Strict topologies and Banach-Steinhaus type theorems

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Abstract. Let X be a completely regular Hausdorff space, E a real Banach space, and let $C_b(X,E)$ be the space of all E-valued bounded continuous functions on X. We study linear operators from $C_b(X,E)$ endowed with the strict topologies β_z $(z = \sigma, \tau, \infty, g)$ to a real Banach space $(Y, \|\cdot\|_Y)$. In particular, we derive Banach-Steinhaus type theorems for $(\beta_z, \|\cdot\|_Y)$ continuous linear operators from $C_b(X,E)$ to Y. Moreover, we study σ-additive and τ-additive operators from $C_b(X,E)$ to Y.

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1. Introduction

We denote by $\sigma(L,K)$ and $\tau(L,K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L,K\rangle$. Given a Hausdorff locally convex space (L,ξ) by L'_ξ we will denote its topological dual. Recall that (L,ξ) is said to be a *strong Mackey space* if every relatively countably $\sigma(L'_\xi, L)$-compact subset of L'_ξ is ξ-equicontinuous (see [K, p. 196], [KO$_2$, p. 482]). Clearly, if (L,ξ) is a strong Mackey space, then $\xi = \tau(L, L'_\xi)$.

Let X be a completely regular Hausdorff space and $(E, \|\cdot\|_E)$ a real Banach space. Let $C_b(X,E)$ be the Banach space of all E-valued bounded continuous functions on X provided with supremum norm $\|\cdot\|_\infty$. We will write $C_b(X)$ instead of $C_b(X,\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers. For a function $f \in C_b(X,E)$ we will write $\|f\|(x) = \|f(x)\|_E$ for all $x \in X$. Then $\|f\| \in C_b(X)$.

A subset H of $C_b(X,E)$ is said to be *solid* whenever $\|f_1\| \leq \|f_2\|$ (i.e., $\|f_1(x)\|_E \leq \|f_2(x)\|_E$ for all $x \in X$) and $f_1 \in C_b(X,E)$, $f_2 \in H$ implies $f_1 \in H$. A linear topology ξ on $C_b(X,E)$ is said to be *locally solid* if it has a local base at 0 consisting of solid sets (see [K], [KO$_2$], [NR]).

In the topological measure theory a number of locally convex topologies β_z on $C_b(X,E)$, called strict topologies have been studied. Definitions of strict topologies β_z base in some natural way on the topology of X, or perhaps its Stone-Čech compactification βX. Then $(C_b(X,E),\beta_z)'$ can be identified with some natural spaces $M_z(X,E')$ of E'-valued measures (see [F], [K], [KO$_1$], [KO$_2$], [KO$_3$], [KV], [NR]). In this paper we consider the strict topologies $\beta_\sigma, \beta_\tau, \beta_g$ and β_∞ on

$C_b(X,E)$. Note that in [F] and [K] the topologies β_σ and β_τ are denoted by β_1 and β respectively. It is well known that the strict topologies β_z ($z=\sigma,\tau,g$ and ∞) are locally solid (see [K, Theorem 8.1] for $z=\sigma,\tau,\infty$ and [KO$_2$, Theorem 6] for $z=g$). Moreover, $(C_b(X,E),\beta_z)$ is a strong Mackey space for $z=\sigma$ (see [KO$_1$, Corollary 6]); $z=\tau$ and X paracompact (see [K, Theorem 6.1]); $z=\infty$ (see [K, Theorem 3.7]) and $z=g$ (see [KO$_2$, Theorem 7]).

From now on $(Y,\|\cdot\|_Y)$ is a real Banach space, and let Y' stand for its Banach dual. Let $\mathcal{L}(C_b(X,E),Y)$ stand for the space of all bounded ($=(\|\cdot\|_\infty,\|\cdot\|_Y)$-continuous) linear operators from $C_b(X,E)$ to Y. The *strong operator topology* (briefly SOT) is the locally convex topology on $\mathcal{L}(C_b(X,E),Y)$ defined by the family of seminorms $\{p_f : f\in C_b(X,E)\}$, where $p_f(T)=\|T(f)\|_Y$ for all $T\in\mathcal{L}(C_b(X,E),Y)$. The *weak operator topology* (briefly WOT) is the locally convex topology on $\mathcal{L}(C_b(X,E),Y)$ defined by the family of seminorms $\{p_{f,y'} : f\in C_b(X,E),\ y'\in Y'\}$, where $p_{f,y'}(T)=|\langle T(f),y'\rangle|$ for all $T\in\mathcal{L}(C_b(X,E),Y)$. In view of the Banach-Steinhaus theorem the space $\mathcal{L}(C_b(X,E),Y)$ provided with SOT is sequentially complete. By $\mathcal{L}_{\beta_z}(C_b(X,E),Y)$ (for $z=\sigma,\tau,g$ and ∞) we will denote the subspace of $\mathcal{L}(C_b(X,E),Y)$ consisting of all those $T\in\mathcal{L}(C_b(X,E),Y)$ which are $(\beta_z,\|\cdot\|_Y)$-continuous.

In Section 2 we study topological properties of the spaces $\mathcal{L}_{\beta_z}(C_b(X,E),Y)$, where $z=\sigma,\tau,g$ and ∞. In particular, we derive Banach-Steinhaus type theorems for $(\beta_z,\|\cdot\|_Y)$-continuous linear operators from $C_b(X,E)$ to Y (see Theorem 2.5 and Corollary 2.6 below). In Section 3 we consider σ-additive and τ-additive operators from $C_b(X,E)$ to Y.

2. Linear operators on $C_b(X,E)$ with strict topologies

For a bounded linear operator $T : C_b(X,E)\longrightarrow Y$ let $T' : Y'\longrightarrow C_b(X,E)'$ denote its conjugate, i.e., $\langle f,T'(y')\rangle=\langle T(f),y'\rangle$ for $f\in C_b(X,E)$ and $y'\in Y'$.

Proposition 2.1. *Let $T : C_b(X,E)\longrightarrow Y$ be a bounded linear operator and let $z=\sigma,g,\infty$ ($z=\tau$ and X is paracompact). Then the following statements are equivalent:*

(i) *$T'(Y')\subset C_b(X,E)'_{\beta_z}$, i.e., $y'\circ T\in C_b(X,E)'_{\beta_z}$ for each $y'\in Y'$;*
(ii) *T is $(\sigma(C_b(X,E),C_b(X,E)'_{\beta_z}),\ \sigma(Y,Y'))$-continuous;*
(iii) *T is $(\beta_z,\|\cdot\|_Y)$-continuous.*

Proof: (i)$\Longleftrightarrow$(ii) General fact; see [AB, Proposition 9.26].

(ii)$\Longleftrightarrow$(iii) It is known that T is $(\sigma(C_b(X,E),C_b(X,E)'_{\beta_z}),\sigma(Y,Y'))$-continuous if and only if T is $(\tau(C_b(X,E),C_b(X,E)'_{\beta_z}),\tau(Y,Y'))$-continuous (see [AB; Ex. 11, p. 149]). Since $\beta_z=\tau(C_b(X,E),C_b(X,E)'_{\beta_z})$ and $\tau(Y,Y')$ coincides with the $\|\cdot\|_Y$-topology, the proof is complete.

(iii)$\Longrightarrow$(i) It is obvious. $\square$

Proposition 2.2. *$\mathcal{L}_{\beta_z}(C_b(X,E),Y)$ is a sequentially closed subspace of $\mathcal{L}(C_b(X,E),Y)$ for WOT, where $z=\sigma,g,\infty$ (resp. $z=\tau$ and X is paracompact).*

PROOF: Let (T_n) be a sequence in $\mathcal{L}_{\beta_z}(C_b(X,E),Y)$ such that $T_n \longrightarrow T$ for WOT, where $T\in\mathcal{L}(C_b(X,E),Y)$. Given $y'_o\in Y'$, for each $f\in C_b(X,E)$ we get $(y'_o\circ T)(f)=\lim(y'_o\circ T_n)(f)$, where $y'_o\circ T_n\in C_b(X,E)'_{\beta_z}$ for $n\in\mathbb{N}$, and $y'_o\circ T\in C_b(X,E)'$. It follows that $(y'_o\circ T_n)$ is a $\sigma(C_b(X,E)'_{\beta_z},C_b(X,E))$-Cauchy sequence in $C_b(X,E)'_{\beta_z}$. Since the space $(C_b(X,E)'_{\beta_z},\sigma(C_b(X,E)'_{\beta_z},C_b(X,E)))$ is sequentially complete (see [KO$_3$, Theorem 3]), there exists $\Phi_o\in C_b(X,Y)'_{\beta_z}$ such that $\Phi_o(f)=\lim(y'_o\circ T_n)(f)$ for each $f\in C_b(X,E)$. Hence $y'_o\circ T=\Phi_o\in C_b(X,E)'_{\beta_z}$, and by Proposition 2.1 we get $T\in\mathcal{L}_{\beta_z}(C_b(X,E),Y)$. □

Corollary 2.3. *Let $z=\sigma,g,\infty$ ($z=\tau$ and X is paracompact). Then*

(i) *$\mathcal{L}_{\beta_z}(C_b(X,E),Y)$ is a sequentially closed subspace of $\mathcal{L}(C_b(X,E),Y)$ for* SOT*;*

(ii) *the space* $(\mathcal{L}_{\beta_z}(C_b(X,E),Y)$, SOT) *is sequentially complete.*

PROOF: (i) It follows from Proposition 2.2 because WOT $\subset$ SOT.

(ii) It follows from (i) because the space $(\mathcal{L}(C_b(X,E),Y),SOT)$ is sequentially complete. □

The following general result will be of importance (see [SZ, Theorem 2]).

Proposition 2.4. *Let $\mathcal{K}$ be a SOT-compact subset of $\mathcal{L}_{\beta_z}(C_b(X,E),Y)$, where $z=\sigma,g,\infty$ (resp. $z=\tau$ and X is paracompact). If H is a $\sigma(Y',Y)$-closed and $\|\cdot\|_Y$-equicontinuous subset of Y', then the set $\bigcup\{T'(H):T\in\mathcal{K}\}$ is a $\sigma(C_b(X,E)'_{\beta_z},\,C_b(X,E))$-compact subset of $C_b(X,E)'_{\beta_z}$.*

Now we are ready to prove Banach-Steinhaus type theorems for $(\beta_z,\|\cdot\|_Y)$-continuous linear operators from $C_b(X,E)$ to Y.

Theorem 2.5. *Let $\mathcal{K}$ be a SOT-compact subset of $\mathcal{L}_{\beta_z}(C_b(X,E),Y)$, where $z=\sigma,\ g,\infty$ (resp. $z=\tau$ and X is paracompact). Then $\mathcal{K}$ is a $(\beta_z,\|\cdot\|_Y)$-equicontinuous.*

PROOF: Since the closed unit ball $B_{Y'}$ in Y' is $\sigma(Y',Y)$-closed and $(\|\cdot\|_Y)$-equicontinuous (see [AB, Theorem 9.21]), by Proposition 2.4 the set $Z=\bigcup\{T'(B_{Y'}):T\in\mathcal{K}\}$ is $\sigma(C_b(X,E)'_{\beta_z},C_b(X,E))$-compact subset of $C_b(X,E)'_{\beta_z}$. Since $(C_b(X,E),\beta_z)$ is a strong Mackey space, the set $Z=\{y'\circ T:T\in\mathcal{K},\ y'\in B_{Y'}\}$ is $(\beta_z,\|\cdot\|_Y)$-equicontinuous. Then for given $\varepsilon>0$ there exists a neighbourhood V_ε of 0 for β_z such that for all $f\in V_\varepsilon$,

$$\sup_{T\in\mathcal{K}}\|T(f)\|_Y=\sup\{|(y'\circ T)(f)|:T\in\mathcal{K},\ y'\in Y'\}\le\varepsilon.$$

This means that $\mathcal{K}$ is $(\beta_z,\|\cdot\|)$-equicontinuous, as desired. □

Corollary 2.6. *Let $T_k : C_b(X,E) \longrightarrow Y$ be $(\beta_z, \|\cdot\|_Y)$-continuous linear operators for $z = \sigma, g, \infty$ (resp. $z = \tau$ and X is paracompact) and $k \in \mathbb{N}$. Assume that $T(f) := \lim T_k(f)$ exists in $(Y, \|\cdot\|_Y)$ for each $f \in C_b(X,E)$. Then T is $(\beta_z, \|\cdot\|_Y)$-continuous and the set $\{T_k : k \in \mathbb{N}\}$ is $(\beta_z, \|\cdot\|_Y)$-equicontinuous.*

Proof: In view of the Banach-Steinhaus theorem T is bounded. Hence by Corollary 2.3 T is $(\beta_z, \|\cdot\|_Y)$-continuous. Then $T_k \longrightarrow T$ in $\mathcal{L}_{\beta_z}(C_b(X,E), Y)$ for SOT, so $\{T_k : k \in \mathbb{N}\} \cup \{T\}$ is a SOT-compact subset of $\mathcal{L}_{\beta_z}(C_b(X,E), Y)$. Hence by Theorem 2.5 the set $\{T_k : k \in \mathbb{N}\}$ is $(\beta_z, \|\cdot\|_Y)$-equicontinuous. □

3. σ-additive and τ-additive operators on $C_b(X,E)$

We start by recalling definitions of σ-Dini and Dini topologies on $C_b(X,E)$ (see [NR, Definition 5.1]). For a net (u_α) in $C_b(X)$ we will write $u_\alpha \downarrow 0$ whenever $u_\alpha(x) \downarrow 0$ for all $x \in X$.

Definition 3.1. Let ξ be a locally solid topology on $C_b(X,E)$.

(i) ξ is said to be a *σ-Dini topology* if, whenever (f_n) is a sequence in $C_b(X,E)$ such that $\|f_n\| \downarrow 0$, then $f_n \longrightarrow 0$ for ξ.

(ii) ξ is said to be a *Dini topology* if, whenever (f_α) is a net in $C_b(X,E)$ such that $\|f_\alpha\| \downarrow 0$, then $f_\alpha \longrightarrow 0$ for ξ.

We have (see [NR, Theorem 5.2]):

Proposition 3.1. *β_σ (resp. β_τ) is the finest locally convex σ-Dini (resp. Dini) topology on $C_b(X,E)$.*

Following [F, Definition 2.1] we can extend the definition of σ-additive and τ-additive linear functionals on $C_b(X,E)$ to linear operators from $C_b(X,E)$ to Y.

Definition 3.2. Let $T : C_b(X,E) \longrightarrow Y$ be a bounded linear operator, and let $B_\infty = \{g \in C_b(X,E) : \|g\|_\infty \leq 1\}$.

(i) T is said to be *σ-additive* if $\sup_{g \in B_\infty} \|T(u_n\, g)\|_Y \xrightarrow[n]{} 0$ for each sequence (u_n) in $C_b(X)$ such that $u_n \downarrow 0$.

(ii) T is said to be *τ-additive* if $\sup_{g \in B_\infty} \|T(u_\alpha\, g)\|_Y \xrightarrow[\alpha]{} 0$ for each net (u_α) in $C_b(X)$ such that $u_\alpha \downarrow 0$.

By $L_\sigma(C_b(X,E), Y)$ (resp. $L_\tau(C_b(X,E), Y)$) we will denote the set of all σ-additive (resp. τ-additive) operators from $C_b(X,E)$ to Y.

Proposition 3.2. *For a bounded linear operator $T : C_b(X,E) \longrightarrow Y$ the following statements are equivalent:*

(i) *$y' \circ T \in C_b(X,E)'_{\beta_\sigma}$ for each $y' \in Y'$;*
(ii) *T is $(\beta_\sigma, \|\cdot\|_Y)$-continuous;*
(iii) *T is $(\beta_\sigma, \|\cdot\|_Y)$-sequentially continuous;*
(iv) *T is σ-additive.*

PROOF: (i)$\Longleftrightarrow$(ii) It follows from Proposition 2.1.

(ii)$\Longrightarrow$(iii) It is obvious.

(iii)$\Longrightarrow$(iv) Assume that T is $(\beta_z, \|\cdot\|_Y)$-sequentially continuous, and let $u_n \downarrow 0$ in $C_b(X)$. Note that $\sup_{g\in B_\infty} \|T(u_n\, g)\|_Y \le \|T\| \cdot \|u_1\|_\infty < \infty$. Let $\varepsilon > 0$ be given. Then for each $n \in \mathbb{N}$ there exists $g_n \in B_\infty$ such that

$$\sup_{g\in B_\infty} \|T(u_n\, g)\|_Y \le \|T(u_n\, g_n)\|_Y + \frac{\varepsilon}{2}\,.$$

For a fixed $e_o \in S_E$ (= the closed unit sphere in E), we have $\|(u_n g)(x)\|_E \le \|(u_n \otimes e_o)(x)\|_E = \|u_n(x)e_o\|_E = u_n(x) \downarrow 0$ for all $x \in X$. Since β_σ is a σ-Dini topology, we obtain that $u_n \otimes e_o \longrightarrow 0$ for β_σ. Hence $u_n\, g \longrightarrow 0$ for β_σ because β_σ is locally solid. It follows that $\|T(u_n g_n)\|_Y \longrightarrow 0$. Choose $n_\varepsilon \in \mathbb{N}$ such that $\|T(u_n g_n)\|_Y \le \frac{\varepsilon}{2}$ for $n \ge n_\varepsilon$. It follows that $\sup_{g\in B_\infty} \|T(u_n g)\|_Y \le \varepsilon$ for $n \ge n_\varepsilon$, i.e., T is σ-additive.

(iv)$\Longrightarrow$(i) It follows from [F, Theorem 2.3]. □

Hence we have

$$\mathcal{L}_{\beta_\sigma}(C_b(X,E), Y) = L_\sigma(C_b(X,E), Y).$$

Now we are in position to state a Banach-Steinhaus type theorem for σ-additive operators from $C_b(X,E)$ to Y.

Theorem 3.3. *Let $\mathcal{K}$ be a SOT-compact subset of $L_\sigma(C_b(X,E), Y)$. Then $\mathcal{K}$ is uniformly σ-additive, i.e.,*

$$\sup_{T\in\mathcal{K}} \big(\sup_{g\in B_\infty} \|T(u_n\, g)\|_Y \big) \xrightarrow[n]{} 0 \quad \textit{whenever} \quad u_n \downarrow 0 \quad \textit{in} \quad C_b(X).$$

PROOF: By Theorem 2.5 the set $\mathcal{K}$ is $(\beta_\sigma, \|\cdot\|_Y)$-equicontinuous. Let $\varepsilon > 0$ be given. Then there exists a solid neighbourhood V_ε of 0 for β_σ such that $\|T(f)\|_Y \le \varepsilon$ for all $f \in V_\varepsilon$ and each $T \in \mathcal{K}$. Now let $u_n \downarrow 0$ in $C_b(X)$, and let $e_o \in S_E$ be fixed. Then for each $g \in B_\infty$ we have $\|(u_n\, g)(x)\|_E \le \|(u_n \otimes e_o)(x)\|_E = u_n(x) \downarrow 0$ for all $x \in X$. Since β_σ is a σ-Dini topology, there exists $n_\varepsilon \in \mathbb{N}$ such that $u_n \otimes e_o \in V_\varepsilon$ for $n \ge n_\varepsilon$. Hence for each $g \in B_\infty$, we get $u_n\, g \in V_\varepsilon$ for $n \ge n_\varepsilon$, because V_ε is a solid subset of $C_b(X,E)$. It follows that for $n \ge n_\varepsilon$, $\sup_{T\in\mathcal{K}}(\sup_{g\in B_\infty} \|T(u_n\, g)\|_Y) \le \varepsilon$. □

As an application of Corollary 2.6 and Theorem 3.3 we have

Corollary 3.4. *Let $T_k : C_b(X,E) \longrightarrow Y$ be σ-additive operators for $k \in \mathbb{N}$, and assume that $T(f) := \lim_k T_k(f)$ exists in $(Y, \|\cdot\|_Y)$ for each $f \in C_b(X,E)$. Then T is a σ-additive operator and the set $\{T_k : k \in \mathbb{N}\}$ is uniformly σ-additive, i.e., $\sup_k(\sup_{g\in B_\infty} \|T_k(u_n\, g)\|_Y) \xrightarrow[n]{} 0$ whenever $u_n \downarrow 0$ in $C_b(X)$.*

Following the proofs of Proposition 3.2, Theorem 3.3 and Corollary 3.4 we can derive analogous results for τ-additive operators from $C_b(X,E)$ to Y.

Proposition 3.5. *Assume that X is paracompact. Then for a bounded linear operator $T : C_b(X,E) \longrightarrow Y$ the following statements are equivalent:*

(i) $y' \circ T \in C_b(X,E)'_{\beta_\tau}$ *for each* $y' \in Y'$;

(ii) T *is* $(\beta_\tau, \|\cdot\|_Y)$*-continuous;*

(iii) T *is* τ*-additive.*

Hence, if X is paracompact, then

$$\mathcal{L}_{\beta_\tau}(C_b(X,E),Y) = L_\tau(C_b(X,E),Y).$$

Theorem 3.6. *Assume that X is paracompact. Let $\mathcal{K}$ be a SOT-compact subset of $L_\tau(C_b(X,E),Y)$. Then $\mathcal{K}$ is uniformly τ-additive, i.e.,*

$$\sup_{T\in\mathcal{K}}\Big(\sup_{g\in B_\infty}\|T(u_\alpha g)\|_Y\Big) \xrightarrow[\alpha]{} 0 \quad \textit{whenever} \quad u_\alpha \downarrow 0 \quad \textit{in} \quad C_b(X).$$

Corollary 3.7. *Assume that X is paracompact. Let $T_k : C_b(X,E) \longrightarrow Y$ be τ-additive operators for $k \in \mathbb{N}$, and assume that $T(f) := \lim T_k(f)$ exists in $(Y, \|\cdot\|_Y)$ for each $f \in C_b(X,E)$. Then $T : C_b(X,E) \longrightarrow Y$ is a τ-additive operator and the set $\{T_k : k \in \mathbb{N}\}$ is uniformly τ-additive, i.e., $\sup_k \left(\sup_{g\in B_\infty}\|T_k(u_\alpha g)\|_Y\right) \xrightarrow[\alpha]{} 0$ whenever $u_\alpha \downarrow 0$ in $C_b(X)$.*

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