

## Perturbed Hammerstein integral inclusions with solutions that change sign

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*Dedicated to Professor Espedito De Pascale on the occasion of his retirement.*

*Abstract.* We establish new existence results for nontrivial solutions of some integral inclusions of Hammerstein type, that are perturbed with an affine functional. In order to use a theory of fixed point index for multivalued mappings, we work in a cone of continuous functions that are positive on a suitable subinterval of  $[0, 1]$ . We also discuss the optimality of some constants that occur in our theory. We improve, complement and extend previous results in the literature.

*Keywords:* fixed point index, cone, nontrivial solution

*Classification:* Primary 45G10; Secondary 34A60, 34B10, 47H04, 47H10, 47H30

### 1. Introduction

Quite often differential inclusions arise in the study of problems in applied mathematics, engineering and economics, since some mathematical models utilize multivalued maps instead of single-valued maps [6], [12]. One approach to finding solutions of a boundary value multivalued problem is to re-write the problem as an integral inclusion and then to investigate the existence of solutions via different tools of nonlinear analysis, see for example [2], [18], [19], [36], [37].

When the problem is to find positive solutions of the corresponding integral operator, some authors use various generalizations [1], [3], [38], [39] of the well-known Guo-Krasnosel'skiĭ theorem on cone-compressions and expansions (see for example [17]). This is usually done by seeking the solutions in the cone of positive functions or in smaller cone included in it. This idea relies on the fact that the integral operator leaves the cone of positive functions invariant.

Here we establish new results for the existence of nontrivial solutions of integral inclusion of the type

$$(1.1) \quad u(t) \in \gamma(t)\alpha[u] + \int_0^1 k(t, s)F(s, u(s)) ds,$$

where  $\alpha[u]$  is a positive functional and both  $\gamma$  and  $k$  are allowed to change sign, so a positive solution need not exist. The integral equation corresponding to (1.1) has been studied in [24], motivated from the fact that it arises in the study of some heat flow problem.

Our approach relies on the theory of fixed point index for multivalued mappings [14], which is an extension of the well-known classical fixed point index. For our index calculations we work on the cone of functions that are positive on a suitable interval  $[a, b] \subset [0, 1]$ , but are allowed to change sign elsewhere. This type of cone has never been used before in the multivalued case, not even in the simpler case of integral Hammerstein inclusions, that is when  $\alpha[u] \equiv 0$ . In order to obtain the existence of nontrivial solutions for multivalued boundary value problems we extend the theoretical results of [24] to the context of set-valued mappings.

We also establish new results for the multivalued, nonlocal BVP

$$(1.2) \quad -u''(t) \in F(t, u(t)), \quad \text{a.e. on } [0, 1],$$

$$(1.3) \quad u'(0) + \alpha[u] = 0, \quad \beta u'(1) + u(\eta) = 0, \quad \eta \in [0, 1],$$

when  $\alpha[u]$  is a positive functional given by  $\alpha[u] = A_0 + \int_0^1 u(s) dA(s)$ , involving a Lebesgue-Stieltjes integral. This type of BC includes

$$\alpha[u] = \sum_{i=1}^m \alpha_i u(\xi_i) \quad \text{and} \quad \alpha[u] = \int_0^1 \alpha(s) u(s) ds,$$

that is, multi-point and integral BCs, that are widely studied objects both in the single-valued (see for example [26], [35], [43]) and in the multivalued case (see for example [7], [8], [9], [10], [13]). We point out that the BVP (1.2)–(1.3) occurs, in the single-valued case, in the study of a thermostat model [24], [25], [41], [42].

By a more careful analysis of some of the constants that occur in the theory, we are able to obtain values that are, in some sense, optimal for this method. This follows the study done, for different ranges of the parameters and different BCs, for the single-valued case, in [30], [31], [32], [41], [42].

Our results are an improvement valid also in the single-valued case.

## 2. Preliminaries

We recall some definitions and facts here and refer to [5], [6], [11], [12], [14] for further information. Let  $(X, \|\cdot\|)$  be a Banach space and  $\mathcal{KC}(X)$  be the set of non-empty compact convex subsets of  $X$ .

A multivalued function  $F : X \multimap X$  is said to be upper semicontinuous (u.s.c.) on  $X$  if for each  $x \in X$ , the set  $F(x)$  is a non-empty closed subset of  $X$  and for each open set  $V \subset X$ , with  $F(x) \subset V$ , there exists an open neighborhood  $U$  of  $x$  such that  $F(U) \subset V$ . A multivalued map  $F$  is said to have closed graph if  $x_n \rightarrow x, y_n \rightarrow y, y_n \in F(x_n)$  imply that  $y \in F(x)$ .

It is well-known that for a multivalued map  $F$  with non-empty compact values, the upper semicontinuity of  $F$  is equivalent to the fact that  $F$  maps compact sets into relatively compact sets and has closed graph.

Let  $J$  be a closed bounded interval of the real line. A multivalued map  $F : J \rightarrow \mathcal{KC}(X)$  is said to be measurable if for each  $x \in X$  the function  $D : J \rightarrow \mathbb{R}$  defined by  $D(t) := d(x, F(t)) = \inf \{\|x - z\| \mid z \in F(t)\}$  is measurable.

A multivalued map  $F : J \times \mathbb{R} \multimap \mathbb{R}$  is said to be an upper-Carathéodory map if for each  $u \in \mathbb{R}$ ,  $t \mapsto F(t, u)$  is measurable and for almost every  $t \in J$ ,  $u \mapsto F(t, u)$  is upper semicontinuous.

For each  $u \in C(J, \mathbb{R})$ , the set of  $L^1$ -selections of a multivalued map  $F : J \times \mathbb{R} \multimap \mathbb{R}$  is given by

$$S_{F,u}^1 := \{f_u \in L^1(J, \mathbb{R}) : f_u(t) \in F(t, u(t)) \text{ for a.e. } t \in J\}.$$

The following result is due to Lasota-Opial [33].

**Lemma 2.1.** *Let  $J$  be a closed bounded interval of the real line,  $F : J \times \mathbb{R} \rightarrow \mathcal{KC}(\mathbb{R})$ ,  $F$  be an upper-Carathéodory map,  $S_{F,u}^1$  be non-empty for each  $u \in C(J, \mathbb{R})$  and  $\Theta : L^1(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$  be a linear continuous operator. Then  $\Theta \circ S_{F,\cdot}^1 : C(J, \mathbb{R}) \multimap C(J, \mathbb{R})$ ,  $u \mapsto \Theta(S_{F,u}^1)$  is a closed graph map in  $C(J, \mathbb{R}) \times C(J, \mathbb{R})$ .*

Let  $K$  be a cone in a Banach space  $X$ , that is,  $K$  is a closed convex set such that  $\lambda x \in K$  for  $x \in K$  and  $\lambda \geq 0$  and  $K \cap (-K) = \{0\}$ . If  $\Omega$  is a bounded open subset of  $K$  (in the relative topology) we denote by  $\bar{\Omega}$  and  $\partial\Omega$  the closure and the boundary relative to  $K$ . When  $\Omega$  is an open bounded subset of  $X$  we write  $\Omega_K = \Omega \cap K$ , an open subset of  $K$ .

We utilize an extension of the well-known fixed point index for single-valued compact maps (see for example [4], [17]) to the case of multivalued maps due to Fitzpatrick and Petryshyn [14]. The formal definition of this index is rather technical and involves a topological degree introduced by Ma [34]. We refer to the paper [14] for the details regarding the construction of the index and summarize in Theorem 2.2, in a similar way as in [40], its main properties. Fixed point indices for multivalued mappings in more general settings can be found in the monograph by Andres and Górniewicz [5].

**Theorem 2.2.** *Let  $K$  be a cone in  $X$  and let  $\Omega$  be a bounded open set of  $X$  such that  $\Omega_K \neq \emptyset$ . Let  $T : \bar{\Omega}_K \multimap K$  be an upper semicontinuous compact map. Suppose that  $x \notin T(x)$  for all  $x \in \partial_K\Omega$ , the boundary of  $\Omega$  relative to  $K$ . The fixed point index has the following properties:*

- (P<sub>1</sub>) (Existence) *If  $i_K(T, \Omega_K) \neq 0$ , then  $T$  has a fixed point in  $\Omega_K$ .*
- (P<sub>2</sub>) (Normalisation) *If  $u \in \Omega_K$ , then  $i_K(\hat{u}, \Omega_K) = 1$ , where  $\hat{u}(x) = u$  for  $x \in \bar{\Omega}_K$ .*
- (P<sub>3</sub>) (Additivity) *If  $V_1, V_2$  are disjoint relatively open subsets of  $\Omega_K$  such that  $x \notin T(x)$  for  $x \in \bar{\Omega}_K \setminus (V_1 \cup V_2)$ , then*

$$i_K(T, \Omega_K) = i_K(T, V_1) + i_K(T, V_2).$$

- (P<sub>4</sub>) (Homotopy) *Let  $h : [0, 1] \times \bar{\Omega}_K \multimap K$  be an upper semicontinuous compact map such that  $x \notin h(t, x)$  for  $x \in \partial_K\Omega$  and  $t \in [0, 1]$ . Then*

$$i_K(h(0, \cdot), \Omega_K) = i_K(h(1, \cdot), \Omega_K).$$

Moreover, from the properties  $(P_1)$ – $(P_4)$ , we have that

- (1) if there exists  $e \in K \setminus \{0\}$  such that  $x \notin Tx + \lambda e$  for all  $x \in \partial_K \Omega$  and all  $\lambda > 0$ , then  $i_K(T, \Omega_K) = 0$ ,
- (2) if  $\lambda x \notin Tx$  for all  $x \in \partial_K \Omega$  and all  $\lambda > 1$ , then  $i_K(T, \Omega_K) = 1$ .

### 3. Existence of nontrivial solutions of perturbed Hammerstein integral inclusions

We study the existence of nonzero solutions of the integral inclusion

$$(3.1) \quad u(t) \in \gamma(t)\alpha[u] + \int_0^1 k(t, s)F(s, u(s)) ds,$$

in the space  $C[0, 1]$  of continuous functions, endowed with the usual supremum norm. Although it is not standard, we use the notation

$$[\Upsilon] := \sup \Upsilon, \quad \lfloor \Upsilon \rfloor := \inf \Upsilon, \quad \text{where } \Upsilon \subset \mathbb{R},$$

because we believe that this improves the readability of the paper.

From now on, we assume that  $F, \alpha, \gamma$  and the kernel  $k$  have the following properties:

- $(C_1)$   $F : [0, 1] \times \mathbb{R} \rightarrow \mathcal{KC}([0, +\infty))$  is an upper-Carathéodory map such that for every  $r > 0$ , there exists a  $L^1$  function  $g_r : [0, 1] \rightarrow [0, \infty)$  such that

$$\lceil F(t, u) \rceil \leq g_r(t) \quad \text{for almost all } t \in [0, 1] \quad \text{and all } u \in [-r, r].$$

- $(C_2)$   $k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  is measurable, and for every  $\tau \in [0, 1]$  we have

$$\lim_{t \rightarrow \tau} \int_0^1 |k(t, s) - k(\tau, s)|g_r(s) ds = 0.$$

- $(C_3)$  There exist  $[a, b] \subset [0, 1]$ , a  $L^\infty$  function  $\Phi : [0, 1] \rightarrow [0, \infty)$  and a constant  $c_1 \in (0, 1]$  such that

$$\begin{aligned} |k(t, s)| &\leq \Phi(s) \quad \text{for } t \in [0, 1] \quad \text{and almost every } s \in [0, 1] \\ k(t, s) &\geq c_1\Phi(s) \quad \text{for } t \in [a, b] \quad \text{and almost every } s \in [0, 1]. \end{aligned}$$

- $(C_4)$   $\gamma : [0, 1] \rightarrow \mathbb{R}$  is continuous and there exist a constant  $c_2 \in (0, 1]$  such that

$$\gamma(t) \geq c_2 \|\gamma\| \quad \text{for } t \in [a, b].$$

- $(C_5)$   $\alpha : K \rightarrow \mathbb{R}^+$  is a continuous functional with

$$\alpha[u] = A_0 + \int_0^1 u(s) dA(s),$$

where  $dA$  is a Lebesgue-Stieltjes measure with  $A_1 := \int_0^1 dA(s) < \infty$ .

(C<sub>6</sub>) The function  $t \mapsto k(t, s)$  is integrable with respect to the measure  $dA$ , that is

$$\mathcal{K}(s) := \int_0^1 k(t, s) dA(t)$$

is well defined.

$$(3.2) \quad K = \{u \in C([0, 1]) : \min\{u(t) : t \in [a, b]\} \geq c\|u\|\},$$

where  $c = \min\{c_1, c_2\}$ . This type of cone was introduced in [23] and later used in [15], [16], [20], [21], [22], [24].

This is similar to, but larger than, a cone of non-negative functions used by Lan [28], which is a type of cone first used by Krasnosel'skiĭ, see e.g. [27], and D. Guo, see e.g. [17]. Note that functions in  $K$  are positive on the interval  $[a, b]$  but may change sign on  $[0, 1]$ . We write  $K_r = \{u \in K : \|u\| < r\}$ ,  $\overline{K}_r = \{u \in K : \|u\| \leq r\}$ , and define

$$\Gamma = \int_0^1 \gamma(t) dA(t).$$

We consider now the multivalued map  $T : K \rightarrow C([0, 1])$  defined for  $u \in K$  by

$$T(u) := \left\{ h \in C([0, 1]) : \text{there exists } f_u \in S_{F,u}^1 \text{ such that,} \right. \\ \left. \text{for every } t \in [0, 1], h(t) = \gamma(t)\alpha[u] + \int_0^1 k(t, s)f_u(s) ds \right\}.$$

We point out that  $T$  is well-defined, since (C<sub>1</sub>) implies that the set  $S_{F,u}^1$  is non-empty (see [12]).

**Theorem 3.1.** *If the hypotheses (C<sub>1</sub>)–(C<sub>6</sub>) hold for some  $r > 0$ , then  $T$  maps  $\overline{K}_r$  into  $K$ . When these hypotheses hold for each  $r > 0$ ,  $T$  maps  $K$  into  $K$ . Moreover,  $T$  is an u.s.c. compact map with non-empty convex compact values.*

PROOF: Let  $u \in \overline{K}_r$ ,  $t \in [0, 1]$  and  $h \in T(u)$ , Then we have,

$$|h(t)| \leq |\gamma(t)|\alpha[u] + \int_0^1 |k(t, s)|f_u(s) ds \leq |\gamma(t)|\alpha[u] + \int_0^1 \Phi(s)f_u(s) ds$$

so that

$$(3.3) \quad \|h\| \leq \|\gamma\|\alpha[u] + \int_0^1 \Phi(s)f_u(s) ds.$$

Also

$$\begin{aligned} \min_{t \in [a,b]} \{h(t)\} &\geq c_2 \|\gamma\| \alpha[u] + c_1 \int_0^1 \Phi(s) f_u(s) ds \\ &\geq c \left[ \|\gamma\| \alpha[u] + \int_0^1 \Phi(s) f_u(s) ds \right] \geq c \|h\|, \end{aligned}$$

where  $c = \min\{c_1, c_2\}$ . Hence  $Tu \subset K$  for every  $u \in \overline{K}_r$ .

To see that  $Tu$  is a convex set for all  $u \in K$ , let  $h_1, h_2 \in Tu$ . Then there are  $f_u^1$  and  $f_u^2$  in  $S_{F,u}^1$  such that for all  $t \in [0, 1]$

$$h_1(t) = \gamma(t)\alpha[u] + \int_0^1 k(t, s) f_u^1(s) ds$$

and

$$h_2(t) = \gamma(t)\alpha[u] + \int_0^1 k(t, s) f_u^2(s) ds.$$

Let  $\lambda \in (0, 1)$ . We have

$$\lambda h_1(t) + (1 - \lambda)h_2(t) = \gamma(t)\alpha[u] + \int_0^1 k(t, s) [\lambda f_u^1(s) + (1 - \lambda)f_u^2(s)] ds,$$

with  $\lambda f_u^1(s) + (1 - \lambda)f_u^2(s) \in S_{F,u}^1$  because  $S_{F,u}^1$  is convex (see [33]).

Now, we show that the multimap  $T$  is compact. Firstly, we show that  $T$  sends bounded sets into bounded sets. It is enough to see that  $T(\overline{K}_r)$  is bounded. Let  $u \in \overline{K}_r$  and  $h \in T(u)$ . Then, for all  $t \in [0, 1]$  and some  $f_u \in S_{F,u}^1$ , from (3.3) we have

$$\|h\| \leq \|\gamma\| \alpha[u] + \int_0^1 \Phi(s) g_r(s) ds \leq \|\gamma\| (A_0 + rA_1) + M_r,$$

for some  $0 \leq M_r < \infty$ .

We prove now that  $T$  sends bounded sets into equicontinuous sets. Let  $t_1, t_2 \in [0, 1]$ ,  $t_1 < t_2$ ,  $u \in \overline{K}_r$  and  $h \in T(u)$ . Then

$$\begin{aligned} |h(t_1) - h(t_2)| &\leq |\gamma(t_1) - \gamma(t_2)| \alpha[u] + \int_0^1 |k(t_1, s) - k(t_2, s)| f_u(s) ds \\ &\leq |\gamma(t_1) - \gamma(t_2)| \alpha[u] + \int_0^1 |k(t_1, s) - k(t_2, s)| g_r(s) ds. \end{aligned}$$

Then  $|h(t_1) - h(t_2)| \rightarrow 0$  when  $t_1 \rightarrow t_2$ . By the Ascoli-Arzelà Theorem we can conclude that  $T$  is a compact map.

Finally, we show that  $T$  has closed graph. Let  $u_n, u_0 \in K$ ,  $u_n \rightarrow u_0$ ,  $h_n \in T(u_n)$ ,  $h_n \rightarrow h_0$ . We have to prove that  $h_0 \in T(u_0)$ , that is, there exists  $f_{u_0} \in S_{F,u_0}^1$  such that for all  $t \in [0, 1]$ ,  $h_0(t) = \gamma(t)\alpha[u_0] + \int_0^1 k(t, s) f_{u_0}(s) ds$ . Since  $h_n \in$

$T(u_n)$ , there exists  $f_{u_n} \in S_{F,u_n}^1$  such that for all  $t \in [0, 1]$ ,  $h_n(t) = \gamma(t)\alpha[u_n] + \int_0^1 k(t, s)f_{u_n}(s) ds$ . From Lemma 2.1, applied to the integral operator

$$\Theta : L^1([0, 1], R) \rightarrow C([0, 1], R), \quad w \mapsto \int_0^1 k(t, s)w(s) ds,$$

we obtain that the operator

$$\Theta \circ S_{F,\cdot}^1 : C([0, 1], R) \rightarrow C([0, 1], R), \quad u \mapsto \left\{ \int_0^1 k(t, s)f_u(s) ds : f_u \in S_{F,u}^1 \right\}$$

has closed graph. So, because  $h_n - \gamma\alpha[u_n] \in \Theta \circ S_{F,u_n}^1$  and  $h_n - \gamma\alpha[u_n] \rightarrow h_0 - \gamma\alpha[u_0]$ , we have that  $h_0 - \gamma\alpha[u_0] \in \Theta \circ S_{F,u_0}^1$ , that is, there exists  $f_{u_0} \in S_{F,u_0}^1$  such that for all  $t \in [0, 1]$ ,

$$h_0(t) = \gamma(t)\alpha[u_0] + \int_0^1 k(t, s)f_{u_0}(s) ds.$$

□

Let  $q : C([0, 1]) \rightarrow \mathbb{R}$  denote the continuous function  $q(u) = \min\{u(t) : t \in [a, b]\}$ . We shall use the open set  $V_\rho = \{u \in K : q(u) < \rho\}$ .  $V_\rho$  is equal to the set called  $\Omega_{\rho/c}$  in [22]. Note that  $K_\rho \subset V_\rho \subset K_{\rho/c}$ .

Firstly we prove that the index is 0 on the set  $V_\rho$ ; this is an extension of Lemma 2.4 of [24] to the multivalued case.

**Lemma 3.2.** *Assume that there exists  $\rho > 0$  such that  $u \notin Tu$  for  $u \in \partial V_\rho$  and*

(I $^0_\rho$ ) *there exists a measurable function  $\psi_\rho : [a, b] \rightarrow \mathbb{R}_+$  such that*

$$[F(t, u)] \geq \rho \psi_\rho(t) \text{ for all } u \in [\rho, \rho/c] \text{ and almost all } t \in [a, b],$$

$$(3.4) \quad \alpha[u] \geq \alpha_0 \rho \text{ for } u \in \partial V_\rho,$$

and

$$(3.5) \quad c_2 \|\gamma\| \alpha_0 + \inf_{t \in [a, b]} \int_a^b k(t, s)\psi_\rho(s) ds \geq 1.$$

Then we have  $i_K(T, V_\rho) = 0$ .

PROOF: Let  $e(t) \equiv 1$  for  $t \in [0, 1]$ . Then  $e \in K$ . We prove that

$$u \notin T(u) + \lambda e \text{ for all } u \in \partial V_\rho \text{ and } \lambda > 0,$$

which ensures, by Theorem 2.2, that the index is 0 on the set  $V_\rho$ . In fact, if not, there exist  $u \in \partial V_\rho$  and  $\lambda > 0$  such that  $u - \lambda e \in Tu$ . Then there exists  $f_u \in S_{F,u}^1$

such that for all  $t \in [a, b]$

$$\begin{aligned} u(t) &= \gamma(t)\alpha[u] + \int_0^1 k(t, s)f_u(s) ds + \lambda \\ &\geq c_2\|\gamma\|\alpha_0\rho + \rho \int_a^b k(t, s)\psi_\rho(s) ds + \lambda. \end{aligned}$$

By  $(I_\rho^0)$ , this implies that  $q(u) \geq \rho + \lambda > \rho$  contradicting the fact that  $u \in \partial V_\rho$ .  $\square$

We next prove that the index is 1 on the set  $K_\rho$ ; this is an extension of Lemma 2.6 of [24] to the multivalued case.

**Lemma 3.3.** *Suppose  $\Gamma < 1$  and assume that there exists  $\rho > 0$  such that  $u \notin Tu$  for all  $u \in \partial K_\rho$  and*

$(I_\rho^1)$  *there exists a measurable function  $\phi_\rho : [0, 1] \rightarrow \mathbb{R}_+$  such that  $\mathcal{K}\phi_\rho \in L^1([0, 1])$ ,*

$$[F(t, u)] \leq \rho \phi_\rho(t) \quad \text{for all } u \in [-\rho, \rho] \quad \text{and almost all } t \in [0, 1],$$

and

$$(3.6) \quad \frac{A_0\|\gamma\|}{(1-\Gamma)\rho} + \frac{\|\gamma\|}{(1-\Gamma)} \int_0^1 \mathcal{K}(s)\phi_\rho(s) ds + \sup_{t \in [0, 1]} \int_0^1 |k(t, s)|\phi_\rho(s) ds \leq 1.$$

Then we have  $i_K(T, K_\rho) = 1$ .

PROOF: We show that  $\lambda u \notin Tu$  for every  $u \in \partial K_\rho$  and for every  $\lambda > 1$ ; this ensures, by Theorem 2.2, that the index is 1 on  $K_\rho$ . In fact, if there exists  $\lambda > 1$  and  $u \in \partial K_\rho$  such that  $\lambda u \in Tu$  then for some  $f_u \in S_{F, \lambda u}^1$

$$(3.7) \quad \lambda u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t, s)f_u(s) ds.$$

Therefore we have

$$(3.8) \quad \lambda \int_0^1 u(t)dA(t) = \alpha[u]\Gamma + \int_0^1 \mathcal{K}(s)f_u(s) ds.$$

Hence

$$(\lambda - \Gamma)\alpha[u] = \lambda A_0 + \int_0^1 \mathcal{K}(s)f_u(s) ds.$$

Substituting into (3.7) gives

$$\lambda u(t) = \frac{\lambda A_0 \gamma(t)}{\lambda - \Gamma} + \frac{\gamma(t)}{\lambda - \Gamma} \int_0^1 \mathcal{K}(s)f_u(s) ds + \int_0^1 k(t, s)f_u(s) ds.$$



Taking the absolute value and then the supremum for  $t \in [0, 1]$  yields

$$\lambda\rho \leq \frac{\lambda A_0 \|\gamma\|}{\lambda - \Gamma} + \frac{\|\gamma\|}{\lambda - \Gamma} \int_0^1 \mathcal{K}(s) f_u(s) ds + \sup_{t \in [0,1]} \int_0^1 |k(t, s)| f_u(s) ds.$$

Thus we have, since  $\lambda > 1$ ,

$$(3.9) \quad \rho < \frac{A_0 \|\gamma\|}{1 - \Gamma} + \frac{\|\gamma\|}{1 - \Gamma} \int_0^1 \mathcal{K}(s) \rho \phi_\rho(s) ds + \sup_{t \in [0,1]} \int_0^1 |k(t, s)| \rho \phi_\rho(s) ds.$$

This contradicts (3.6) and proves the result. □

We can now state the following new result on the existence of multiple nonzero solutions for equation (3.1).

**Theorem 3.4.** *Equation (3.1) has a nonzero solution in  $K$  if either of the following conditions hold.*

- (H<sub>1</sub>) *There exist  $\rho_1, \rho_2 \in (0, \infty)$  with  $\rho_1 < \rho_2$  such that  $(I_{\rho_1}^1), (I_{\rho_2}^0), u \notin Tu$  for  $u \in \partial V_{\rho_2}$ .*
- (H<sub>2</sub>) *There exist  $\rho_1, \rho_2 \in (0, \infty)$  with  $\rho_1 < c\rho_2$  such that  $(I_{\rho_1}^0), (I_{\rho_2}^1), u \notin Tu$  for  $u \in \partial K_{\rho_2}$ .*

Equation (3.1) has two nonzero solutions in  $K$  if one of the following conditions hold.

- (D<sub>1</sub>) *There exist  $\rho_1, \rho_2, \rho_3$  with  $\rho_1 < \rho_2$  and  $\rho_2 < c\rho_3$  such that  $(I_{\rho_1}^1), (I_{\rho_2}^0), u \notin Tu$  for  $u \in \partial V_{\rho_2}$  and  $(I_{\rho_3}^1)$  hold.*
- (D<sub>2</sub>) *There exist  $\rho_1, \rho_2, \rho_3$  with  $\rho_1 < c\rho_2 < c\rho_3$  such that  $(I_{\rho_1}^0), (I_{\rho_2}^1), u \notin Tu$  for  $u \in \partial K_{\rho_2}$  and  $(I_{\rho_3}^0)$  hold.*

Moreover, if in (D<sub>1</sub>), strict inequality holds in  $(I_{\rho_1}^1)$ , then equation (3.1) has a third solution  $u_0 \in K_{\rho_1}$  (possibly zero).

We omit the proof which follows simply from properties of fixed point index, for details of similar proofs see [22], [28].

**Remark 3.5.** It is possible to state results for three or more nonzero solutions by expanding the lists in conditions (D<sub>1</sub>), (D<sub>2</sub>), see [29] for the type of result that may be stated.

#### 4. Nonzero solutions of some BVP

We now consider the BVP

$$(4.1) \quad -u''(t) \in F(t, u(t)), \text{ a.e. on } [0, 1],$$

with boundary conditions

$$(4.2) \quad u'(0) + \alpha[u] = 0, \beta u'(1) + u(\eta) = 0, \eta \in [0, 1].$$

The solution of  $-u'' = y$  under these BCs can be written

$$u(t) = (\beta + \eta - t)\alpha[u] + \beta \int_0^1 y(s) ds + \int_0^\eta (\eta - s)y(s) ds - \int_0^t (t - s)y(s) ds.$$

By a solution of the BVP (4.1)–(4.2) we mean a solution  $u \in C[0, 1]$  of the corresponding integral inclusion

$$u(t) \in (\beta + \eta - t)\alpha[u] + \int_0^1 k(t, s)F(s, u(s)) ds,$$

where

$$(4.3) \quad k(t, s) = \beta + \begin{cases} \eta - s, & s \leq \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t - s, & s \leq t \\ 0, & s > t. \end{cases}$$

Note that  $k(t, s)$  in (4.3) is the kernel for the special case  $\alpha[u] \equiv 0$ , studied in the case of  $F(t, u)$  single-valued in [25].

Here we discuss the case  $\beta > 0$  and  $\beta + \eta < 1$ . When  $0 < \beta + \eta < 1$  there cannot exist positive solutions for all positive right-hand sides, but there are nonzero solutions that are positive on an interval  $[0, b]$  for any  $b$  with  $0 < b < \beta + \eta < 1$ .

We point out that, with the same technique one can prove, for different ranges of the parameter  $\beta$  the existence of solutions that are positive, negative and negative on an interval, see Remark 3.4 of [24] for the results that can be stated.

**Upper bounds.** Upper bounds for  $|k(t, s)|$  and  $\gamma(t)$  were given in [24], [25] as follows

$$\Phi(s) = \|\gamma\| = \begin{cases} \beta + \eta, & \text{for } \beta + \eta \geq \frac{1}{2}, \\ 1 - (\beta + \eta), & \text{for } \beta + \eta < \frac{1}{2}. \end{cases}$$

**Lower bounds.** We take  $[a, b]$  with  $0 \leq a < b < \eta + \beta$ . Note that in  $[a, b]$ ,  $\gamma(t)$  is a decreasing function of  $t$  and  $\min_{t \in [a, b]} \gamma(t) = \beta + \eta - b$ . A simple calculation shows that

$$\min_{t \in [a, b]} k(t, s) = k(b, s) \geq \begin{cases} \beta, & \text{for } b \leq \eta, \\ \beta + \eta - b, & \text{for } b > \eta. \end{cases}$$

Thus we can choose

$$(4.4) \quad c = \begin{cases} \beta/(\beta + \eta), & \text{for } b \leq \eta, \beta + \eta \geq \frac{1}{2}, \\ \beta/(1 - (\beta + \eta)), & \text{for } b \leq \eta, \beta + \eta < \frac{1}{2}, \\ (\beta + \eta - b)/(\beta + \eta), & \text{for } b > \eta, \beta + \eta \geq \frac{1}{2}, \\ (\beta + \eta - b)/(1 - (\beta + \eta)), & \text{for } b > \eta, \beta + \eta < \frac{1}{2}. \end{cases}$$

Henceforth we work on the cone

$$K = \{u \in C[0, 1], \min_{t \in [a, b]} u(t) \geq c\|u\|\},$$

with  $c$  as in (4.4).

For brevity and clarity, we state a result for the existence of one nontrivial solution when  $F(t, u) = g(t)H(u)$ , for which the hypotheses are easier to check. Of course there are more general results, including existence of multiple nonzero solutions, analogous to Theorem 3.4.

When  $F(t, u) = g(t)H(u)$  where  $\Phi g \in L^1[0, 1]$  and  $H$  is upper semicontinuous, then  $\psi_\rho(s) = g(s)H_{\rho, \rho/c}$ , where  $H_{\rho, \rho/c} = \inf\{[H(u)]/\rho : \rho \leq u \leq \rho/c\}$ . Then (3.5) reads more simply

$$(4.5) \quad c_2\|\gamma\|\alpha_0 + H_{\rho, \rho/c} \cdot \frac{1}{M} \geq 1,$$

where

$$(4.6) \quad \frac{1}{M} = \inf_{t \in [a, b]} \int_a^b k(t, s)g(s) ds.$$

Moreover we have  $\phi_\rho(s) = g(s)H^{-\rho, \rho}$  where  $H^{-\rho, \rho} = \sup\{[H(u)]/\rho : -\rho \leq u \leq \rho\}$ . Then (3.6) reads more simply

$$(4.7) \quad \frac{A_0\|\gamma\|}{\rho(1-\Gamma)} + \left(\frac{\|\gamma\|}{1-\Gamma} \int_0^1 \mathcal{K}(s)g(s) ds + \frac{1}{m}\right)H^{-\rho, \rho} \leq 1,$$

where

$$(4.8) \quad \frac{1}{m} = \sup_{t \in [0, 1]} \int_0^1 |k(t, s)|g(s) ds.$$

**Remark 4.1.** We point out that all the constants that appear in (4.5) and (4.7) can be computed. This was shown, in a single-valued case, for the four-point BVP corresponding to  $\alpha[u] = \alpha u(\xi)$  with  $\xi \in [0, b]$ , see Remark 3.1 and Example 3.2 of [24].

**Theorem 4.2.** *Let  $0 \leq a < b < \beta + \eta < 1$ , and suppose that  $\int_a^b \Phi(s)g(s) ds > 0$ . Let  $c$  be as in (4.4). Let  $m$  be as in (4.8) and  $M$  as in (4.6). Then the BVP (4.1)–(4.2) has at least one nonzero solution, positive on  $[0, b]$ , if either one of the following conditions hold.*

- (S<sub>1</sub>) *There exist  $\rho_1, \rho_2 \in (0, \infty)$  with  $\rho_1 < \rho_2$  such that  $H^{-\rho_1, \rho_1}$  satisfies (4.7) and  $H_{\rho_2, \rho_2/c}$  satisfies  $(I_{\rho_2}^0)$ , with (3.5) replaced by (4.5).*
- (S<sub>2</sub>) *There exist  $\rho_1, \rho_2 \in (0, \infty)$  with  $\rho_1 < c\rho_2$  such that  $H_{\rho_1, \rho_1/c}$  satisfies  $(I_{\rho_1}^0)$ , with (3.5) replaced by (4.5) and  $H^{-\rho_2, \rho_2}$  satisfies (4.7).*

### 5. Optimal constants

We now assume that  $g \equiv 1$  and we seek the ‘optimal’  $[a, b]$  for which  $M(a, b)$  is a minimum. This type of problem has been tackled in the past for different BCs [30], [31], [32], [41], [42]. In particular, Webb [41], [42] studied the case when  $\beta + \eta \geq 1$ , that leads to positive solutions. Here we assume that  $\beta + \eta < 1$ ,  $\beta > 0$  and  $\eta > 0$ . The value  $1/m$  was given in [24], as follows

$$1/m = \sup_{t \in [0,1]} \int_0^1 |k(t, s)| ds = \max \left\{ \beta + \frac{1}{2}\eta^2, \beta^2 - \beta + \frac{1}{2}(1 - \eta^2) \right\}.$$

For arbitrary  $0 \leq a < b < \beta + \eta$ , the kernel  $k(t, s)$  is a positive, non-increasing function of  $t$ . Thus

$$1/M(a, b) = \min_{t \in [a,b]} \int_a^b k(t, s) ds = \int_a^b k(b, s) ds.$$

Note that  $\inf_{0 \leq a < b} M(a, b) = M(0, b)$  and

$$1/M(0, b) = \int_0^b k(b, s) ds = \begin{cases} -b^2 + b\beta + \eta b, & \text{if } b \leq \eta, \\ -b^2/2 + \beta b + \eta^2/2, & \text{if } b \geq \eta. \end{cases}$$

Now we have

$$\max_{0 < b \leq \eta} \{\beta b + \eta b - b^2\} = \max\{(\beta + \eta)^2/4, \beta\eta\} = \begin{cases} (\beta + \eta)^2/4, & \text{if } \beta \leq \eta, \\ \beta\eta, & \text{if } \beta \geq \eta, \end{cases}$$

and

$$\max_{0 < \eta \leq b} \{-b^2/2 + \beta b + \eta^2/2\} = \max\{(\beta^2 + \eta^2)/2, \beta\eta\} = \begin{cases} \beta\eta, & \text{if } \beta \leq \eta, \\ (\beta^2 + \eta^2)/2, & \text{if } \beta \geq \eta. \end{cases}$$

This implies that

$$1/M_{opt} = \begin{cases} (\beta + \eta)^2/4, & \text{if } \beta \leq \eta, \\ (\beta^2 + \eta^2)/2, & \text{if } \beta \geq \eta, \end{cases}$$

and therefore we choose

$$[a, b] = \begin{cases} [0, (\beta + \eta)/2], & \text{if } \beta \leq \eta, \\ [0, \beta], & \text{if } \beta \geq \eta. \end{cases}$$

Finally, in the following table, we compute some values for the constants  $m, M_{opt}, c$  for the optimal  $[a, b]$ , when  $\eta$  and  $\beta$  vary.

$\eta$	$\beta$	$m$	$M_{opt}$	$[a, b]$	$c$
1/5	1/5	25/8	25	$[0, 1/5]$	1/3
	2/5	50/21	10	$[0, 2/5]$	1/3
	3/5	50/31	5	$[0, 3/5]$	1/4
2/5	1/5	25/7	100/9	$[0, 3/10]$	1/2
	2/5	25/12	25/4	$[0, 2/5]$	1/2
3/5	1/5	50/19	25/4	$[0, 2/5]$	1/4

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