Corrigendum to "Realizations of Loops and Groups defined by short identities"

A.D. Keedwell

In a recent paper [1], we considered all those quasigroup identities (or "laws") which force a quasigroup to be a loop or group and, for a subset of these, provided proofs.

In several cases, we appealed to the Belousov/Taylor theorem (which says that any quasigroup which satisfies a balanced identity (or law) is isotopic to a group) but without making it clear that the quasigroup is itself a group only if it has an identity element (that is, only if it is a loop). We now provide the omitted part of these proofs using the notation of [1].

Theorem 1. *A quasigroup which satisfies any one of the balanced laws* $x \cdot yz = xz \cdot y$ ($L8.4$), $x \cdot yz = y \cdot zx$ ($L9.2$) *or* $xy \cdot z = y \cdot zx$ ($L9.3$) *has a two-sided identity element* (*and is an abelian group*).

Proof: Put $y = f_z$ in ($L8.4$). This gives $xz = xz \cdot f_z$, whence $f_z = e_{xz}$. But, for all $w \in Q$, there exists $x \in Q$ such that $xz = w$. Thus, $f_z = e_w$ for all $z, w \in Q$. Since both z and w are arbitrary, this implies that all left local identities are equal to all right local identities and so there is an universal two-sided identity.

Next put $x = e_z$ in ($L9.2$). This gives $e_z \cdot yz = yz$ so $e_z = f_{yz}$. Given any $w \in Q$, there exists a unique $y \in Q$ such that $yz = w$. Thus, $e_z = f_w$ for all $z, w \in Q$. Since both z and w are arbitrary, this implies that all left local identities are equal to all right local identities and so, as before, there is an universal two-sided identity.

Finally, put $x = e_z$ in ($L9.3$). This gives $e_z y \cdot z = yz$ and so $e_z y = y$ by right cancellation of z. Thus, $e_z = f_y$ for all $z, y \in Q$. Hence, a quasigroup which satisfies ($L9.3$) has a universal two-sided identity. □

We also give a complete proof of Theorem 5.6(C) of [1].

Theorem 2. *A quasigroup which satisfies the law* $x(yx \cdot z) = yz$ ($L5.3$) *is a group.*

Proof: Put $x = z$ in $L5.3$. This gives $z(yz \cdot z) = yz$. But, given $z \in Q$, we may choose $y \in Q$ such that $yz = w$ for any element $w \in Q$. Thus, $z(wz) = w$ for all $w, z \in Q$. We note that

$$z(wz) = w \Leftrightarrow R_z L_z = \mathrm{Id} \Leftrightarrow L_z R_z = \mathrm{Id} \Leftrightarrow (zw)z = w.$$

Now multiply $L5.3$ on the right by x. We get $[x(yx \cdot z)]x = (yz)x$ and so, using the equality just proved, $yx \cdot z = yz \cdot x$. This equality is a balanced identity and so $(Q, \cdot)$ is group isotopic[1]. To show that it is a group, we must show the existence of a two-sided identity element.

First put $x = e_y$ in the relation $yx \cdot z = yz \cdot x$. This gives $yz = yz \cdot e_y$ and so $e_y = e_{yz}$. Since, for any $w \in Q$, there exists an element z for which $w = yz$, $e_y = e_w$ for all $y, w \in Q$. Thus, e_y is a universal right identity.

Now, put $y = x$ in $L5.3$. We get $x(xx \cdot z) = xz$ so, by left cancellation of x, $xx \cdot z = z$ and so $xx = f_z$ for all $x, z \in Q$. Therefore, in particular, $f_z = e_y e_y = e_y$ for all $y, z \in Q$. That is, the identity is two-sided as required. $\square$

Remark. In Theorem 5.4 of [1], it is necessary to use the method of the footnote of that paper to show the existence of an identity element before using the Belousov/Taylor theorem because the balanced identity $x \cdot zy = z \cdot xy$ is again not one of those which forces the quasigroup to be a loop[2].

References

[1] Keedwell A.D., *Realizations of Loops and Groups defined by short identities*, Comment. Math. Univ. Carolin. **50** (2009), no. 3, 373–383.

Department of Mathematics, University of Surrey, Guildford,
Surrey GU2 7XH, United Kingdom

(*Received* September 7, 2009)

[1] Although this identity is balanced, it is not equivalent to one of those (listed in Theorem 1 above) which forces the quasigroup to have a two-sided identity element.

[2] In fact, it is the dual of the one which appears in Theorem 2 above.