i) such that, in the space (X, τ_0), either (a′) all subsequences of (z_i) are summable to a common limit, with respect to $\langle a_{ij}\rangle$; or

(b′) no subsequence of (z_i) is summable, with respect to $\langle a_{ij}\rangle$.

In case (b′) the sequence $(y_i) = (z_i)$ proves the theorem. Consider now the case (a′), and let $x \in X$ be the limit to which are summable all the subsequences of (z_i). There exists a family $\mathcal{P}$ of seminorms on X, which induces the topology of X with $\mathrm{card}(\mathcal{P}) \leq \omega_1$. Consider the set:

$$\mathcal{A} = \{A = \{k_1 < k_2 < \ldots\} \in [\mathbb{N}]^\omega \mid (z_{k_i}) \text{ is summable to } x \text{ with respect to } \langle a_{ij}\rangle\}\,.$$

Observe that

$$A = \{k_1 < k_2 < \ldots\} \in \mathcal{A} \Leftrightarrow$$

$$\Leftrightarrow (z_{k_i}) \text{ is summable to } x$$

$$\Leftrightarrow \text{the sequence } (z_i'),\ z_i' = \sum_{j=1}^{\infty} \alpha_{ij} z_{k_j}, \quad \text{converges to } x$$

$$\Leftrightarrow \big(\forall\, p \in \mathcal{P}\big)\big(\forall m \in \mathbb{N}\big)\big(\exists\, s \in \mathbb{N}\big)\Big[\big(\forall\, n \geq s\big)\Big(p\big(z_n' - x\big) < \frac{1}{m+1}\Big)\Big].$$

Therefore,

$$\mathcal{A} = \bigcap_{p\in\mathcal{P}} \bigcap_{m\in\mathbb{N}} \bigcup_{s\in\mathbb{N}} \bigcap_{n\geq s} \mathcal{D}_{p,m,n},$$

where

$$\mathcal{D}_{p,m,n} = \Big\{ \{k_1 < k_2 < \ldots\} \in [\mathbb{N}]^\omega \mid p\Big(\sum_{j=1}^{\infty} a_{nj} z_{k_j} - x\Big) < \frac{1}{m+1}\Big\}.$$

The set $\mathcal{D}_{p,m,n}$ is open, being the inverse image of some open set by a continuous function. Hence the set

$$\bigcap_{m\in\mathbb{N}} \bigcup_{s\in\mathbb{N}} \bigcap_{n\geq s} \mathcal{D}_{p,m,n}$$

is Borel. By Theorem 1.3 it follows that the set $\mathcal{A}$ is completely Ramsey. Thus, there exists $M = \{k_1 < k_2 < \ldots\} \in [\mathbb{N}]^\omega$ such that: either (I) $[M]^\omega \subseteq \mathcal{A}$ or (II) $[M]^\omega \cap \mathcal{A} = \emptyset$. We set $(y_i) = (z_{k_i})$. In case (I) all the subsequences of (y_i) are summable to x, with respect to $\langle a_{ij}\rangle$, and in case (II), no subsequence of (y_i) is summable, with respect to $\langle a_{ij}\rangle$. □

Remarks 3.1. (1) Theorem 3.1, in the case of a sequentially complete locally convex space X of local weight ω, coincides with Theorem 2.1.

Theorem 3.2, in the case where the family $\mathcal{T}$ is countable and for each $\tau \in \mathcal{T}$ the local weight of (X, τ) is ω, is proved in ZFC set theory and, clearly, gives a generalization of Theorem 2.1 when $\langle a_{ij}\rangle$ is such that for every i, $a_{ij} \neq 0$ only for finitely many j.

(2) If the local weight of X is equal to ω_1, we do not know whether Theorems 3.1 and 3.2 can be proved in ZFC set theory. However, we think that these theorems are independent of the ZFC axioms.

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References

[1] Balcar B., Pelant J., Simon P., *The space of ultrafilters on* $\mathbb{N}$ *covered by nowhere dense sets*, Fund. Math. **110** (1980), 11–24.
[2] Dunford N., Schwartz J.T., *Linear Operators I: General Theory*, Pure and Applied Mathematics, vol. 7, Interscience, New York, 1958.
[3] Ellentuck E., *A new proof that analytic sets are Ramsey*, J. Symbolic Logic **39** (1974), 163–165.
[4] Erdös P., Magidor M., *A note on regular methods of summability and the Banach-Saks property*, Proc. Amer. Math. Soc. **59** (1976), 232–234.
[5] Galvin F., Prikry K., *Borel sets and Ramsey's theorem*, J. Symbolic Logic **38** (1973), 193–198.
[6] Kechris A., *Classical Descriptive Set Theory*, Springer, New York, 1995.
[7] Plewik S., *On completely Ramsey sets*, Fund. Math. **127** (1986), 127–132.
[8] Rudin W., *Functional Analysis*, McGraw-Hill, New York, 1973.
[9] Tsarpalias A., *A note on the Ramsey property*, Proc. Amer. Math. Soc. **127** (1999), 583–587.

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