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# Symmetric difference on orthomodular lattices and $Z_2$-valued states

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*Abstract.* The investigation of orthocomplemented lattices with a symmetric difference initiated the following question: Which orthomodular lattice can be embedded in an orthomodular lattice that allows for a symmetric difference? In this paper we present a necessary condition for such an embedding to exist. The condition is expressed in terms of $Z_2$-valued states and enables one, as a consequence, to clarify the situation in the important case of the lattice of projections in a Hilbert space.

*Keywords:* orthomodular lattice, quantum logic, symmetric difference, Boolean algebra, group-valued state

*Classification:* 06A15, 03G12, 28E99, 81P10

## 1. Introduction and preliminaries

In the paper [11] the author introduces algebras that can be viewed as "orthomodular lattices with a symmetric difference". Their definition is as follows (the standard definition of an orthocomplemented lattice can be found in [9], [10], [16], etc.).

**Definition 1.1.** Let $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$, where $(X, \wedge, \vee, ^\perp, 0, 1)$ is an orthocomplemented lattice and $\triangle : X^2 \to X$ is a binary operation. Then $L$ is said to be an *orthocomplemented difference lattice* (abbr., an ODL) if the following formulas hold in $L$:

(D$_1$) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,
(D$_2$) $x \triangle 1 = x^\perp$, $1 \triangle x = x^\perp$,
(D$_3$) $x \triangle y \le x \vee y$.

Let us first formulate basic properties of ODLs as we shall use them in the sequel (see also [11]). We shall adopt the convention that in writing a formula with $\triangle$ and $^\perp$, we give the preference to the operation $^\perp$ over the operation $\triangle$. Thus, for instance, $x \triangle y^\perp$ means $x \triangle (y^\perp)$, etc.

**Proposition 1.2.** *Let $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$ be an ODL. Then the following statements hold true:*

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(1) $x \triangle 0 = x$, $0 \triangle x = x$,
(2) $x \triangle x = 0$,
(3) $x \triangle y = y \triangle x$,
(4) $x \triangle y^\perp = x^\perp \triangle y = (x \triangle y)^\perp$,
(5) $x^\perp \triangle y^\perp = x \triangle y$,
(6) $x \triangle y = 0 \Leftrightarrow x = y$,
(7) $(x \wedge y^\perp) \vee (y \wedge x^\perp) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^\perp$.

Proof: Suppose that $x, y \in L$ and verify the properties (1)–(7).

(1) Let us first see that the property ($D_2$) yields $1 \triangle 1 = 1^\perp = 0$. Using this, we have $x \triangle 0 = x \triangle (1 \triangle 1) = (x \triangle 1) \triangle 1 = x^\perp \triangle 1 = (x^\perp)^\perp = x$. Analogously, $0 \triangle x = (1 \triangle 1) \triangle x = 1 \triangle (1 \triangle x) = 1 \triangle x^\perp = (x^\perp)^\perp = x$.

(2) Let us first show that $x^\perp \triangle x^\perp = x \triangle x$. We consecutively obtain $x^\perp \triangle x^\perp = (x \triangle 1) \triangle (1 \triangle x) = (x \triangle (1 \triangle 1)) \triangle x = (x \triangle 0) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \leq x$ as well as $x \triangle x = x^\perp \triangle x^\perp \leq x^\perp$. This implies that $x \triangle x \leq x \wedge x^\perp = 0$.

(3) $x \triangle y = (x \triangle y) \triangle 0 = (x \triangle y) \triangle [(y \triangle x) \triangle (y \triangle x)] = x \triangle (y \triangle y) \triangle x \triangle (y \triangle x) = x \triangle 0 \triangle x \triangle (y \triangle x) = x \triangle x \triangle (y \triangle x) = 0 \triangle (y \triangle x) = y \triangle x$.

(4) $x \triangle y^\perp = x \triangle (y \triangle 1) = (x \triangle y) \triangle 1 = (x \triangle y)^\perp$. The equality $x^\perp \triangle y = (x \triangle y)^\perp$ follows from $x \triangle y^\perp = (x \triangle y)^\perp$ by using the equality (3).

(5) Applying (4), we obtain $x^\perp \triangle y^\perp = (x^\perp \triangle y)^\perp = (x \triangle y)^{\perp\perp} = x \triangle y$.

(6) If $x = y$, then $x \triangle y = 0$ by the condition (2). Conversely, suppose that $x \triangle y = 0$. Then $x = x \triangle 0 = x \triangle (y \triangle y) = (x \triangle y) \triangle y = 0 \triangle y = y$.

(7) The property ($D_3$) together with the properties (4), (5) imply that $x \triangle y \leq x \vee y$, $x \triangle y \leq x^\perp \vee y^\perp = (x \wedge y)^\perp$, $x \wedge y^\perp \leq x \triangle y$, $x^\perp \wedge y \leq x \triangle y$. □

Our interest in this paper is the relationship of ODLs to orthomodular lattices (OMLs). Let us recall the definition of OML (the acquaintance with basic facts about OMLs will be helpful in the sequel — see [1], [9], [10], etc.).

**Definition 1.3.** Let $L$ be an orthocomplemented lattice. If $L$ satisfies the orthomodular law,

$$x \leq y \Rightarrow y = x \vee (y \wedge x^\perp),$$

then $L$ is said to be an *orthomodular lattice* (abbr., an OML).

Though the orthomodular law is not explicitly stated among the axioms of ODL, it can be easily shown ([11]) that an ODL is automatically orthomodular. More precisely, if $K$ is an ODL and $K_{\mathrm{supp}}$ is the orthocomplemented lattice obtained from $K$ by forgetting $\triangle$, then $K_{\mathrm{supp}}$ is an OML. A question arises: Given an OML, $L$, can $L$ be made an ODL? Or, in case the above question answers in the negative too often, can $L$ be at least enlarged to an ODL? If $L$ allows for such an enlargement, the algebraic "calculus" of $L$ would be enriched and these ODL-enlargeable OMLs might find an application in quantum logic theory, or elsewhere (see [3], [6], [18], etc.).

Let us comment on "the state of art" in this line of problems and agree on some terminology. In [11] the author shows that several OMLs are *ODL-convertible*, i.e. they are such OMLs that can be endowed with $\triangle$ to become ODLs. Such are, for instance, the lattices $MO_\kappa$ for $\kappa = 2^n - 1$, the lattice $MO_\kappa$ for any infinite cardinal $\kappa$, certain pastings of Boolean algebras (this will also be commented on later), several "non-concrete" OMLs, etc. On the other hand, there are OMLs that are far from being ODL-convertible (such as, for instance, each finite OML the cardinality of which differs from $2^n$). In fact, there are even OMLs that are not ODL-embeddable (an OML, $L$, is said to be *ODL-embeddable* if there is an ODL, $K$, such that $L$ is a sub-OML of $K_{\text{supp}}$) — a rather elaborate construction presented in [12] provides such an example. In considering the ODL-embeddable OMLs a rather interesting connection came into existence. It turned out that if $L$ is ODL-embeddable then it has to possess an abundance of $Z_2$-states. This allows us to show, in an interplay with [15], that if $n \geq 4$ then the projection lattice $L(R^n)$ is not ODL-embeddable. The same question about $L(R^3)$ remains open (see also [8], [15]). However, a purely ODL consideration (Theorem 3.10) clarifies the ODL-convertibility of $L(R^3)$: The lattice $L(R^3)$ is not ODL-convertible (Theorem 3.11). The lattice $L(R^2)$ is ODL-convertible and, of course, so is $L(R^1)$.

Let $L$ be an OML. Let us recall that two elements $a, b \in L$ are called *compatible* in $L$ ($a \; C \; b$) if they lie in a Boolean subalgebra of $L$ (see [1] and [9] for the properties of compatible pairs). If $a, b \in L$ are not compatible, we write $a \neg \, C \; b$. Further, let us recall that by *a block* in $L$ we mean a maximal Boolean subalgebra of $L$. Finally, let us call the set $C(L) = \{c \in L;\ c \; C \; a \text{ for any } a \in L\}$ the *centre* of $L$ (i.e., $C(L)$ is the set of all "absolutely compatible" elements of $L$). Obviously, $C(L)$ is the intersection of all blocks of $L$.

It is convenient to adopt the following convention.

**Convention 1.4.** Let $L$ be an ODL. Then any OML notion can be referred to $L$ as well by applying this notion to the corresponding OML $L_{\text{supp}}$.

**Proposition 1.5.** *Let $L$ be an ODL and let $a, b \in L$ with $a \; C \; b$. Then $a \triangle b = (a \wedge b^\perp) \vee (b \wedge a^\perp) = (a \vee b) \wedge (a \wedge b)^\perp$. A corollary: If $a \; C \; b$, then $a \; C \; a \triangle b$.*

Proof: It follows from Proposition 1.2(7). □

In concluding this paragraph let us observe the following consequence of Proposition 1.5: For each block $B$ of $L$, the operation $\triangle$ on $L$ acts on $B$ as the standard symmetric difference.

## 2. OMLs with 8-element blocks

In this section we shall be interested in some intrinsic properties of the OMLs whose blocks are of cardinality 8 and whose pairs of atoms, $a$ and $b$, satisfy the inequality $a \vee b < 1$. We will then apply the results obtained in the constructions enabling us to prove our main result formulated in Theorem 3.10. (It should be noted that the class of OMLs considered in this section contains, as an important example, the lattice $L(R^3)$ of projections of $R^3$. The paper [17] studies, with the motivation coming from theoretical physics, the existence of sub-orthoposets of $L(R^3)$. Incidentally, our result of Theorem 2.5 adds to Proposition 6.5 of [17].)

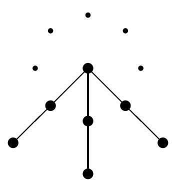
**Proposition 2.1.** *Let $L$ be an OML such that the cardinality of each block of $L$ is 8. Then*

(i) *for any pair $a$ and $b$ of atoms in $L$, the following statement holds true: $a \vee b < 1$ if and only if there is an atom $c$ such that $a\ C\ c$ and $b\ C\ c$;*
(ii) *for any pair of distinct atoms $a$ and $b$ in $L$ there is at most one atom $c$ such that $a\ C\ c$ and $b\ C\ c$.*

Proof: The statement (i) is trivial. For the statement (ii) suppose that $a, b$ are atoms and $a \neq b$. Suppose that $c, d$ are such atoms that $c\ C\ a$, $c\ C\ b$, $d\ C\ a$ and $d\ C\ b$. Then we have $0 < a < a \vee b \leq c^\perp \wedge d^\perp \leq c^\perp < 1$. Since each block of $L$ has 8 elements, we infer that $c^\perp \wedge d^\perp = c^\perp$. Thus, $c^\perp \leq d^\perp$ and therefore $d \leq c$. As $c, d$ are atoms, it follows that $c = d$. □

**Definition 2.2.** An OML $L$ is said to be a 3-*star* if $L$ is isomorphic with the product $\{0,1\} \times MO_\kappa$ for $\kappa \geq 1$.

The figure below indicates the Greechie diagram of the 3-star $\{0,1\} \times MO_\kappa$. Note that the number of blocks of this 3-star is $\kappa$.



**Proposition 2.3.** *Let $L$ be an OML. Then $L$ is a 3-star if and only if the cardinality of each block of $L$ is 8 and $C(L) \neq \{0,1\}$.*

Proof: The proof is evident. □

Prior to the main result of this section, let us recall some notions of orthomodular combinatorics (see also [4] and [16]).

**Definition 2.4.** Let $L$ be an OML such that the cardinality of each block of $L$ is 8. For three mutually distinct and compatible atoms $a_1, a_2, a_3$ of $L$, let us denote by $[a_1, a_2, a_3]_L$ the block of $L$ generated by these atoms.

An *n-path* in $L$ ($n \geq 1$) is a sequence $B_1, \ldots, B_n$ of blocks of $L$ such that there are pairwise distinct atoms $b_1, a_1, b_2, \ldots, a_n, b_{n+1} \in L$ with $B_i = [b_i, a_i, b_{i+1}]_L$, $i = 1, \ldots, n$.

An *n-loop* in $L$ ($n \geq 3$) is a sequence $B_1, \ldots, B_n$ of blocks of $L$ such that there are pairwise distinct atoms $b_1, a_1, b_2, \ldots, a_n \in L$ with $B_i = [b_i, a_i, b_{i+1}]_L$, $i = 1, \ldots, n-1$, $B_n = [b_n, a_n, b_1]_L$.

We shall also need the following corollary of Greechie's lemma ([4]): An OML satisfying the assumptions of Def. 2.4 cannot contain any $n$-loop for $n \leq 4$.

**Theorem 2.5.** *Let $L$ be an OML. Let the cardinality of each block of $L$ be 8 and let $C(L) = \{0,1\}$. Let for any pair $a, b$ of atoms in $L$ the inequality $a \vee b < 1$ hold true. Then any block of $L$ is contained in a 5-loop.*

PROOF: We shall need three lemmas (the OML $L$ dealt with in the lemmas satisfies the assumptions of Theorem 2.5).

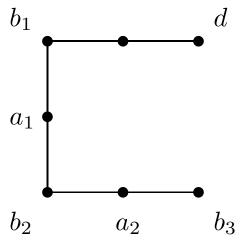
**Lemma 1.** *Each block in $L$ is contained in a 2-path.*

PROOF: Consider a block $B = [a_1, a_2, a_3]_L$. Since $L$ is not a Boolean algebra, we see that $L \neq B$. Hence there is an atom $b \in L$ with $b \notin B$. The assumptions required for $L$ obviously guarantee the existence of an atom $c \in L$ such that $a_1\ C\ c$ and $b\ C\ c$. Let us complete the lemma arguing by cases. If $c \in \{a_1, a_2, a_3\}$, then the couple $[a_1, a_2, a_3]_L, [c, b, c^\perp \wedge b^\perp]_L$ is a 2-path. If $c \notin \{a_1, a_2, a_3\}$, then the couple $[a_1, a_2, a_3]_L, [c, a_1, c^\perp \wedge a_1^\perp]_L$ is a 2-path. The proof is done. □

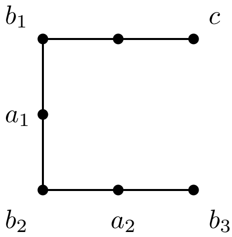
**Lemma 2.** *Each 2-path in $L$ is contained in a 3-path.*

PROOF: Consider a 2-path, some $B_1 = [b_1, a_1, b_2]_L, B_2 = [b_2, a_2, b_3]_L$. Since $b_2 \notin C(L)$, there is an atom $d \in L$ such that $b_2 \neg\ C\ d$. It follows that $d \notin \{b_1, a_1, a_2, b_3\}$. We have two possibilities to argue.

(I) First, $d$ is compatible with some of the atoms $b_1, a_1, a_2, b_3$. Without any loss of generality, suppose that $d\ C\ b_1$. Then $a_1 \neg\ C\ d$, $a_2 \neg\ C\ d$ and $b_3 \neg\ C\ d$. Indeed, if $a_1\ C\ d$ then $d = b$. If $a_2\ C\ d$ or $b_3\ C\ d$ then $L$ contains a 4-loop which is excluded by the Greechie lemma. Thus, we obtain the following Greechie diagram:



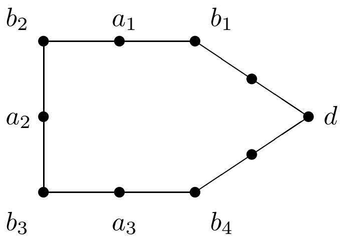

(II) Second, $d$ is not compatible with any of the elements $b_1, a_1, a_2, b_3$. By our assumption, there is an atom $c \in L$ such that $b_1\ C\ c$ and $d\ C\ c$. Since $d$ is not compatible with any of the elements $b_1, a_1, b_2, a_2, b_3$ and since $d\ C\ c$, we see that $c \notin \{b_1, a_1, b_2, a_2, b_3\}$. Mimicking the reasoning of the part (I) we obtain a 3-path portrayed below:




This completes the proof of Lemma 2. $\square$

**Lemma 3.** *Each 3-path in $L$ is contained in a 5-loop.*

PROOF: Consider a 3-path, some $B_1 = [b_1, a_1, b_2]_L$, $B_2 = [b_2, a_2, b_3]_L$, $B_3 = [b_3, a_3, b_4]_L$. By our assumption on $L$, there is an atom $d \in L$ such that $d\ C\ b_1$ and $d\ C\ b_4$. Obviously, $d \notin \{a_1, b_2, a_2, b_3, a_3\}$. In other words, we have completed the proof of Lemma 3 by constructing a 5-loop in $L$ with the following Greechie diagram:




$\square$

Let us return to the proof of Theorem 2.5. Let us choose a block $B$ of $L$. Then a consecutive application of Lemma 1, Lemma 2 and Lemma 3 allows us to obtain the desired 5-loop. $\square$

## 3. Results

Let $Z_2$ stand for the group $\{0, 1\}$ understood with the modulo 2 addition $\oplus$ (thus, $1 \oplus 1 = 0 \oplus 0 = 0$, $1 \oplus 0 = 0 \oplus 1 = 1$). Let $L$ be an OML and let $s : L \to Z_2$ be a mapping. Then $s$ is said to be a *$Z_2$-valued state* (abbr., a *$Z_2$-state*) provided $s(1) = 1$ and $s(x \vee y) = s(x) \oplus s(y)$ whenever $x, y \in L$, $x \leq y^{\perp}$. The following definition is a variant of "fullness" dealt with in the quantum logic theory ([7]) and it is crucial in our consideration.

**Definition 3.1.** Let $L$ be an OML. Then $L$ is called *$Z_2$-full* if for any $x, y \in L$, $x \neq y$, $x \neq 0$, $y \neq 1$ there exists a $Z_2$-state, $s$, on $L$ such that $s(x) = 1$ and $s(y) = 0$.

Our first result reads as follows.

**Theorem 3.2.** *Let $L$ be an OML. If $L$ is ODL-embeddable then $L$ is $Z_2$-full.*

The proof of Theorem 3.2 will be obtained in a series of propositions. Let us first examine a certain type of ideals in ODLs. They will correspond to $Z_2$-states.

**Definition 3.3.** Let $K$ be an ODL and let $I$ be a subset of $K$. Then $I$ is said to be a $\triangle$-*ideal* if $0 \in I$ and whenever $a, b \in I$, then $a \triangle b \in I$. Further, if $1 \notin I$, then $I$ is called a *proper* $\triangle$-*ideal*. Finally, $I$ is called *maximal* if $I$ is proper and for any proper $\triangle$-ideal $J$ with $I \subseteq J$ we have $I = J$.

**Proposition 3.4.** *Suppose that $K$ is an ODL and $I$ is a proper $\triangle$-ideal in $K$. Suppose that $x \in K$ and neither $x$ nor $x^\perp$ belongs to $I$. Let us write $J = I \cup \{a \triangle x;\ a \in I\}$. Then $J$ is also a proper $\triangle$-ideal in $K$ and, moreover, $x \in J$ and $x^\perp \notin J$.*

Proof: The set $J$ is obviously a $\triangle$-ideal. Let us see that $1 \notin J$. Suppose on the contrary that $1 \in J$. Then $1 = a \triangle x$ for some element $a \in I$. The equality $1 = a \triangle x$ implies that $a = x^\perp$ (indeed, by Proposition 1.2 we have $0 = (a \triangle x)^\perp = a \triangle x^\perp$ and therefore $a = x^\perp$). But $x^\perp$ does not belong to $I$ which is a contradiction. Thus, $1 \notin J$. Further $x = 0 \triangle x \in J$. If $x^\perp \in J$, then $1 = x \triangle x^\perp \in J$ — a contradiction again. □

**Proposition 3.5.** *Let $K$ be an ODL and let $I$ be a maximal $\triangle$-ideal in $K$. Then* $\mathrm{card}(\{x, x^\perp\} \cap I) = 1$ *for any $x \in K$.*

Proof: Suppose that $I$ is maximal and $x \in K$. Suppose further that $x \notin I$ and, also $x^\perp \notin I$. Then (Proposition 3.4) there is a $\triangle$-ideal, $J$, such that $I \subseteq J$ and $I \neq J$. As a result, at least one element of the set $\{x, x^\perp\}$ belongs to $I$. Looking for a contradiction, suppose that $\{x, x^\perp\} \subseteq I$. Then $x \triangle x^\perp = 1$ which means that $1 \in I$ — a contradiction ($I$ is supposed to be proper). □

**Proposition 3.6.** *Let $K$ be an ODL and let $a, b \in K$, $a \neq b$, $a < 1$ and $0 < b$. Then there is a maximal $\triangle$-ideal, $J$, such that $a \in J$ and $b \notin J$.*

Proof: Write $\mathcal{I} = \{I \subseteq K;\ I \text{ is a proper } \triangle\text{-ideal},\ a \in I \text{ and } b \notin I\}$. Then $\{0, a\} \in \mathcal{I}$ and therefore $\mathcal{I} \neq \emptyset$. By a standard application of Zorn's lemma, the set $\mathcal{I}$ ordered by inclusion contains a maximal element, $J$. Of course, $J$ is a proper $\triangle$-ideal. Moreover, $b^\perp \in J$ (otherwise the $\triangle$-ideal $J' = J \cup \{c \triangle b^\perp;\ c \in J\}$ extends $J$, Proposition 3.4, and $J'$ belongs to the system $\mathcal{I}$). Let us show that $J$ is maximal. Suppose therefore that $J \subseteq I$ for a proper $\triangle$-ideal $I$, $J \neq I$. Thus, $I$ is strictly larger than $J$ and therefore $I \notin \mathcal{I}$. Therefore $b \in I$ and since $b^\perp \in J \subseteq I$, we see that $1 = b \triangle b^\perp \in I$. This means that $I$ is not proper and the proof is complete. □

**Proposition 3.7.** *Let $K$ be an ODL and $I$ be a maximal $\triangle$-ideal in $K$. Let us define a mapping $s : K \to Z_2$ as follows: $s(a) = 0$ (resp., $s(a) = 1$) if $a \in I$ (resp., $a \notin I$). Then $s(x \triangle y) = s(x) \oplus s(y)$ for any $x, y \in L$. A consequence: The mapping $s$ is a $Z_2$-state on $K_{\mathrm{supp}}$.*

Proof: Let us consider two elements $x, y \in K$. We are to prove the equality $s(x \bigtriangleup y) = s(x) \oplus s(y)$. We will argue by cases. If both $x$ and $y$ belong to $I$, then $x \bigtriangleup y \in I$ and therefore $s(x \bigtriangleup y) = 0 = 0 \oplus 0 = s(x) \oplus s(y)$. If $x \in I$ and $y \notin I$, then $x \bigtriangleup y \notin I$ (indeed, should $x \bigtriangleup y$ be an element of $I$, then $y = x \bigtriangleup (x \bigtriangleup y) \in I$ which is a contradiction). Hence $s(x \bigtriangleup y) = 1 = 0 \oplus 1 = s(x) \oplus s(y)$. The case of $x \notin I$ and $y \in I$ argues analogously. Let us suppose that $x \notin I$ and $y \notin I$. Since $I$ is a maximal $\bigtriangleup$-ideal, we infer that $x^\perp \in I$ and $y^\perp \in I$. Then $x^\perp \bigtriangleup y^\perp \in I$. But $x^\perp \bigtriangleup y^\perp = x \bigtriangleup y$ (Proposition 1.2(5)) and therefore $x \bigtriangleup y \in I$. Hence $s(x \bigtriangleup y) = 0 = 1 \oplus 1 = s(x) \oplus s(y)$.

It remains to show that the mapping $s$ defined above is a $Z_2$-state on $K_{\mathrm{supp}}$. Of course, $s(1) = 1$. Let us take $x, y \in K$ with $x \leq y^\perp$. Then $x\, C\, y$ and therefore (Proposition 1.5) we see that $x \bigtriangleup y = (x \vee y) \wedge (x \wedge y)^\perp = (x \vee y) \wedge 0^\perp = x \vee y$. Then $s(x \vee y) = s(x \bigtriangleup y) = s(x) \oplus s(y)$ by the analysis above. The proof of Proposition 3.7 is complete. □

Proof of Theorem 3.2: Let $L$ be an ODL-embeddable OML. Then there is an ODL, $K$, such that $L$ is a sub-OML of $K_{\mathrm{supp}}$. Let $x, y$ be elements of $L$ with $x \neq y$, $x \neq 0$ and $y \neq 1$. According to Proposition 3.6 there is a maximal $\bigtriangleup$-ideal $J$ in $K$ such that $y \in J$ and $x \notin J$. Let us set $s(a) = 0$ for $a \in J$ and $s(a) = 1$ for $a \in K$, $a \notin J$. Then, according to Proposition 3.7, the mapping $s$ is a $Z_2$-state on $K_{\mathrm{supp}}$. If we denote by $s_1$ the restriction of $s$ to the OML $L$, then $s_1$ is a $Z_2$-state on $L$. Moreover, $s_1(x) = s(x) = 1$ and $s_1(y) = s(y) = 0$. □

The link of ODL-embeddable OMLs with $Z_2$-states revealed in Theorem 3.2 allows us to shed light on the ODL embeddability of the lattice $L(H)$ of projections in a (real) Hilbert space $H$.

**Theorem 3.8.** *Let $H$ be a Hilbert space. If $\dim H \geq 4$, then $L(H)$ is not ODL-embeddable.*

Proof: In [15] it is shown that for $\dim H \geq 4$ the OML $L(H)$ does not allow for any $Z_2$-state. The rest follows from Theorem 3.2. □

The case of $L(R^3)$ remains open — it seems still open whether or not $L(R^3)$ possesses a $Z_2$-state (see [8] and [15]). However, it is not difficult to show that $L(R^3)$ cannot be made an ODL (i.e., it can be proved that $L(R^3)$ is not ODL-convertible). In fact, even relatively mild lattice-theoretic conditions shared by $L(R^3)$ prevent us from introducing $\bigtriangleup$ on $L(R^3)$. We are going to prove this by deriving a characterization of 3-stars — a result which may be of separate interest in the theory of ODLs.

Recall first a result already referred to in the introduction (for a detailed proof, see [11]; let us provide a sketch for the convenience of the reader).

**Proposition 3.9.** *Let $\kappa$ be a cardinal number. Let $\kappa = 2^n - 1$ for a natural number $n \in \mathbb{N}$ or let $\kappa$ be infinite. Then the horizontal sum $MO_\kappa$ is, up to an ODL-isomorphism, uniquely ODL-convertible.*

Proof: Let $\kappa = 2^n - 1$ (resp. $\kappa$ be infinite). Then there is a Boolean algebra, $B$, with $\mathrm{card}(B) = 2^{n+1}$ (resp. $\mathrm{card}(B) = \kappa$). Take a prime-ideal on $B$, some $I$ and set, for any $a \in I \setminus \{0\}$, $B_a = \{0, a, a^\perp, 1\}$. Since $\mathrm{card}(I \setminus \{0\}) = \kappa$, we see that $MO_\kappa$ is OML-isomorphic with the horizontal sum of $B_a$, $a \in I \setminus \{0\}$. Moreover, $MO_\kappa$ and $B$ have the same underlying set. Thus, elements $c, d \in MO_\kappa$ can be viewed as elements of $B$ and hence we can define $c \triangle d$ as the corresponding symmetric difference in $B$ (understood in $MO_\kappa$ this time). It can be shown that $MO_\kappa$ endowed with this symmetric difference is an ODL and that $\triangle$ is (up to an ODL-isomorphism) the only one which converts $MO_\kappa$ to an ODL. □

Before we formulate the main result of this section let us again make use of Convention 1.4 allowing ourselves to call an ODL $K$ a 3-star provided so is $K_{\mathrm{supp}}$.

**Theorem 3.10.** *Let $K$ be an ODL. Then the following two statements are equivalent:*

(i) *$K$ is a 3-star,*
(ii) *the cardinality of each maximal Boolean subalgebra of $K$ is 8, and for any pair $a, b \in K$ of atoms in $K$ the inequality $a \vee b < 1$ holds true.*

Proof: The implication (i)⇒(ii) is obvious. Let us launch on (ii)⇒(i). Let us first formulate and prove a few auxiliary propositions.

**Lemma 1.** *Suppose that $K$ is as in Theorem 3.10(ii). Let $a, b$ be atoms of $K$. Then*

(i) *$a \triangle b$ is a co-atom of $K$ if and only if $a \neq b$ and $a\ C\ b$,*
(ii) *if $a$ is not compatible with $b$, then $a \triangle b$ is an atom of $K$.*

Proof: (i) If $a \neq b$ and $a\ C\ b$, then $a \leq b^\perp$ and therefore $a \triangle b = a \vee b$. Since both $a, b$ belong to an 8-element Boolean subalgebra of $K$, the element $a \triangle b$ must be a co-atom.

Suppose for the reverse implication that $a \triangle b = d^\perp$ for an atom $d \in K$. Choose an atom, $c$, such that $a\ C\ c$ and $b\ C\ c$. Then $a \leq c^\perp$ and $b \leq c^\perp$. It follows that $a \triangle b \leq a \vee b \leq c^\perp$. Thus, $d^\perp \leq c^\perp$ and therefore $c \leq d$. Since $c, d$ are atoms, we see that $c = d$. The equality $a \triangle b = c^\perp$ gives us $a \triangle a \triangle b = a \triangle c^\perp$. According to Proposition 1.2 we have $b = a \triangle c^\perp$. Since $a\ C\ c^\perp$, we see in view of Proposition 1.5 that $a\ C\ a \triangle c^\perp$. Hence $a\ C\ b$.

(ii) Suppose that $a \neg\ C\ b$. As known ([1] and [9]), $a\ C\ b$ precisely when $a\ C\ b^\perp$. It follows that $a \neq b^\perp$ and $a \neq b$. Then $a \triangle b \neq 1$ and $a \triangle b \neq 0$. If $a \triangle b$ were a co-atom, the part (i) gives us $a\ C\ b$. This implies that $a \triangle b$ is an atom in $K$. □

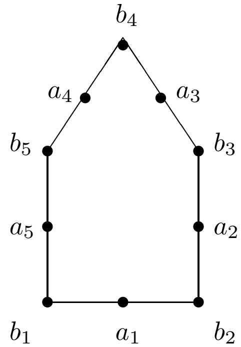
**Lemma 2.** *Suppose that $K$ is as in Theorem 3.10(ii). Let $a, b, c$ be atoms in $K$. Then $a \triangle b \triangle c = 1$ if and only if the atoms $a, b, c$ are pairwise distinct and pairwise compatible.*

Proof: If $a, b, c$ are pairwise distinct and pairwise compatible, they must be the atoms of a block of $K$. In this case $a \triangle b \triangle c = 1$.

Suppose that $a \triangle b \triangle c = 1$. Then $a, b, c$ are pairwise distinct. Indeed, if e.g. $a = b$, then $a \triangle b \triangle c = a \triangle a \triangle c = 0 \triangle c = c \neq 1$. Further, $a \triangle b = c^{\perp}$ and therefore $a \triangle b$ is a co-atom. It follows that $a\ C\ b$ (Lemma 1). Analogously, $a\ C\ c$ and $b\ C\ c$ and this completes the proof. □

**Lemma 3.** *Suppose that $K$ is as in Theorem 3.10(ii). Then $K$ does not contain a 5-loop.*

Proof: Suppose that it is not the case. Then there must be a configuration of blocks indicated by the following figure.




We see that we obtain the following collection of identities:
$b_1 \triangle a_1 \triangle b_2 = 1$, $b_2 \triangle a_2 \triangle b_3 = 1$, $b_3 \triangle a_3 \triangle b_4 = 1$, $b_4 \triangle a_4 \triangle b_5 = 1$, and $b_5 \triangle a_5 \triangle b_1 = 1$.
As a result, we have the equality
$(b_1 \triangle a_1 \triangle b_2) \triangle (b_2 \triangle a_2 \triangle b_3) \triangle (b_3 \triangle a_3 \triangle b_4) \triangle (b_4 \triangle a_4 \triangle b_5) \triangle (b_5 \triangle a_5 \triangle b_1) = 1 \triangle 1 \triangle 1 \triangle 1 \triangle 1$. Since $x \triangle x = 0$ for any $x$ in $K$, the right-hand side of the equality above equals to 1 and the left-hand side equals to $a_1 \triangle a_2 \triangle a_3 \triangle a_4 \triangle a_5$. Thus, $a_1 \triangle a_2 \triangle a_3 \triangle a_4 \triangle a_5 = 1$. Let us rewrite the last equality as follows: $(a_1 \triangle a_2) \triangle (a_3 \triangle a_4) \triangle a_5 = 1$. Lemma 1 gives us that $a_1 \triangle a_2$ as well as $a_3 \triangle a_4$ are atoms in $K$. Further, Lemma 2 implies that $a_1 \triangle a_2$ and $a_5$ are compatible atoms. Moreover, $a_1 \leq b_2^{\perp}$ and $a_2 \leq b_2^{\perp}$. This means that $a_1 \triangle a_2 \leq a_1 \vee a_2 \leq b_2^{\perp}$. We therefore see that $b_2\ C\ (a_1 \triangle a_2)$. But then $b_1$ and $a_1 \triangle a_2$ are distinct atoms that are compatible with $a_5$ and $b_2$. This contradicts Proposition 2.1(ii). The proof of Lemma 3 is complete. □

Proof of Theorem 3.10: It is easily seen that the proof of Theorem 3.10 can be obtained as an interplay of the Lemma 3 and Theorem 2.5. Indeed, suppose $K$ satisfies the conditions of Theorem 3.10(ii). Then as $K$ does not contain a 5-loop, to avoid a contradiction with Theorem 2.5 we must have $C(K) \neq \{0, 1\}$. But this means that $K$ is a 3-star (Proposition 2.3). □

**Theorem 3.11.** *The OML $L(R^3)$ is not ODL-convertible.*

PROOF: Suppose that $L(R^3)$ is ODL-convertible. Then $L(R^3)$ must be a 3-star (Theorem 3.10). But $C(L(R^3)) = \{0,1\}$ and we have reached a contradiction. The proof is complete. □

**Theorem 3.12.** *The OMLs $L(R^2)$ and $L(R^1)$ are ODL-convertible.*

PROOF: Of course, $L(R^1) = \{0,1\}$ and there is nothing to prove. Let us consider $L(R^2)$. Obviously, $L(R^2)$ is nothing but $MO_\kappa$, where $\kappa = 2^{\omega_0}$ (= the cardinality of continuum). This OML is ODL-convertible (Proposition 3.9). □

We have seen that a lack of $Z_2$-states on $L$ prevents $L$ from being ODL-embeddable (and, in turn, from being ODL-convertible). It should be noted that in [14] and [19] the authors construct finite OMLs without any group-valued state at all. Their technique therefore provides another type of OMLs that are not ODL-embeddable. However, the technique is very involved and even computer-proved in places. A relatively simple OML without any $Z_2$-states can be constructed on the ground of the following proposition. This proposition allows us to extend the class of non-embeddable OMLs, and it also slightly adds to the area of orthomodular peculiarities (see [4], [13], etc.). It should be noted that the result generalizes Proposition 7.2 of the paper [12].

**Proposition 3.13.** *Suppose that $L$ is an OML. Suppose that there are blocks $B_1, B_2, \dots, B_n$ of $L$ such that the following two conditions are satisfied:*

(1) *each $B_i$, $1 \le i \le n$ is finite and $n$ is an odd number,*
(2) *if $a \in L$ is an atom in $L$, then $a$ lies in an even number of blocks $B_1, B_2, \dots, B_n$ (i.e. the cardinality of the set $\{i;\ a \in B_i\}$ is even).*

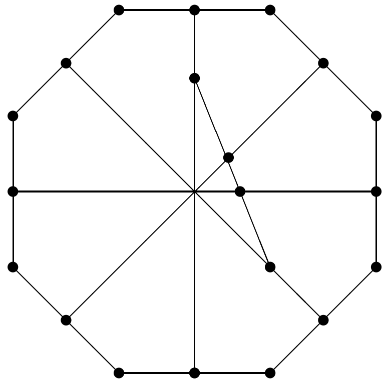
*Then there is no $Z_2$-state on $L$.*

PROOF: Seeking a contradiction, let $s : L \to Z_2$ be a $Z_2$-state. Let $\{a_{i,1}, \dots, a_{i,k_i}\}$ be the set of all atoms of the algebra $B_i$, $i = 1, \dots, n$. Then the elements $a_{i,1}, \dots, a_{i,k_i}$ are mutually orthogonal and, moreover, $a_{i,1} \vee \dots \vee a_{i,k_i} = 1_L$. Since $s$ is a $Z_2$-state, we have $s(a_{i,1} \vee \dots \vee a_{i,k_i}) = s(a_{i,1}) \oplus \dots \oplus s(a_{i,k_i})$. Since $a_{i,1} \vee \dots \vee a_{i,k_i} = 1_L$, we obtain $s(a_{i,1} \vee \dots \vee a_{i,k_i}) = s(1_L) = 1$. Summarizing, $s(a_{i,1}) \oplus \dots \oplus s(a_{i,k_i}) = 1$ for any $i \in \{1, \dots, n\}$. As a consequence,

$$(s(a_{1,1}) \oplus \dots \oplus s(a_{1,k_1})) \oplus \dots \oplus (s(a_{n,1}) \oplus \dots \oplus s(a_{n,k_n})) = 1 \oplus \dots \oplus 1.$$

The right-hand side of the latter identity contains the element 1 exactly $n$-many times. Since $n$ is odd, the right-hand side equals to 1. Moreover, if $a$ is an arbitrary atom of $L$, then the assumption of Proposition 3.13 gives us that the left-hand side of the identity contains the expression $s(a)$ an even number of times. By the property of the operation $\oplus$, the left-hand side must be equal to 0. We have derived a contradiction and the proof is complete. □

This result enables us to construct OMLs that do not possess a $Z_2$-state (and, as a consequence, the OMLs that are not ODL-embeddable). Let us conclude our paper by exhibiting a simple example of an OML in this class (the OML portrayed below by its Greechie diagram obviously satisfies the assumptions of Proposition 3.13; a proper class of such OMLs can be constructed in an analogous manner).



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