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# Pseudo-amenability of Brandt semigroup algebras

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*Abstract.* In this paper it is shown that for a Brandt semigroup $S$ over a group $G$ with an arbitrary index set $I$, if $G$ is amenable, then the Banach semigroup algebra $\ell^1(S)$ is pseudo-amenable.

## 1. Introduction

The concept of amenability for Banach algebras was introduced by Johnson in 1972 [6]. Several modifications of this notion, such as approximate amenability and pseudo-amenability, were introduced in [2] and [4]. In the current paper we investigate the pseudo-amenability of Brandt semigroup algebras. It was shown in [2] and [4] that for the group algebra $L^1(G)$, amenability, approximate amenability and pseudo-amenability coincide and are equivalent to the amenability of locally compact group $G$. In the semigroup case we know that, if $S$ is a discrete semigroup, then amenability of $\ell^1(S)$ implies that $S$ is regular and amenable [1]. Ghahramani et al. [3] have shown that, if $\ell^1(S)$ is approximately amenable, then $S$ is regular and amenable. The present author and Pourabbas in [9] have shown that for a Brandt semigroup $S$ over a group $G$ with an index set $I$, the following are equivalent.

  (i) $\ell^1(S)$ is amenable.
  (ii) $\ell^1(S)$ is approximately amenable.
  (iii) $I$ is finite and $G$ is amenable.

This result corrects [7, Theorem 1.8]. In the present paper we show that for a Brandt semigroup $S$ over a group $G$ with an arbitrary (finite or infinite) index set $I$, amenability of $G$ implies pseudo-amenability of $\ell^1(S)$.

## 2. Preliminaries

Throughout $\hat{\otimes}$ denotes the completed projective tensor product. For an element $x$ of a set $X$, $\delta_x$ is its point mass measure in $\ell^1(X)$. Also, we frequently use the identification $\ell^1(X \times Y) = \ell^1(X)\hat{\otimes}\ell^1(Y)$ for the sets $X$ and $Y$.

A Banach algebra $A$ is called (approximately) amenable, if for any *dual* Banach $A$-bimodule $E$, every bounded *derivation* from $A$ to $E$ is (*approximately*) *inner*. It is well known that amenability of $A$ is equivalent to existence of a *bounded approximate diagonal*, that is a bounded net $(m_i) \in A\hat{\otimes}A$ such that for every

$a \in A$, $a \cdot m_i - m_i \cdot a \longrightarrow 0$ and $\pi(m_i)a \longrightarrow a$, where $\pi : A \hat{\otimes} A \longrightarrow A$ is the continuous bimodule homomorphism defined by $\pi(a \otimes b) := ab$ $(a, b \in A)$, and called the *diagonal map*. The famous Johnson Theorem [6], says that, for any locally compact group $G$, amenability of $G$ and $L^1(G)$ are equivalent. For a modern account on amenability see [8] and for approximate amenability see the original papers [2] and [3].

A Banach algebra $A$ is called pseudo-amenable ([4]) if there is a net $(n_i) \in A \hat{\otimes} A$, called an *approximate diagonal* for $A$, such that $a \cdot n_i - n_i \cdot a \longrightarrow 0$ and $\pi(n_i)a \longrightarrow a$ for each $a \in A$.

Let $I$ be a nonempty set and let $G$ be a discrete group. Consider the set $T := I \times G \times I$, add a null element ø to $T$, and define a semigroup multiplication on $S := T \cup \{ø\}$, as follows. For $i, i', j, j' \in I$ and $g, g' \in G$, let

$$(i, g, j)(i', g', j') = \begin{cases} (i, gg', j') & \text{if } j = i', \\ ø & \text{if } j \neq i', \end{cases}$$

also let $ø(i, g, j) = (i, g, j)ø = ø$ and $øø = ø$. Then $S$ becomes a semigroup that is called Brandt semigroup over $G$ with index $I$, and usually denoted by $B(I, G)$. For more details see [5].

The Banach space $\ell^1(T)$, with the convolution product,

$$(ab)(i, g, j) = \sum_{k \in I, h \in G} a(i, gh^{-1}, k)b(k, h, j),$$

for $a, b \in \ell^1(T)$, $i, j \in I$, $g \in G$, becomes a Banach algebra. (Note that if $G$ is the one point group, and $I$ is finite, then $\ell^1(T)$ is an ordinary matrix algebra.) We have a closed relation between the Banach algebra $\ell^1(T)$ and the Banach semigroup algebra $\ell^1(S)$:

**Lemma 1.** *There exists a homeomorphic isomorphism $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$ of Banach algebras, where the multiplication of $\ell^1(T) \oplus \mathbb{C}$ is coordinatewise.*

PROOF: Consider the following short exact sequence of Banach algebras and continuous algebra homomorphisms:

$$0 \longrightarrow \ell^1(T) \longrightarrow \ell^1(S) \longrightarrow \mathbb{C} \longrightarrow 0,$$

where the second arrow $\Psi : \ell^1(T) \longrightarrow \ell^1(S)$ is defined by $\Psi(b)(t) := b(t)$ and $\Psi(b)(ø) := -\sum_{s \in T} b(s)$, for $b \in \ell^1(T)$ and $t \in T \subset S$, and the third arrow $\Phi : \ell^1(S) \longrightarrow \mathbb{C}$ is the integral functional, $\Phi(a) := \sum_{s \in S} a(s)$ $(a \in \ell^1(S))$. Now, let $\Theta : \ell^1(S) \longrightarrow \ell^1(T)$ be the restriction map, $\Theta(a) := a \mid_T$. Then $\Theta$ is a continuous algebra homomorphism and $\Theta\Psi = \text{Id}_{\ell^1(T)}$. Thus the exact sequence splits and we have $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$. □

**Lemma 2.** *If $\ell^1(T)$ is pseudo-amenable, then so is $\ell^1(S)$.*

PROOF: Suppose that $\ell^1(T)$ is pseudo-amenable. Then by Lemma 1 and [4, Proposition 2.1], $\ell^1(S)$ is pseudo-amenable. □

### 3.   The main result

Let $S, T, G$ and $I$ be as above. We need some other notations and computations:

For $a \in \ell^1(T)$ and every $u, v \in I$, let $a_{(u,v)}$ be an element of $\ell^1(G)$ defined by $a_{(u,v)}(g) := a(u, g, v)$ $(g \in G)$. Note that

$$\|a\|_{\ell^1(T)} = \sum_{u,v \in I} \|a_{(u,v)}\|_{\ell^1(G)}.$$

For $b \in \ell^1(G \times G)$, $c \in \ell^1(G)$ and any $i, j, i', j' \in I$, let $E^b_{(i,j,i',j')}$ and $H^c_{(i,j)}$ be elements of $\ell^1(T \times T)$ and $\ell^1(T)$ respectively, defined by

$$E^b_{(i,j,i',j')}(u, g, v, u', g', v') = \begin{cases} b(g, g') & \text{if } u = i, v = j, u' = i', v' = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$H^c_{(i,j)}(u, g, v) = \begin{cases} c(g) & \text{if } u = i, v = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $u, v, u', v' \in I$ and $g, g' \in G$. Also note that

(1)        $\|E^b_{(i,j,i',j')}\|_{\ell^1(T \times T)} = \|b\|_{\ell^1(G \times G)}, \qquad \|H^c_{(i,j)}\|_{\ell^1(T)} = \|c\|_{\ell^1(G)}.$

For $u, v \in I$ and $g \in G$, the module action of $\ell^1(T)$ on $\ell^1(T \times T)$ becomes

(2)                $\delta_{(u,g,v)} \cdot E^b_{(i,j,i',j')} = \begin{cases} E^{\delta_g \cdot b}_{(u,j,i',j')} & \text{if } i = v, \\ 0 & \text{if } i \neq v, \end{cases}$

(3)                $E^b_{(i,j,i',j')} \cdot \delta_{(u,g,v)} = \begin{cases} E^{b \cdot \delta_g}_{(i,j,i',v)} & \text{if } j' = u, \\ 0 & \text{if } j' \neq u. \end{cases}$

For the multiplication of $\ell^1(T)$ we have

(4)   $\delta_{(u,g,v)} H^c_{(i,j)} = \begin{cases} H^{\delta_g c}_{(u,j)} & \text{if } i = v, \\ 0 & \text{if } i \neq v, \end{cases} \qquad H^c_{(i,j)} \delta_{(u,g,v)} = \begin{cases} H^{c \delta_g}_{(i,v)} & \text{if } j = u, \\ 0 & \text{if } j \neq u. \end{cases}$

And finally, the diagonal maps $\pi : \ell^1(T \times T) \longrightarrow \ell^1(T)$ and $\pi : \ell^1(G \times G) \longrightarrow \ell^1(G)$ have the relation

(5)                        $\pi(E^b_{(i,j,i',j')}) = \begin{cases} H^{\pi(b)}_{(i,j')} & \text{if } j = i', \\ 0 & \text{if } j \neq i'. \end{cases}$

We are now ready to prove our main result:

**Theorem 3.** *Suppose that $G$ is amenable. Then $\ell^1(S)$ is pseudo-amenable.*

PROOF: Let $(m_\lambda)_{\lambda \in \Lambda} \in \ell^1(G \times G)$ be a bounded approximate diagonal for the amenable Banach algebra $\ell^1(G)$. For any finite nonempty subset $F$ of $I$ and $\lambda \in \Lambda$, let

$$W_{F,\lambda} := \frac{1}{\#F} \sum_{i,j \in F} E^{m_\lambda}_{(i,j,j,i)},$$

where $\#F$ denotes the cardinal of $F$. We show that the net $(W_{F,\lambda}) \in \ell^1(T \times T)$ over the directed set $\Gamma \times \Lambda$, where $\Gamma$ is the directed set of finite subsets of $I$ ordered by inclusion, is an approximate diagonal for $\ell^1(T)$.

For any $u, v \in I$ and $g \in G$, by equations (2) and (3), we have,

$$\delta_{(u,g,v)} \cdot W_{F,\lambda} = \begin{cases} \frac{1}{\#F} \sum_{j \in F} E^{\delta_g \cdot m_\lambda}_{(u,j,j,v)} & \text{if } v \in F, \\ 0 & \text{if } v \notin F, \end{cases}$$

$$W_{F,\lambda} \cdot \delta_{(u,g,v)} = \begin{cases} \frac{1}{\#F} \sum_{j \in F} E^{m_\lambda \cdot \delta_g}_{(u,j,j,v)} & \text{if } u \in F, \\ 0 & \text{if } u \notin F, \end{cases}$$

and thus,

$$\delta_{(u,g,v)} \cdot W_{F,\lambda} - W_{F,\lambda} \cdot \delta_{(u,g,v)} = \begin{cases} \frac{1}{\#F} \sum_{j \in F} E^{\delta_g \cdot m_\lambda - m_\lambda \cdot \delta_g}_{(u,j,j,v)} & \text{if } u \in F, v \in F, \\ \frac{1}{\#F} \sum_{j \in F} E^{\delta_g \cdot m_\lambda}_{(u,j,j,v)} & \text{if } v \in F, u \notin F, \\ -\frac{1}{\#F} \sum_{j \in F} E^{m_\lambda \cdot \delta_g}_{(u,j,j,v)} & \text{if } u \in F, v \notin F, \\ 0 & \text{if } v \notin F, u \notin F. \end{cases}$$

Then, for $a = \sum_{u,v \in I, g \in G} a(u,g,v) \delta_{(u,g,v)}$ in $\ell^1(T)$ we have

$$a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a = \frac{1}{\#F} \sum_{j,u,v \in F} E^{a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)}}_{(u,j,j,v)}$$

$$+ \frac{1}{\#F} \sum_{j,v \in F, u \in I-F} E^{a_{(u,v)} \cdot m_\lambda}_{(u,j,j,v)}$$

$$- \frac{1}{\#F} \sum_{j,u \in F, v \in I-F} E^{m_\lambda \cdot a_{(u,v)}}_{(u,j,j,v)},$$

and thus, by (1),

(6)
$$\left\| a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a \right\| \leq \sum_{u,v \in F} \left\| a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)} \right\|$$

$$+ \sum_{v \in F, u \in I-F} \left\| a_{(u,v)} \cdot m_\lambda \right\|$$

$$+ \sum_{u \in F, v \in I-F} \left\| m_\lambda \cdot a_{(u,v)} \right\|.$$

Now, suppose that $M > 0$ is a bound for the norms of $m_\lambda$'s. Let $\epsilon > 0$ be arbitrary, and let $F_0$ be an element of $\Gamma$ such that

$$\sum_{(u,v)\in J_0, g\in G} |a(u,g,v)| = \sum_{(u,v)\in J_0} \|a_{(u,v)}\| < \epsilon,$$

where $J_0 = (I \times (I - F_0)) \cup ((I - F_0) \times I)$. And choose a $\lambda_0 \in \Lambda$ such that for every $\lambda \geq \lambda_0$,

$$\sum_{u,v\in F_0} \|a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)}\| < \epsilon.$$

Now, if $(F, \lambda) \in \Gamma \times \Lambda$ such that $F_0 \subseteq F$, $\lambda \geq \lambda_0$, then we have,

$$\sum_{u,v\in F} \|a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)}\| \leq \sum_{u,v\in F_0} \|a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)}\|$$
$$+ \sum_{(u,v)\in J_0} \|a_{(u,v)} \cdot m_\lambda\|$$
$$+ \sum_{(u,v)\in J_0} \|m_\lambda \cdot a_{(u,v)}\|$$
$$< \epsilon + \epsilon M + \epsilon M,$$

and analogously,

$$\sum_{v\in F, u\in I-F} \|a_{(u,v)} \cdot m_\lambda\| < \epsilon M$$

and

$$\sum_{u\in F, v\in I-F} \|m_\lambda \cdot a_{(u,v)}\| < \epsilon M.$$

Thus by (6), we have $\|a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a\| < \epsilon + 4\epsilon M$.

Therefore, we proved that $a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a \longrightarrow 0$, for every $a \in \ell^1(T)$.

Now, we prove that $\pi(W_{F,\lambda})a \longrightarrow a$ for any $a \in \ell^1(T)$.

By (5), we have

$$\pi(W_{F,\lambda}) = \frac{1}{\#F} \sum_{i,j\in F} H_{(i,i)}^{\pi(m_\lambda)} = \sum_{i\in F} H_{(i,i)}^{\pi(m_\lambda)}.$$

Thus, (4) implies that

$$\pi(W_{F,\lambda})a = \sum_{i\in F, v\in I} H_{(i,v)}^{\pi(m_\lambda)a_{(i,v)}},$$

since $a = \sum_{u,v \in I} H_{(u,v)}^{a_{(u,v)}}$. Then we have,

(7)
$$\left\| \pi(W_{F,\lambda})a - a \right\| \leq \sum_{i \in F, v \in I} \left\| H_{(i,v)}^{\pi(m_\lambda)a_{(i,v)} - a_{(i,v)}} \right\|$$
$$+ \sum_{v \in I, u \in I-F} \left\| H_{(u,v)}^{a_{(u,v)}} \right\|.$$

Let $\epsilon > 0$ be arbitrary, and let $F_0$ and $J_0$ be as above. Choose a $\lambda_1 \in \Lambda$ such that for every $\lambda \geq \lambda_1$,

$$\sum_{i,j \in F_0} \left\| \pi(m_\lambda)a_{(i,j)} - a_{(i,j)} \right\| < \epsilon.$$

Now, if $(F, \lambda) \in \Gamma \times \Lambda$ is such that $F_0 \subseteq F$, $\lambda \geq \lambda_1$, then by (1) we have,

$$\sum_{i \in F, v \in I} \left\| H_{(i,v)}^{\pi(m_\lambda)a_{(i,v)} - a_{(i,v)}} \right\| \leq \sum_{i,j \in F_0} \left\| \pi(m_\lambda)a_{(i,j)} - a_{(i,j)} \right\|$$
$$+ \sum_{(u,v) \in J_0} \left\| \pi(m_\lambda)a_{(u,v)} \right\| + \sum_{(u,v) \in J_0} \left\| a_{(u,v)} \right\|$$
$$< \epsilon + \epsilon M + \epsilon,$$

and

$$\sum_{v \in I, u \in I-F} \left\| H_{(u,v)}^{a_{(u,v)}} \right\| = \sum_{v \in I, u \in I-F} \left\| a_{(u,v)} \right\| < \epsilon.$$

Thus, by (7) we have

$$\left\| \pi(W_{F,\lambda})a - a \right\| < 3\epsilon + \epsilon M.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We end with a natural question:

**Question 4.** *Does pseudo-amenability of $\ell^1(B(I,G))$ imply amenability of $G$?*

REFERENCES

[1] Duncan J., Namioka I., *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. Edinburgh Sect. A **80** (1978), 309–321.
[2] Ghahramani F., Loy R.J., *Generalized notion of amenability*, J. Funct. Anal. **208** (2004), 229–260.
[3] Ghahramani F., Loy R.J., Zhang Y., *Generalized notions of amenability II*, J. Funct. Anal. **254** (2008), 1776–1810.
[4] Ghahramani F., Zhang Y., *Pseudo-amenable and pseudo-contractible Banach algebras*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 111–123.
[5] Howie J.M., *An Introduction to Semigroup Theory*, Academic Press, London, 1976.
[6] Johnson B.E., *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972), pp 96.
[7] Lashkarizadeh Bami M., Samea H., *Approximate amenability of certain semigroup algebras*, Semigroup Forum **71** (2005), 312–322.

[8] Runde V., *Lectures on amenability*, Lecture Notes in Mathematics, 1774, Springer, Berlin, 2002.

[9] Sadr M.M., Pourabbas A., *Approximate amenability of Banach category algebras with application to semigroup algebras*, Semigroup Forum, to appear.

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