On reflexive closed set lattices

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Abstract. For a topological space X, let $S(X)$ denote the set of all closed subsets in X, and let $C(X)$ denote the set of all continuous maps $f : X \to X$. A family $\mathcal{A} \subseteq S(X)$ is called *reflexive* if there exists $\mathcal{C} \subseteq C(X)$ such that $\mathcal{A} = \{A \in S(X) : f(A) \subseteq A$ for every $f \in \mathcal{C}\}$. Every reflexive family of closed sets in space X forms a sub complete lattice of the lattice of all closed sets in X. In this paper, we continue to study the reflexive families of closed sets in various types of topological spaces. More necessary and sufficient conditions for certain families of closed sets to be reflexive are obtained.

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1. Introduction

Given a topological space X, let $S(X)$ denote the set of all closed subsets of X and $C(X)$ denote the set of all continuous maps $f : X \to X$. For any $\mathcal{A} \subseteq S(X)$ and $\mathcal{F} \subseteq C(X)$ define

$$\mathrm{Alg}(\mathcal{A}) = \{f \in C(X) : f(A) \subseteq A \ \text{ for every } \ A \in \mathcal{A}\} \ \text{ and}$$
$$\mathrm{Lat}(\mathcal{F}) = \{A \in S(X) : f(A) \subseteq A \ \text{ for every } \ f \in \mathcal{F}\}.$$

A family $\mathcal{A} \subseteq S(X)$ is called *reflexive* if there exists $\mathcal{F} \subseteq C(X)$ such that $\mathcal{A} = \mathrm{Lat}(\mathcal{F})$, or equivalently, if $\mathrm{Lat}(\mathrm{Alg}(\mathcal{A})) = \mathcal{A}$.

If $\mathcal{A} \subseteq S(X)$ is reflexive, then the following conditions are satisfied [10]:

(a) $X, \emptyset \in \mathcal{A}$,
(b) $\mathcal{B} \subseteq \mathcal{A}$ implies $\bigcap \mathcal{B} \in \mathcal{A}$, and
(c) $\mathcal{B} \subseteq \mathcal{A}$ implies $\mathrm{cl}(\bigcup \mathcal{B}) \in \mathcal{A}$,

where cl is the closure operator.

A family $\mathcal{A}$ of closed sets satisfying conditions (a), (b) and (c) will be called a *closed set lattice*. In [11] it was proved that in a discrete space every closed set lattice is reflexive. In [10] we showed that a locally compact metric space is zero-dimensional if and only if every closed set lattice in the space is reflexive.

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In the current paper, we continue to consider the problem: under what conditions is a closed set lattice reflexive? In Section 2 we prove some properties of reflexive families of closed sets in a general space. In particular, we give a necessary and sufficient condition for a closed set lattice in an arbitrary space to be reflexive. Section 3 is devoted to the investigation of reflexive closed set lattices in the real line $\mathbb{R}$. For certain classes of finite families we proved a sufficient condition for them to be reflexive. The main result in Section 4 is that every reflexive closed set lattice is a closed subset in the hyperspace of all closed sets with the Vietoris topology. In the last section, we use set-valued maps to establish a necessary and sufficient condition for a closed set lattice in an arbitrary space to be reflexive.

The problem on reflexive families of closed subspaces in a Hilbert space has been studied extensively by Halmos, Longstaff and others (see [2]–[7]).

2. Some general properties of reflexive families of closed sets

Let $\mathcal{A}$ be a closed set lattice in a topological space X. For each $B \subseteq X$, define $\phi_{\mathcal{A}}(B) = \bigcap\{A \in \mathcal{A} : B \subseteq A\}$. We shall write $\phi(B)$ for $\phi_{\mathcal{A}}(B)$ where no confusion occurs; and $\phi(x)$ for $\phi(\{x\})$ for each $x \in X$. From condition (b) it is seen that $\phi_{\mathcal{A}}(B) \in \mathcal{A}$ for all $B \subseteq X$. For every $B \subseteq X$, $B \in \mathcal{A}$ if and only if $\phi_{\mathcal{A}}(B) = \bigcup\{\phi_{\mathcal{A}}(x) : x \in B\}$ (see [10, Lemma 3]). Consequently, for any two closed set lattices $\mathcal{A}$ and $\mathcal{B}$, we have that $\mathcal{A} = \mathcal{B}$ if and only if $\phi_{\mathcal{A}}(x) = \phi_{\mathcal{B}}(x)$ for every $x \in X$. For two closed sets A, B with $A \subseteq B$, define

$$\Phi(A,B) = \bigcup\{C : C \text{ is a connected component of } B \text{ and } C \cap A \neq \emptyset\},$$

and

$$\Phi_P(A,B) = \bigcup\{C : C \text{ is a path-connected component of } B \text{ and } C \cap A \neq \emptyset\}.$$

Then $A \subseteq \Phi_P(A,B) \subseteq \Phi(A,B) \subseteq B$.

Lemma 1. *If $\mathcal{A}$ is a reflexive closed set lattice in a space, then it satisfies the following condition:*

(d) *If $A, B \in \mathcal{A}$ and $A \subseteq B$, then* $\mathrm{cl}(\Phi(A,B)), \mathrm{cl}(\Phi_P(A,B)) \in \mathcal{A}$.

PROOF: As an example, we show that $\mathrm{cl}(\Phi_P(A,B)) \in \mathcal{A}$. By the property of continuous maps, it is enough to show that $f(\Phi_P(A,B)) \subseteq \Phi_P(A,B)$ for any $f \in \mathrm{Alg}(\mathcal{A})$. For each path-connected component C of B, $f(C)$ is path-connected and is contained in B. Thus $f(C) \subseteq C'$ for some path-connected component C' of B. Furthermore, if $C \cap A \neq \emptyset$, then $f(C) \cap A \neq \emptyset$ since $A \in \mathcal{A}$. It implies that $C' \cap A \neq \emptyset$. Therefore, $f(C) \subseteq C' \subseteq \Phi_P(A,B)$. It thus follows that $f(\Phi_P(A,B)) \subseteq \Phi_P(A,B)$. □

The following lemma is an easy observation.

Lemma 2. *Let $\mathcal{A}$ be a family of closed sets in a space X satisfying condition* (d). *If $A, B \in \mathcal{A}$, $A \subseteq B$ and there is no other member of $\mathcal{A}$ lying between A and B, then either* $\mathrm{cl}(\Phi(A,B)) = A$ *or* $\mathrm{cl}(\Phi(A,B)) = B$.

Lemma 3. *Suppose that $X = \mathbb{R}$ or X is a compact Hausdorff space and $\mathcal{A}$ is a reflexive closed set lattice in X. Then the following condition is satisfied:*

(d′) *If $A, B \in \mathcal{A}$ and $A \subseteq B$ then $\Phi(A, B) \in \mathcal{A}$.*

Proof: By Lemma 1 it is enough to verify that $D = \Phi(A, B)$ is closed. We first prove the conclusion for $X = \mathbb{R}$. Suppose that x is an accumulation point of D. Then, without loss of generality, we may assume that there exists a sequence (x_n) such that $x_n \in C_n$ for some connected component C_n of the subset B with $C_n \cap A \neq \emptyset$, $x_1 < x_2 < \cdots$ and $\lim_{n\to\infty} x_n = x$. In addition, we can assume that $C_n \neq C_m$ if $n \neq m$. Then $C_n = [a_n, b_n]$ and $a_n \leq b_n < a_{n+1}$ for every n. It follows from $C_n \cap A \neq \emptyset$ that there exists $y_n \in C_n \cap A$ for every n. Trivially, $x = \lim_{n\to\infty} y_n \in A \subseteq D$. Hence, D is closed.

Now let X be a compact Hausdorff space. For any $x \in \mathrm{cl}(D)$ we need to show that $x \in D$. For this it suffices to show that if C_0 is the connected component of B containing x then $C_0 \cap A \neq \emptyset$. Suppose, by contraposition, that $C_0 \cap A = \emptyset$. By [1, 6.1.23 Theorem], $\bigcap\{C : C \text{ is a clopen set in } B \text{ and } C \ni x\} = C_0$. Since X is compact, there exist clopen sets $C_1, C_2, \ldots, C_k$ in B containing x such that $C_1 \cap C_2 \cap \cdots \cap C_k \cap A = \emptyset$. On the other hand, $C_1 \cap C_2 \cap \cdots \cap C_k$ is clopen in B containing x and $x \in \mathrm{cl}(D)$, thus there is a connected component C of B with $C \cap A \neq \emptyset$ and $C \cap C_1 \cap C_2 \cap \cdots \cap C_k \neq \emptyset$. Since C is connected and $C_1 \cap C_2 \cap \cdots \cap C_k$ is a clopen set in B, we have $C \subseteq C_1 \cap C_2 \cap \cdots \cap C_k$. This contradicts that $C_1 \cap C_2 \cap \cdots \cap C_k \cap A = \emptyset$. The proof is completed. □

Remark 1. The above lemma is not true when $X = \mathbb{R}^2$. For example, $A = \{(\frac{1}{n}, n) : n = 1, 2, \ldots\}$ and $B = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\} \times \mathbb{R}$ are closed sets in $\mathbb{R}^2$ and $A \subseteq B$ but $\Phi(A, B) = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \times \mathbb{R}$ is not closed in $\mathbb{R}^2$.

Lemma 4. *Suppose that $X = \mathbb{R}$ or X is a compact Hausdorff space and $\mathcal{A}$ is a closed set lattice in X which satisfies the condition* (d′). *If $b \in X$, $B \in \mathcal{A}$, $B \subseteq \phi(b)$ and $B \cap C_0 \neq \emptyset$ for some connected component C_0 of $\phi(b)$ containing b, then $B \cap C \neq \emptyset$ for all connected component C of $\phi(b)$.*

Proof: Let $E = \bigcup\{C : C \cap B \neq \emptyset \text{ and } C \text{ is a connected component of } \phi(b)\}$. Then by condition (d′), $E \in \mathcal{A}$, thus $E = \phi(b)$ because $b \in E$. For any connected component C' of $\phi(b)$, if $C' \cap B = \emptyset$ then $C' \cap E = \emptyset$, which is impossible as $C' \subseteq \phi(b) = E$. It follows that every connected component of $\phi(b)$ must intersect B. □

In the sequel, we shall frequently use the following fact: A family $\mathcal{A}$ of closed sets is reflexive if and only if for any closed set B not in $\mathcal{A}$, there exists $f \in \mathrm{Alg}(\mathcal{A})$ such that $f(B) \not\subseteq B$.

Theorem 1. *Let $\mathcal{A}$ be a closed set lattice in a space X. Then $\mathcal{A}$ is reflexive if and only if for each $x \in X$, the set $\{f(x) : f \in \mathrm{Alg}(\mathcal{A})\}$ is dense in $\phi(x)$.*

Proof: Let $\mathcal{A}$ be reflexive. For any $x \in X$ let $D_x = \{f(x) : f \in \mathrm{Alg}(\mathcal{A})\}$. Note that $\mathrm{Alg}(\mathcal{A})$ is closed under composition. So $g(D_x) \subseteq D_x$ for every $g \in \mathrm{Alg}(\mathcal{A})$.

Since each $g \in \mathrm{Alg}(\mathcal{A})$ is continuous, $g(\mathrm{cl}(D_x)) \subseteq \mathrm{cl}(D_x)$. Thus $\mathrm{cl}(D_x) \in \mathcal{A}$. In addition, if $x \in A \in \mathcal{A}$, then $D_x \subseteq A$, thus $\mathrm{cl}(D_x) \subseteq \phi(x)$. It follows from $x \in D_x \subseteq \mathrm{cl}(D_x)$ that $\mathrm{cl}(D_x) = \phi(x)$.

Now assume that for each $x \in X$, $\{f(x) : f \in \mathrm{Alg}(\mathcal{A})\}$ is dense in $\phi(x)$. Let $\mathcal{B} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$. By a general fact, $\mathcal{B}$ is reflexive and $\mathrm{Alg}(\mathcal{B}) = \mathrm{Alg}(\mathcal{A})$ (see [10]). Fix an element $x \in X$. Then $\{f(x) : f \in \mathrm{Alg}(\mathcal{B})\} = \{f(x) : f \in \mathrm{Alg}(\mathcal{A})\}$, so by the assumption and the proved necessity part, this set is dense in $\phi_{\mathcal{B}}(x)$ and $\phi_{\mathcal{A}}(x)$, respectively. Thus $\phi_{\mathcal{B}}(x) = \phi_{\mathcal{A}}(x)$ since these two sets are closed. By the remark at the end of the first paragraph of this section, we deduce that $\mathcal{A} = \mathcal{B}$, so $\mathcal{A}$ is reflexive. $\square$

Trivially, we have the following corollary.

Corollary 1. *Let $\mathcal{A}$ be a closed set lattice of a space X. If for any $b \in X$ and any $c \in \phi(b)$ there is $f \in \mathrm{Alg}(\mathcal{A})$ such that $f(b) = c$, then $\mathcal{A}$ is reflexive.*

The following example shows that the converse of Corollary 1 is not true.

Example 1. Let

$$X = (\{0\} \times [-1, 1]) \cup \left\{ \left(x, \sin \frac{1}{x}\right) \mid x \in (0, 1] \right\}$$

with the topology of subspace of $\mathbb{R}^2$, and let $\mathcal{A} = \{\emptyset, \{(1, \sin 1)\}, X\}$. It follows from the proof of [10, Proposition 1] that $\mathcal{A}$ is reflexive. Observe that $\phi((\frac{1}{2}, \sin \frac{1}{2})) = X$. However for any $f \in \mathrm{Alg}(\mathcal{A})$, $f(1, \sin 1) = (1, \sin 1)$, thus $f(\frac{1}{2}, \sin \frac{1}{2})$ does not belong to the set $\{0\} \times [-1, 1])$ because the image of the set $\{(x, \sin \frac{1}{x}) | \, x \in (0, 1]\}$ under f must be path-connected. Thus there is no $f \in \mathrm{Alg}(\mathcal{A})$ such that $f(\frac{1}{2}, \sin \frac{1}{2}) = (0, 0) \in \phi((\frac{1}{2}, \sin \frac{1}{2}))$.

In next section, we shall prove some results on reflexivity of closed set lattices in the Euclidean space $\mathbb{R}$. These results are not true for $\mathbb{R}^2$. The Example 3 in Section 5 indicates that the converse of Corollary 1 fails even when $X = \mathbb{R}$ or X is a closed interval in $\mathbb{R}$.

3. Families of closed sets of $\mathbb{R}$

In this section we consider the reflexive problem for closed set lattices in the Euclidean space $\mathbb{R}$ of real numbers.

Theorem 2. *A finite chain $\mathcal{A}$ of closed subsets of $\mathbb{R}$ is reflexive if and only if it satisfies conditions* (a) *and* (d$'$).

Proof: We only need to prove the "if" part. Suppose that $\mathcal{A} : \emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = \mathbb{R}$ is a chain of closed sets in $\mathbb{R}$, and $\mathcal{A}$ satisfies conditions (a) and (d$'$). By Corollary 1, it suffices to verify that for every $b \in \mathbb{R}$ and $c \in \phi(b)$, there exists $h \in \mathrm{Alg}(\mathcal{A})$ such that $h(b) = c$.

Fact 1. For every $b \in \mathbb{R}$ and $c \in \phi(b)$, there exists a continuous map $f : (-\infty, b] \to \mathbb{R}$ such that $f(b) = c$ and $f(A_j \cap (-\infty, b]) \subseteq A_j$ for every $j \leq n$.

Fact 2. For every $b \in \mathbb{R}$ and $c \in \phi(b)$, there exists a continuous map $g : [b, +\infty) \to \mathbb{R}$ such that $g(b) = c$ and $g(A_j \cap [b, +\infty)) \subseteq A_j$ for every $j \leq n$.

In order to prove the above facts, we first introduce some symbols. For each $x \in \mathbb{R}$, let $i(x) = \min\{k : x \in A_k\}$. Then $\phi(x) = A_{i(x)}$ for all $x \in \mathbb{R}$. For any $k \geq i(x)$, let $C(x, k)$ be the connected component of A_k containing x and $C_-(x, k) = (-\infty, x] \cap C(x, k)$. Then $C(x, i)$ and $C_-(x, i)$ both are closed intervals in $\mathbb{R}$. Using the above symbols, Fact 1 can be rephrased as the following Fact 3:

Fact 3. For every $k \leq n$, $b \in \mathbb{R}$ with $i(b) = k$ and $c \in A_k$, there exists a continuous map $f : (-\infty, b] \to \mathbb{R}$ such that $f(b) = c$ and $f(A_j \cap (-\infty, b]) \subseteq A_j$ for every $j \leq n$.

We prove Fact 3 by induction on $i(b)$. If $i(b) = 1$, let $f(x) \equiv c$ for every $x \in (-\infty, b]$, then f is a required map. Suppose such an f exists for all b with $i(b) < k$. Let $b \in \mathbb{R}$ with $i(b) = k$ and $c \in A_k$. If $A_{k-1} \cap (-\infty, b] = \emptyset$, the constant map $f(x) \equiv c$ for every $x \in (-\infty, b]$ satisfies the requirement. Now assume that $A_{k-1} \cap (-\infty, b] \neq \emptyset$. Let $b_1 = \max\{x \in A_{k-1} : x < b\}$, $l = \min\{j : k \leq j \leq n$ and $C_-(b, j) \cap A_{i(b_1)} \neq \emptyset\}$ (note that $C_-(b, n) \cap A_{i(b_1)} = (-\infty, b] \cap A_{i(b_1)} = A_{i(b_1)}$, so l exists).

If $l > k$, then $C_-(b, l-1) \cap A_{k-1} = \emptyset$ (otherwise $b_1 \in C_-(b, l-1)$ which implies $C_-(b, l-1) \cap A_{i(b_1)} \neq \emptyset$). Thus there exists a nonempty open interval $(s, t) \subseteq (b_1, b)$ such that

$$(*) \qquad (s, t) \cap A_{l-1} = \emptyset,$$

otherwise we would have $[b_1, b] \subseteq A_{l-1}$ and thus $b_1 \in A_{i(b_1)} \cap C_-(b, l-1)$, a contradiction.

If $l = k$, let $s = b_1$, $t = b$. Then the above condition $(*)$ also holds. By $i(b_1) \leq l$ and condition (d′), $\Phi(A_{i(b_1)}, A_l) \in \mathcal{A}$. From the definition of l, it follows that $b \in \Phi(A_{i(b_1)}, A_l)$ and so $c \in A_{i(b)} \subseteq \Phi(A_{i(b_1)}, A_l)$. Hence $C(c, l) \cap A_{i(b_1)} \neq \emptyset$. Choose $c_1 \in C(c, l) \cap A_{i(b_1)}$. Since $i(b_1) < k$, by the induction assumption, there exists a continuous map $f_1 : (-\infty, b_1] \to \mathbb{R}$ such that $f_1(b_1) = c_1$ and $f_1(A_i \cap (-\infty, b_1]) \subseteq A_i$ for every $i \leq n$. Now we extend f_1 to $(-\infty, b]$ by adjoining the continuous map $f_2 : [b_1, b] \to \mathbb{R}$ defined by

$$f_2(x) = \begin{cases} c_1 & x \in [b_1, s], \\ c & x \in [t, b], \\ \text{linear} & x \in [s, t]. \end{cases}$$

Then $f = f_1 \cup f_2 : (-\infty, b] \to \mathbb{R}$ satisfies our requirement. To see this, we only need to note that if $j < k$ then $A_j \cap (b_1, b] = \emptyset$, if $k \leq j \leq l-1$ then c, c_1 both are in A_j and $(s, t) \cap A_j = \emptyset$, and if $j \geq l$ then the closed interval spanned by c and c_1 is contained in A_j. We are done.

The proof of Fact 2 is similar to that of Fact 1.

Now the map $h = f \cup g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the requirement. The proof is completed. $\square$

From the proof of Theorem 2 it is easily seen that the following more general conclusion is also true. We put it as a corollary because it can be proved using almost the same arguments.

Corollary 2. *If $\mathcal{A} : \emptyset \subset A_1 \subset A_2 \subset \cdots$ is a sequence of closed sets of $\mathbb{R}$ such that $\cup_{i=1}^{\infty} A_i = \mathbb{R}$, then $\mathcal{A} \cup \{\mathbb{R}\}$ is reflexive if and only if it satisfies condition* (d$'$).

Theorem 3. *Let $\mathcal{A} = \{\emptyset, A \cap B, A, B, A \cup B, \mathbb{R}\}$, where A and B are closed sets in $\mathbb{R}$. Then $\mathcal{A}$ is reflexive in $\mathbb{R}$ if and only if it satisfies condition* (d$'$).

Proof: The proof is similar to that one of Theorem 2. We retain all the symbols used in the proof of Theorem 2. Let $b \in \mathbb{R}$ and $c \in \phi(b)$. We show that there exists $f \in \mathrm{Alg}(\mathcal{A})$ such that $f(b) = c$. Again, it is enough to show that there exists a continuous map $f : (-\infty, b] \to \mathbb{R}$ such that $f(A \cap (-\infty, b]) \subseteq A$ and $f(B \cap (-\infty, b]) \subseteq B$.

Case 1: $\phi(b) = A \cap B$. Let $f(x) = c$ for all $x \in (-\infty, b]$.

Case 2: $\phi(b) = A \neq A \cap B$. If $(-\infty, b] \cap B = \emptyset$, let $f(x) = c$ for all $x \in (-\infty, b]$. If $(-\infty, b] \cap B \neq \emptyset$, let $b_0 = \max(-\infty, b] \cap B$. If $[b_0, b] \not\subseteq A$, then there exists an open interval $(s, t) \subseteq [b_0, b]$ such that $(s, t) \cap A = \emptyset$, define

$$f(x) = \begin{cases} c & x \in [t, b], \\ x & x \in (-\infty, s], \\ \text{linear} & x \in [s, t]. \end{cases}$$

If $[b_0, b] \subseteq A$, by condition (d$'$), $\Phi(A \cap B, A) \in \mathcal{A}$. Now $b_0 \in A \cap B$, $b \notin B$ and $[b_0, b] \subseteq \Phi(A \cap B, A)$, so $\Phi(A \cap B, A) \neq A \cap B$, hence, by Lemma 2, $\Phi(A \cap B, A) = A$. Thus there exists a connected component C_0 of A such that $c \in C_0$ and $C_0 \cap B \neq \emptyset$. Choose $c_0 \in C_0 \cap B \subset A \cap B$. Since the connected set C_0 contains both c and c_0, there exists a continuous map $f_1 : [b_0, b] \to C_0$ with $f_1(b_0) = c_0$ and $f_1(b) = c$.

Now define $f(x) = f_1(x)$ for $x \in [b_0, b]$ and $f(x) = c_0$ for $x \leq b_0$.

Case 3: $\phi(b) = B \neq A \cap B$. Similar to Case 2.

Case 4: $\phi(b) = \mathbb{R}$ and $A \cup B \neq \mathbb{R}$. If $A \cap B \neq \emptyset$, choose $a_0 \in A \cap B$ and let $f : (-\infty, b] \to \mathbb{R}$ be a continuous extension of the continuous map $f_0 : (-\infty, b] \cap (A \cup B \cup \{b\}) \to \mathbb{R}$ defined by

$$f_0(x) = \begin{cases} a_0 & x \in A \cup B, \\ c & x = b. \end{cases}$$

If $A \cap B = \emptyset$, choose $a_0 \in A$ and $b_0 \in B$ and let $f : (-\infty, b] \to \mathbb{R}$ be a continuous extension of the continuous map $f_1 : (-\infty, b] \cap (A \cup B \cup \{b\}) \to \mathbb{R}$ defined by

$$f_1(x) = \begin{cases} a_0 & x \in A, \\ b_0 & x \in B, \\ c & x = b. \end{cases}$$

Then the function f defined in each of the 4 cases satisfies our requirements. $\square$

The following example shows that conditions (a)–(d) are not sufficient for a finite closed set lattice to be reflexive in $\mathbb{R}$, nor in a compact interval of $\mathbb{R}$.

Example 2. Let $X = \mathbb{R}$ or $X = [0, 2]$ and let

$$A = \{0,1,2\} \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{2n}, \frac{1}{2n-1}\right] \cup \bigcup_{n=1}^{\infty} \left[1 + \frac{1}{2n}, 1 + \frac{1}{2n-1}\right],$$
$$B = \{0,1,2\} \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{2n+1}, \frac{1}{2n}\right] \cup \bigcup_{n=1}^{\infty} \left[1 + \frac{1}{2n+1}, 1 + \frac{1}{2n}\right],$$
$$C = \{0\}.$$

Now $\mathcal{A} = \{\emptyset, C, A \cap B, A, B, A \cup B, X\}$ satisfies conditions (a)–(d) but is not reflexive. In fact, it is not hard to verify that $\phi(1) = A \cap B \ni 2$. If $\mathcal{A}$ is reflexive, then it follows from Theorem 1 that there exists $f \in \mathrm{Alg}(\mathcal{A})$ such that $|f(1) - 2| < \frac{1}{2}$. Since $f(A \cap B) \subseteq A \cap B$, we have $f(1) = 2$. Hence, $f([\frac{1}{2}, 1]) \subseteq [\frac{3}{2}, 2]$ because $f(A) \subseteq A$ and f is continuous. As $\frac{1}{2} \in A \cap B$, $f(\frac{1}{2}) \in A \cap B$, so $f(\frac{1}{2}) = \frac{3}{2}$ or $f(\frac{1}{2}) = 2$. Repeating this steps, we can show that $f(\frac{1}{n}) \geq 1$ for all n and thus $f(0) \geq 1$. A contradiction occurs since $C \in \mathcal{A}$ implies $f(0) = 0$.

However we have the following result.

Theorem 4. *Let $\mathcal{A}$ be a finite family of closed subsets in $\mathbb{R}$ such that every $A \in \mathcal{A}$ has finitely many connected components. Then $\mathcal{A}$ is reflexive if it satisfies conditions* (a)–(d).

Proof: Let $b \in \mathbb{R}$ and $c \in \phi(b)$. We show that there exists $f \in \mathrm{Alg}(\mathcal{A})$ such that $f(b) = c$. Using the same method as in the proof of Theorem 2 we just show that there is $f : (-\infty, b] \to \mathbb{R}$ such that $f(b) = c$ and for every $A \in \mathcal{A}$, $f(A \cap (-\infty, b]) \subseteq A$. By our assumptions, there exist only finite numbers of connected components of all elements of $\mathcal{A} \cap (-\infty, b] = \{A \cap (-\infty, b] : A \in \mathcal{A}\}$. Let $\{b_k < b_{k-1} < \cdots < b_1 < b_0 = b\}$ be the set of all end points of those connected components. Observe that for any $x \in (b_{i+1}, b_i)$, $[b_{i+1}, b_i] \subseteq \phi(x)$.

For each $i (0 \leq i \leq k)$, we define a continuous map $f_i : [b_{i+1}, b_i] \to \mathbb{R}$, where we take $[b_{k+1}, b_k] = (-\infty, b_k]$, such that:

(i) $f_0(b) = c$;
(ii) $f_i(x) \in \phi(x)$ for every $x \in [b_{i+1}, b_i]$ and $i = 0, 1, \ldots, k$;
(iii) $f_i(b_{i+1}) = f_{i+1}(b_{i+1})$ for every $i = 0, 1, \ldots, k-1$.

We shall only define, as illustrations, $f_1 : [b_2, b_1] \to \mathbb{R}$ and $f_{k+1} : (-\infty, b_k] \to \mathbb{R}$. We assume f_0 has been defined that satisfies all the conditions. Choose any $a \in (b_2, b_1)$. Then, $b_i \in \phi(a)$ and hence $\phi(b_i) \subseteq \phi(a)$ $(i = 1, 2)$. Furthermore, the connected component of $\phi(a)$ containing a must contains b_2 and thus intersects $\phi(b_2)$. By Lemma 4, the connected component C of $\phi(a)$ containing $f_0(b_1)$ also intersects $\phi(b_2)$ since $f_0(b_1) \in \phi(b_1) \subseteq \phi(a)$. Choose $c_2 \in \phi(b_2) \cap C$ and define $f_1 : [b_2, b_1] \to \mathbb{R}$ as follows:

$$f_1(x) = \begin{cases} f_0(b_1) & x \in [a, b_1], \\ c_2 & x = b_2, \\ \text{linear} & x \in [b_2, a]. \end{cases}$$

Then f_1 satisfies the requirements. In fact, by definition (iii) is clearly true. For every $x \in [a, b_1]$, we have $b_1 \in \phi(x)$ and hence $f_1(x) = f_0(b_1) \in \phi(b_1) \subseteq \phi(x)$. It follows that (ii) holds for all $x \in [a, b_1]$. For each $x \in (b_2, a]$, by $f_1(b_2), f_1(a) \in C$ and the connectedness of C, we have $f_1(x) \in C \subseteq \phi(a)$ since $f_1(x)$ is a convex combination of $f_1(b_2)$ and $f_1(a)$. Clearly $a \in \phi(x)$, it follows that $f_1(x) \in \phi(a) \subseteq \phi(x)$. Hence (ii) also holds for $x \in (b_2, a]$. Also $f_1(b_2) = c_2 \in \phi(b_2)$, hence $f_1(x) \in \phi(x)$ holds for every $x \in [b_2, b_1]$.

We define $f_k(x) = f_{k-1}(b_k)$ for every $x \in (-\infty, b_k]$. Again, conditions (ii) and (iii) are satisfied.

Now the map $f = \bigcup_{i=0}^{k} f_i$ is well-defined and satisfies the requirement. □

Note that for every family $\mathcal{A}$ of closed subsets of a topological space, there is a least closed set lattice containing $\mathcal{A}$, which will be called the *closed set lattice generated by* $\mathcal{A}$. Theorem 3 actually says that the closed set lattice generated by two closed sets is reflexive if and only if it satisfies condition (d′).

Corollary 3. *Every closed set lattice in $\mathbb{R}$ generated by finite numbers of closed intervals is reflexive.*

The proofs of the results in this section rely heavily on the special structure of $\mathbb{R}$. They are not true if we replace $\mathbb{R}$ by $\mathbb{R}^2$. But we do not know how these conclusions can be generalized to $\mathbb{R}^2$ or higher dimensional spaces. A simple natural question is: when is a finite chain of closed sets in $\mathbb{R}^2$ reflexive?

4. Closedness of reflexive closed set lattices in hyperspaces

In this section, we prove another property of reflexive closed set lattices in a regular space. We assume all spaces are T_1 in this section.

Given a T_1 space X, let $S_V(X)$ denote the set $S(X)$ of all closed sets of X equipped with the *Vietoris topology* which has a subbase consisting of the sets of the form

$$U^- = \{A \in S(X) : A \cap U \neq \emptyset\} \quad \text{and} \quad U^+ = \{A \in S(X) : A \subseteq U\},$$

where the set U is open in X. It is well known that for every regular space X, $S_V(X)$ is Hausdorff (cf. [1, 2.7.20]).

Lemma 5. *Let X be a regular space. Then for any continuous $f : X \to X$, $\mathrm{Lat}(\{f\})$ is a closed subset of $S_V(X)$.*

Proof: If, by the contraposition, $\mathrm{Lat}(\{f\})$ is not closed in $S_V(X)$, then there exists $E \in \mathrm{cl}(\mathrm{Lat}(\{f\}))\setminus \mathrm{Lat}(\{f\})$. So $f(E) \not\subseteq E$. Choose $x \in E$ with $f(x) \notin E$. Since X is regular, there exists an open neighbourhood U of x such that $\mathrm{cl}(f(U)) \subseteq X\setminus E$ or $E \subseteq W = X\setminus \mathrm{cl}(f(U))$. Now $E \in W^+ \cap U^-$ and $E \in \mathrm{cl}(\{f\})$, thus there exists $F \in \mathrm{Lat}(\{f\})$ such that $F \in W^+ \cap U^-$. Then $F \cap U \neq \emptyset$ but $F \cap \mathrm{cl}(f(U)) = \emptyset$. Choose $y \in F \cap U$, then $f(y) \in f(U)$ and hence $f(y) \notin F$ which implies $f(F) \not\subseteq F$. A contradiction occurs. Therefore $\mathrm{Lat}(\{f\})$ is closed. □

Since $\mathcal{A} \subseteq S(X)$ is reflexive if and only if there are continuous $f_i : X \to X (i \in I)$ such that $\mathcal{A} = \mathrm{Lat}(\{f_i : i \in I\}) = \bigcap_{i\in I} \mathrm{Lat}(\{f_i\})$, thus, by above lemma we obtain the following:

Theorem 5. *For a regular space X, every reflexive family $\mathcal{A}$ of closed subsets satisfies the following condition:*

(e) *$\mathcal{A}$ is a closed subset of $S_V(X)$.*

Corollary 4. *Let $\mathcal{A}$ be a reflexive closed set lattice in a metric space (X, d). Then for any $\{b\} \notin \mathcal{A}$, there exist $\epsilon > 0$ and a neighbourhood U of b such that $\mathrm{diam}(\phi(x)) > \epsilon$ for all $x \in U$, where $\mathrm{diam}(\phi(x))$ is the diameter of the set $\phi(x)$.*

Proof: Suppose that $\mathcal{A}$ is a reflexive closed set lattice of the metric space (X, d) and $b \in X$ such that $\{b\} \notin \mathcal{A}$. If for every $\epsilon > 0$ and any open set U containing b, there is $x_U \in U$ with $\mathrm{diam}(\phi(x_U)) < \epsilon$, then one can deduce easily that $\{b\} \in \mathrm{cl}(\mathcal{A})$ in $S_V(X)$. So $\{b\} \in \mathcal{A}$, a contradiction. □

Remark 2. (1) For any $\mathcal{B} \subseteq \mathcal{A}$, $\mathrm{cl}(\bigcup \mathcal{B})$ belongs to the closure of the set $\mathcal{B}$ in $S_V(X)$, thus for regular spaces, condition (c) follows from condition (e).

(2) If X is a compact Hausdorff space, then condition (b) follows from condition (e) together with the following condition

$$F_1, F_2 \in \mathcal{A} \ \text{ implies } \ F_1 \cap F_2 \in \mathcal{A}.$$

(3) Lemma 5 can be regarded as a generalization of the fact that the set $\mathrm{Fix}(f)$ of all fixed points of a continuous endomap f is a closed subset. In fact, we may think that X is a subspace of $S_V(X)$ if we regard x and $\{x\}$ to be the same. Then $\mathrm{Fix}(f) = X \cap \mathrm{Lat}(\{f\})$. Thus the closedness of $\mathrm{Fix}(f)$ in X follows from Lemma 5.

(4) Since every finite $\mathcal{A} \subseteq S_V(X)$ is closed in $S_V(X)$, Example 2 shows that conditions (a)–(e) are not sufficient for a family of closed sets to be reflexive.

5. Reflexivity of closed set lattices via set-valued maps

Let X and Y be topological spaces. A map α from X to the set of non-empty closed sets in Y is called a set-valued map from X to Y and denoted by

$\alpha : X \rightrightarrows Y$. Furthermore, if $X = Y$ and $\alpha(x) \ni x$ for every $x \in X$, then α is called a *containing set-valued map* on X. A set-valued map $\alpha : X \rightrightarrows Y$ is called *lower semicontinuous* (*LSC* for short) if for every open set V in Y, the set $\{x \in X : \alpha(x) \cap V \neq \emptyset\}$ is open in X, or equivalently, for every closed set F in Y, $\{x \in X : \alpha(x) \subseteq F\}$ is closed in X. A continuous selection of a set-valued map $\alpha : X \rightrightarrows Y$ is a continuous map $f : X \to Y$ such that $f(x) \in \alpha(x)$ for all $x \in X$. In this section, we define two maps between all subfamilies of $S(X)$ satisfying conditions (a) and (b), and all containing set-valued maps on X, and discuss their properties. Using these two maps, we shall present a necessary and sufficient condition for a subfamily of $S(X)$ to be reflexive.

Let $\mathcal{A} \subseteq S(X)$ satisfy conditions (a) and (b). Then $\kappa(\mathcal{A}) = \phi_{\mathcal{A}} : X \rightrightarrows X$, defined by

$$\kappa(\mathcal{A})(x) = \phi_{\mathcal{A}}(x) = \bigcap\{A \in \mathcal{A} : x \in A\},$$

is a containing set-valued map on X. Conversely, for every containing set-valued map $\alpha : X \rightrightarrows X$, let

$$\tau(\alpha) = \{A \in S(X) : A \supseteq \alpha(x) \ \text{ for every } \ x \in A\}.$$

Then $\tau(\alpha)$ is a subfamily of $S(X)$ satisfying conditions (a) and (b).

Lemma 6. (1) *If $\mathcal{A}$ satisfies conditions* (a)–(c)*, then $\kappa(\mathcal{A}) : X \rightrightarrows X$ is LSC and $\alpha = \kappa(\mathcal{A})$ satisfies condition*

(f) *$\alpha(y) \subseteq \alpha(x)$ for any $x, y \in X$ with $y \in \alpha(x)$.*

In addition $\tau(\kappa(\mathcal{A})) = \mathcal{A}$.

(2) *If $\alpha : X \rightrightarrows X$ is a LSC containing map, then $\tau(\alpha)$ satisfies conditions* (a)–(c). *Furthermore, $\kappa(\tau(\alpha)) = \alpha$ if α satisfies condition* (f).

Proof: (1) For every closed set F of X, let

$$E = \{x \in X : \kappa(\mathcal{A})(x) \subseteq F\}.$$

If $a \in \mathrm{cl}(E)$, then $a \in \mathrm{cl}(\bigcup_{x \in E} \kappa(\mathcal{A})(x)) \subseteq F$. By condition (c), $\mathrm{cl}(\bigcup_{x \in E} \kappa(\mathcal{A})(x)) \in \mathcal{A}$, it follows from the definition of $\kappa(\mathcal{A})$ that $\kappa(\mathcal{A})(a) \subseteq F$, i.e. $a \in E$. Hence E is closed in X and thus $\kappa(\mathcal{A})$ is LSC. Condition (f) is trivially satisfied. For every $A \in \mathcal{A}$ and $x \in A$, $\kappa(\mathcal{A})(x) = \bigcap\{E \in \mathcal{A} : E \ni x\} \subseteq A$. It follows that $A \in \tau(\kappa(\mathcal{A}))$. Conversely, for every $A \in \tau(\kappa(\mathcal{A}))$, $A = \bigcup\{\kappa(\mathcal{A})(x) : x \in A\}$. It follows from conditions (b) and (c) that $A \in \mathcal{A}$. We are done.

(2) (a) and (b) are trivial. To prove (c), let $\mathcal{F}_0 \subseteq \tau(\alpha)$. If there exists $x \in \mathrm{cl}(\bigcup \mathcal{F}_0)$ such that $\mathrm{cl}(\bigcup \mathcal{F}_0) \not\supseteq \alpha(x)$, then $\alpha(x) \cap (X \setminus \mathrm{cl}(\bigcup \mathcal{F}_0)) \neq \emptyset$. Since α is LSC, there exists an open neighborhood U of x such that $\alpha(z) \cap (X \setminus \mathrm{cl}(\bigcup \mathcal{F}_0)) \neq \emptyset$ for every $z \in U$. From $x \in \mathrm{cl}(\bigcup \mathcal{F}_0)$ it follows that we may choose $F \in \mathcal{F}_0$ and $z \in U \cap F$. Then $\mathrm{cl}(\bigcup \mathcal{F}_0) \supset F \supset \alpha(z)$. Thus $\alpha(z) \cap (X \setminus \mathrm{cl}(\bigcup \mathcal{F}_0)) = \emptyset$. A contradiction occurs. This shows that for every $x \in \mathrm{cl}(\bigcup \mathcal{F}_0)$, $\mathrm{cl}(\bigcup \mathcal{F}_0) \supset \alpha(x)$. That is, (c) holds. Now, we assume that α also satisfies (f). Then $\alpha(x) \in \tau(\alpha)$

for every $x \in X$. Thus $\kappa(\tau(\alpha))(x) \subseteq \alpha(x)$. The converse conclusion is trivial. We are done. $\square$

Lemma 7. *Let $\mathcal{A}$ be a closed set lattice in X. Then a map $f : X \to X$ is a continuous selection of $\kappa(\mathcal{A}) : X \rightrightarrows X$ if and only if $f \in \mathrm{Alg}(\mathcal{A})$.*

Proof: Let $f : X \to X$ be a continuous selection of $\kappa(\mathcal{A}) : X \rightrightarrows X$. Then, for every $A \in \mathcal{A}$ and $x \in A$, $f(x) \in \kappa(\mathcal{A})(x) = \phi(x) \subseteq A$. Thus $f(A) \subseteq A$. That is, $f \in \mathrm{Alg}(\mathcal{A})$. Conversely, if $f \in \mathrm{Alg}(\mathcal{A})$, then, for every $x \in X$, $f(x) \in \phi(x) = \kappa(\mathcal{A})(x)$. Therefore, $f : X \to X$ is a continuous selection of $\kappa(\mathcal{A})$. $\square$

From Theorem 1 and Lemma 7 we deduce the following

Proposition 1. *For a space X, a subfamily $\mathcal{A}$ of $S(X)$ is reflexive if and only if $\mathcal{A}$ satisfies the conditions* (a)–(c) *and the condition below:*

(g) *For every $x \in X$, the set $\{f(x) : f$ is a continuous selection of $\kappa(\mathcal{A}) : X \rightrightarrows X\}$ is dense in $\kappa(\mathcal{A})(x)$ for every $x \in X$.*

Now we give an example to show that the converse of Corollary 1 is not true for $X = \mathbb{R}$. By a simple modification of the following example, we can show that the converse of Corollary 1 fails even for the closed interval $X = [1, 3]$.

Example 3. Define functions $f_n : \mathbb{R} \to \mathbb{R}$ as follows: $f_0(x) = x$ for any $x \in \mathbb{R}$, and

$$f_n(x) = \begin{cases} x & x \in (-\infty, 2], \\ 2 & x \in [2, 2 + \frac{n-1}{n}], \\ \frac{n-1}{n} & x \in [3, +\infty), \\ \text{linear} & x \in [2 + \frac{n-1}{n}, 3] \end{cases}$$

for $n = 1, 2, \ldots$. Let $\kappa(x) = \mathrm{cl}(\{f_n(x) : n = 0, 1, 2, \ldots\})$ for every $x \in \mathbb{R}$. Then $\kappa : X \rightrightarrows X$ is a LSC containing and satisfies (g). Thus, by Lemma 6, $\tau(\kappa)$ is a closed set lattices and $\kappa(\tau(\kappa)) = \kappa$. Trivially, every f_n is a continuous selection of $\kappa : X \rightrightarrows X$ and $\{f_n(x) : n = 0, 1, \ldots\}$ is dense in $\kappa(x)$ for every $x \in \mathbb{R}$. Therefore, by Theorem 1, $\tau(\kappa)$ is reflexive but for $x \geq 3$, $\{f(x) : f \in \mathrm{Alg}(\tau(\kappa))\} = \{f_n(x) : n = 0, 1, \ldots\} = \{0, \frac{1}{2}, \frac{2}{3}, \ldots\}$ is not closed.

Theorem 6. *Let X be a subset of a normed space $(L, \|\cdot\|)$. If $\mathcal{A} \subseteq S(X)$ satisfies conditions* (a)–(c) *and $\kappa(\mathcal{A})(x)$ is convex and complete with respect to $\|\cdot\|$ for every $x \in X$, then $\mathcal{A}$ is reflexive.*

Proof: By our assumptions, $\kappa(\mathcal{A})(x)$ is closed in L. Thus, from Lemma 6, we have $\kappa(\mathcal{A}) : X \rightrightarrows L$ is LSC. For every $x \in X$ and $y \in \kappa(\mathcal{A})(x)$, by the Michael Selection Theorem ([8], cf. [9, Theorem 1.4.9]), there exists a continuous selection $f : X \to L$ of $\kappa(\mathcal{A}) : X \rightrightarrows L$ such that $f(x) = y$. Trivially, $f(X) \subseteq X$. It follows from Corollary 1 and Lemma 7 that $\mathcal{A}$ is reflexive. $\square$

Corollary 5. *Let X be a convex complete set in a normed space $(L, \|\cdot\|)$ and $\mathcal{A}$ be a closed set lattice which is generated by a family consisting of convex and complete sets, where "complete" means they are complete with respect to $\|\cdot\|$. Then $\mathcal{A}$ is reflexive.*

The following theorem summarizes the main results in this section.

Theorem 7. *There exists a bijection κ from all subfamilies $\mathcal{F}$ of $S(X)$ satisfying* (a)–(c) *to all LSC set-valued maps α from X to $S(X)$ satisfying* (f). *Furthermore, $\mathcal{A}$ is reflexive if and only if $\kappa(\mathcal{A})$ satisfies condition* (g).

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