Martin's Axiom and ω-resolvability of Baire spaces

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Abstract. We prove that, assuming MA, every crowded T_0 space X is ω-resolvable if it satisfies one of the following properties: (1) it contains a π-network of cardinality $< \mathfrak{c}$ constituted by infinite sets, (2) $\chi(X) < \mathfrak{c}$, (3) X is a T_2 Baire space and $c(X) \leq \aleph_0$ and (4) X is a T_1 Baire space and has a network $\mathcal{N}$ with cardinality $< \mathfrak{c}$ and such that the collection of the finite elements in it constitutes a σ-locally finite family.

Furthermore, we prove that the existence of a T_1 Baire irresolvable space is equivalent to the existence of a T_1 Baire ω-irresolvable space, and each of these statements is equivalent to the existence of a T_1 almost-ω-irresolvable space.

Finally, we prove that the minimum cardinality of a π-network with infinite elements of a space $\mathrm{Seq}(u_t)$ is strictly greater than $\aleph_0$.

Keywords: Martin's Axiom, Baire spaces, resolvable spaces, ω-resolvable spaces, almost resolvable spaces, almost-ω-resolvable spaces, infinite π-network

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1. Introduction

Every space in this article is T_0 and crowded (that is, without isolated points) and so it is infinite. A space X is *resolvable* if it contains two dense disjoint subsets. A space which is not resolvable is called *irresolvable*. Resolvable and irresolvable spaces were studied extensively first by Hewitt [14]. Later, El'kin and Malykhin published a number of papers on these subjects and their connections with various topological problems. One of the problems considered by Malykhin in [22] refers to the existence of irresolvable spaces satisfying the Baire Category Theorem. Kunen, Symański and Tall in [19] afterwards proved that there is such a space if and only if there is a space X on which every real-valued function is continuous at some point. (The question about the existence of a –Hausdorff– space on which every real-valued function is continuous at some point was posed by M. Katětov in [16].) They also proved (see [18] as well):

1. if we assume $V = L$, there is no Baire irresolvable space,

2. the conditions "there is a measurable cardinal" and "there is a Baire irresolvable space" are equiconsistent.

Bolstein introduced in [5] the spaces X in which it is possible to define a real-valued function f with countable range and such that f is discontinuous at every point of X (he called these spaces *almost resolvable*), and proved that every resolvable space satisfies this condition. It was proved in [12] that X is almost resolvable iff there is a function $f : X \to \mathbb{R}$ such that f is discontinuous at every point of X. *Almost-ω-resolvable* spaces were introduced in [26]; these are spaces in which it is possible to define a real-valued function f with countable range, and such that $r \circ f$ is discontinuous in every point of X, for every real-valued finite-to-one function r. It was proved in that article that for a Tychonoff space X, the space of real continuous functions with the box topology, $C_{\square}(X)$, is discrete if and only if X is almost-ω-resolvable. It was also proved that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space without isolated points which is not almost-ω-resolvable, and that, on the contrary, if $V = L$ then every crowded space is almost-ω-resolvable. Later, it was pointed out in [2, Corollary 5.4] that a Baire space is resolvable if and only if it is almost resolvable; so,

1.1 Theorem. *A Baire almost-ω-resolvable space is resolvable.*

It is unknown if every Baire almost-ω-resolvable space is 3-resolvable. With respect to this problem we have the following theorems.

1.2 Theorem ([24])**.** *Gödel's axiom of constructibility, $V = L$, implies that every Baire space is ω-resolvable.*

1.3 Theorem ([2])**.** *Every T_1 Baire space such that each of its dense subsets is almost-ω-resolvable is ω-resolvable.*

These last two results transform our problem to that of finding subclasses of Baire spaces such that each of its crowded dense subsets is almost-ω-resolvable, assuming axioms consistent with ZFC which contrast with $V = L$. Of course, a classic axiom with these characteristics is MA+¬CH. This bet is strengthened by the following result due to V.I. Malykhin ([23, Theorem 1.2]):

1.4 Theorem [$\mathrm{MA}_{\mathrm{countable}}$]**.** *Let a topology on a countable set X have a π-network of cardinality less than $\mathfrak{c}$ consisting of infinite subsets. Then this topology is ω-resolvable.*

It was proved in [2] that every space with countable tightness, every space with π-weight $\leq \aleph_1$ and every σ-space are hereditarily almost-ω-resolvable. So, by Theorem 1.3, every T_1 Baire space with either countable tightness or π-weight $\leq \aleph_1$ or σ is ω-resolvable.

In this article we are going to continue the study of almost-ω-resolvable and Baire resolvable spaces, and we will solve some problems related to these posed in [2]. Section 2 is devoted to establishing basic definitions and results. In Section 3 we prove that under MA every space with either π-weight $< \mathfrak{c}$ or $\chi(X) < \mathfrak{c}$ is ω-resolvable. Furthermore, we are going to see in Section 4 that under SH every T_2 Baire space with countable cellularity is ω-resolvable. Section 5 is devoted to analyse almost-ω-irresolvable spaces. We prove in this section that there is a T_1 Baire irresolvable space iff there is a T_1 Baire ω-irresolvable space, iff there is a T_1 almost-ω-irresolvable space. Finally in Section 6, we prove that the minimum cardinality of a π-network with infinite elements of a space $\mathrm{Seq}(u_t)$ is strictly greater than $\aleph_0$. Moreover, we propose several problems related to our matter through the article.

2. Basic definitions and preliminaries

A space X is *resolvable* if it is the union of two disjoint dense subsets. We say that X is *irresolvable* if it is not resolvable. For a cardinal number $\kappa > 1$, we say that X is *κ-resolvable* if X is the union of κ pairwise disjoint dense subsets.

The *dispersion character* $\Delta(X)$ of a space X is the minimum of the cardinalities of non-empty open subsets of X. If X is $\Delta(X)$-resolvable, then we say that X is *maximally resolvable*. A space X is *hereditarily irresolvable* if every subspace of X is irresolvable. And X is *open-hereditarily irresolvable* if every open subspace of X is irresolvable.

We call a space (X, t) *maximal* if (X, t') contains at least one isolated point when t' strictly contains the topology t. And a space X is *submaximal* if every dense subset of X is open. Moreover, maximal spaces are submaximal, and these are hereditarily irresolvable spaces, which in turn are open-hereditarily irresolvable.

It is possible to prove that a space X is *almost resolvable* if and only if X is the union of a countable collection of subsets each of them with an empty interior [5].

It was proved in [26] that the following formulation can be given as a definition of almost-ω-resolvable space: A space X is called *almost-ω-resolvable* if X is the union of a countable collection $\{X_n : n < \omega\}$ of subsets in such a way that for each $m < \omega$, $\mathrm{int}(\bigcup_{i \le m} X_i) = \emptyset$. In particular, every almost-ω-resolvable space is almost resolvable, every ω-resolvable space is almost-ω-resolvable, every almost resolvable space is infinite, and every T_1 separable space is almost-ω-resolvable.

We are going to say that a space X is *hereditarily almost-ω-resolvable* if each crowded subspace of X is almost-ω-resolvable, and X is *dense-hereditarily almost-ω-resolvable* if each crowded dense subspace of X is almost-ω-resolvable.

Let X be a κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) space. A *κ-resolution* (resp., an *almost resolution*, an *almost-ω-resolution*) for X is a partition $\{V_\alpha : \alpha < \kappa\}$ (resp., a partition $\{V_n : n < \omega\}$) of X such that each V_α is a dense subset of X (resp., $\mathrm{int}(V_n) = \emptyset$ for every $n < \omega$, $\mathrm{int}(\bigcup_{i=0}^{n} V_i) = \emptyset$ for every $n < \omega$).

Finally, a space X is *almost-ω-irresolvable* (resp., κ-irresolvable) if X is not almost-ω-resolvable (resp., X is not κ-resolvable). The hereditary version of almost-ω-irresolvability or κ-irresolvability is that which states that every crowded subspace of X is not almost-ω-resolvable, and, respectively, is not κ-resolvable.

2.1 Example. There are non-T_0 topological spaces which are almost resolvable but not almost-ω-resolvable. In fact, let X be an infinite set and $x, y \in X$ with $x \neq y$. We define a collection $\mathcal{T}$ of subsets of X as follows: $A \in \mathcal{T}$ if either A is the empty set or $x, y \in A$. The family $\mathcal{T}$ is a topology in X and $(X, \mathcal{T})$ satisfies the required conditions.

2.2 Example. It was proved in Theorem 4.4 of [19] that the existence of an ω_1-complete ideal $\mathcal{I}$ over ω_1 which has a dense set of size ω_1 implies the existence of a T_2 Baire strongly irresolvable topology $\mathcal{T}$ on ω_1. On the other hand, it was observed in [26, Corollary 4.9] that every Baire irresolvable space is not almost resolvable. Therefore, $(\omega_1, \mathcal{T})$ is not almost resolvable.

2.3 Example. If there is a measurable cardinal κ, then there is a resolvable Baire space X which is not almost-ω-resolvable and $\Delta(X) = \kappa$. Indeed, let κ be a non-countable Ulam-measurable cardinal, and let p be a free ultrafilter on κ ω_1-complete. Let $X = \kappa \cup \{p\}$. We define a topology t for X as follows: $A \in t \setminus \{\emptyset\}$ if and only if $p \in A$ and $A \cap \kappa \in p$. This space is a Baire resolvable non-almost-ω-resolvable space with $\Delta(X) = \alpha$. Now, let $\mathcal{T}$ be equal to $\{A \subseteq X : A \cap \kappa \in p\}$; $\mathcal{T}$ is a topology in X too, and $(X, \mathcal{T})$ is T_1 submaximal, Baire with $\Delta(X) = \alpha$, but it is not almost resolvable.

Related to the last examples we have:

2.4 Question. *Is there a T_2 resolvable Baire space which is not almost-ω-resolvable?*

2.5 Examples. In ZFC, there are almost-ω-resolvable spaces which are not resolvable. Indeed, the union of Tychonoff crowded topologies in $\mathbb{Q}$ generates a Tychonoff crowded topology. By Zorn's Lemma, we can consider a maximal Tychonoff topology $\mathcal{T}$ in $\mathbb{Q}$. The space $(\mathbb{Q}, \mathcal{T})$ is countable (so, almost-ω-resolvable) hereditarily irresolvable ([14, Theorems 15 and 26], [8, Example 3.3]). $(\mathbb{Q}, \mathcal{T})$ is Tychonoff.

In [1], the authors construct by transfinite recursion a "concrete" (in the sense that we can say how its open sets look) example of a countable dense subset X of the space $2^{\mathfrak{c}}$ which is irresolvable. Since X is countable, it is almost-ω-resolvable.

2.6 Example. For every cardinal number κ, there exists a Tychonoff space X which is almost-ω-resolvable, hereditarily irresolvable and $\Delta(X) \geq \kappa$. In fact, let λ be a cardinal number such that $\kappa \leq \lambda$ and $\mathrm{cof}(\lambda) = \aleph_0$. Let H, G and τ be the topological groups and the topology in G, respectively, defined in [11, pp. 33 and 34], with $|H| = \lambda$. L. Feng proved there that $(H, \tau|_H)$ is an irresolvable card-homogeneous (every open subset of H has the same cardinality as H) Tychonoff space, and each subset $S \subseteq H$ with cardinality strictly less than λ is a nowhere dense subset of H. Let $(\lambda_n)_{n<\omega}$ be a sequence of cardinal numbers such that $\lambda_n < \lambda_{n+1}$ for every $n < \omega$ and $\sup\{\lambda_n : n < \omega\} = \lambda$. We take subsets H_n of H with the properties $H_n \subseteq H_{n+1}$ and $|H_n| = \lambda_n$ for each $n < \omega$, and

$H = \bigcup_{n<\omega} H_n$. We have that each H_n is nowhere dense in H; so $\{H_n : n < \omega\}$ is an almost-ω-resolvable sequence on H. That is, H is almost-ω-resolvable. By the Hewitt Decomposition Theorem (see [14, Theorem 28]), there exists a non-empty open subset U of H which is hereditarily irresolvable. Besides, $\Delta(U) = \Delta(H) \geq \kappa$ and U is almost-ω-resolvable.

2.7 Examples. The first example of a Hausdorff maximal group was constructed by Malykhin in [21] under Martin's Axiom. Malykhin also constructed in [23], in the BK model M_{ω_1} (see [3]) a topological group topology $\mathcal{T}'$ in the infinite countable Boolean group Ω of all finite subsets of ω with symmetric difference as the group operation, such that $(\Omega, \mathcal{T}')$ is T_2, irresolvable and its weight is ω_1 (compare with Corollary 3.6 below). Moreover, in M_{ω_1}, $\omega_1 < \mathfrak{c}$. Moreover, he constructed in M_{ω_1} a countable irresolvable dense subset in 2^{ω_1}. This space has of course weight ω_1.

On the other hand, the class of resolvable spaces includes spaces with well known properties:

2.8 Theorem. (1) *If X has a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \Delta(X)$ and each $N \in \mathcal{N}$ satisfies $|N| \geq \Delta(X)$, then X is maximally resolvable* [9].

(2) *Hausdorff k-spaces are maximally resolvable* [25].

(3) *Countably compact regular T_1 spaces are ω-resolvable* [7].

(4) *Arc connected spaces are ω-resolvable.*

(5) *Every biradial space is maximally resolvable* [29].

(6) *Every homogeneous space containing a non-trivial convergent sequence is ω-resolvable* [28].

(7) *If G is an uncountable $\aleph_0$-bounded topological group, then G is $\aleph_1$-resolvable* [29].

(8) *T_1 Baire spaces with either countable tightness or π-weight $\leq \aleph_1$ are ω-resolvable* [2].

The following basic results will be very helpful (see, for example, [6]).

2.9 Propositions. (1) *If X is the union of κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) subspaces, then X has the same property.*

(2) *Every open and every regular closed subset of a κ-resolvable (resp., almost resolvable, almost-ω-resolvable) space shares this property.*

(3) *Let X be a space which contains a dense subset which is κ-resolvable (resp., almost resolvable, almost-ω-resolvable). Then, X satisfies this property too.*

The following results are easy to prove and are well known.

2.10 Proposition. *Let Y be a κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) space. If $f : X \to Y$ is a continuous and onto function, and for each* *open subset A of X the interior of $f[A]$ is not empty, then X is κ-resolvable (resp., almost-resolvable, almost-ω-resolvable).*

2.11 Proposition. *Let $f : X \to Y$ be continuous and bijective. If X is κ-resolvable (resp., almost-resolvable, almost-ω-resolvable), so is Y.*

2.12 Proposition. (1) *If X is κ-resolvable (resp., almost resolvable, almost-ω-resolvable) and Y is an arbitrary topological space, then $X \times Y$ is κ-resolvable (resp., almost resolvable, almost-ω-resolvable).*

(2) [2] *If X and Y are almost resolvable, then $X \times Y$ is resolvable.*

(3) (O. Masaveu) *If X is the product space $\prod_{\alpha<\kappa} X_\alpha$ where $\kappa \geq \omega$ and each X_α has more than one point, then X is 2^κ-resolvable.*

The following lemmas will be useful later.

2.13 Proposition. *If X is a crowded space such that $\mathrm{cof}(|X|) = \aleph_0$ and every open subset of X has cardinality $|X|$, then X is almost-ω-resolvable.*

2.14 Proposition. *If X has tightness equal to κ, then each point $x \in X$ is contained in a crowded subset of X of cardinality $\leq \kappa$.*

Proof: Let $x_0 \in X$ be an arbitrary fixed point. Since X is crowded, $x_0 \in \mathrm{cl}_X[X \setminus \{x_0\}]$; so there is a subset $F_1 \subseteq X \setminus \{x_0\}$ of cardinality $\leq \kappa$ such that $x_0 \in \mathrm{cl}_X F_1$. If $F_0 \cup F_1$ is crowded, where $F_0 = \{x_0\}$, then we have finished. Otherwise, for each isolated point x of $F_0 \cup F_1$, there is a subset $F_x^2 \subseteq X \setminus (\{x_0\} \cup F_1)$ of cardinality $\leq \kappa$ such that $x \in \mathrm{cl}_X F_x^2$. Let $F_2 = \bigcup_{x\in G_1} F_x^2$ where G_1 is the set of isolated points of $F_0 \cup F_1$. Again, there are two possible situations: either $F_0 \cup F_1 \cup F_2$ is a crowded subspace of cardinality $\leq \kappa$ containing x_0, or $G_2 = \{x \in F_2 : x$ is an isolated point of $F_0 \cup F_1 \cup F_2\}$ is not empty. In this last case, for each $x \in G_2$ we take a subset $F_x^3 \subseteq X \setminus (F_0 \cup F_1 \cup F_2)$ of cardinality $\leq \kappa$ for which $x \in \mathrm{cl}_X F_x^3$. We write $F_3 = \bigcup_{x\in G_2} F_x^3$. Continuing this process if necessary, we obtain either a finite sequence $F_0, \dots ,F_n$ of subsets of X such that $x_0 \in F = \bigcup_{0\leq i\leq n} F_n$ and F has cardinality $\leq \kappa$ and is crowded, or we have to go further: $x_0 \in F = \bigcup_{n<\omega} F_n$. In this last case too, F has cardinality $\leq \kappa$ and is crowded. □

3. Martin's Axiom, π-netweight and ω-resolvable spaces

First, in this section we are going to present, by using Martin's Axiom, a generalization of Theorem 1.4. As usual, if I and J are two sets, $\mathrm{Fn}(I, J)$ stands for the collection of the finite functions with domain contained in I and range contained in J. We define a partial order $\leq$ in $\mathrm{Fn}(I, J)$ by letting $p \leq q$ iff $p \supseteq q$. The partial order set $(\mathrm{Fn}(I, J), \leq)$ is ccc if and only if $|J| \leq \aleph_0$ (Lemma 5.4, p. 205 in [17]).

Let (X, τ) be a topological space. A collection $\mathcal{N} \subseteq \mathcal{P}(X)$ is a π-network of X if each element $U \in \tau \setminus \{\emptyset\}$ contains an element of $\mathcal{N}$.

3.1 Definitions. Let κ be an infinite cardinal.

(1) A space X is almost-κ-resolvable if X can be partitioned as $X = \bigcup_{\alpha<\kappa'} X_\alpha$ where $\omega \leq \kappa' \leq \kappa$, $X_\alpha \neq \emptyset$, and $X_\alpha \cap X_\xi = \emptyset$ if $\alpha \neq \xi$, such that every non-empty open subset of X has a non-empty intersection with an infinite collection of elements in $\{X_\alpha : \alpha < \kappa\}$.

(2) Let $\mathcal{X} = \{X_\alpha : \alpha < \kappa\}$ be a partition of X. A collection $\mathcal{N} = \{N_\xi : \xi < \tau\}$ of infinite subsets of κ is a π-network of $\mathcal{X}$ if for each open set U of X, $\{\alpha < \kappa : X_\alpha \cap U \neq \emptyset\} \supseteq N_\xi$ for a $\xi < \tau$.

(3) A space X is called precisely almost-κ-resolvable if X contains a resolution with a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \kappa$.

The following well known result is due to K. Kuratowski.

3.2 Lemma (The disjoint refinement lemma). *Let $\{A_\xi : \xi < \kappa\}$ be a collection of sets such that, for each $\xi < \kappa$, $|A_\xi| \geq \kappa$. Then, there is a collection $\{B_\xi : \xi < \kappa\}$ of sets satisfying:*

(1) *$B_\xi \subseteq A_\xi$ for all $\xi < \kappa$,*
(2) *$|B_\xi| = \kappa$ for all $\xi < \kappa$,*
(3) *$B_\xi \cap B_\zeta = \emptyset$ for $\xi, \zeta < \kappa$ with $\xi \neq \zeta$.*

3.3 Proposition. *A space X is precisely almost-ω-resolvable if and only if X is ω-resolvable.*

PROOF: Let X be a precisely almost-ω-resolvable space. Let $\mathcal{X} = \{X_\xi : \xi < \tau\}$ be a precise partition of X, and $\mathcal{M} = \{M_n : n < \omega\}$ be a π-network of $\mathcal{X}$. Because of Lemma 3.2, there are infinite and pairwise disjoint sets $T_0, T_1, \ldots, T_n, \ldots$ such that $T_i \subseteq M_i$ for all $i < \omega$.

For each $n < \omega$, we faithfully enumerate T_n: $\{k_i^n : i < \omega\}$. Now we define for each $i < \omega$, $D_i = \bigcup_{j<\omega} X_{k_i^j}$. Each D_n is dense in X and $D_i \cap D_j = \emptyset$ if $i \neq j$.

Moreover, if X is ω-resolvable and $\mathcal{D} = \{D_n : n < \omega\}$ is a collection of pairwise disjoint dense subsets of X, then $\mathcal{D}$ is a precise partition of X and $\mathcal{M} = \{\omega\}$ is a π-network of $\mathcal{D}$. □

When we assume Martin's Axiom, we can generalize Proposition 3.3:

3.4 Theorem. *Let $\mathcal{X} = \{X_\alpha : \alpha < \tau\}$ be an almost-τ-resolvable partition of X. Let $\mathcal{N} = \{N_\xi : \xi < \kappa\}$ be a π-network of $\mathcal{X}$ such that $\kappa < \mathfrak{c}$. If we assume Martin's Axiom, then X is ω-resolvable. In particular, MA implies that ω-resolvability and almost-κ-resolvability precise coincide when $\kappa < \mathfrak{c}$.*

PROOF: In this case, we put $\mathbb{P} = (\mathrm{Fn}(\kappa, \omega), \leq)$ where $\leq$ is defined at the beginning of this section. For each $k \in \omega$ and $N \in \mathcal{N}$, we take the set

$$D_N^k = \{p \in \mathbb{P} : \exists \xi \in N \ \text{ such that } \ p(\xi) = k\}.$$

It happens that each D^k_N is dense in $\mathbb{P}$. In fact, let q be an arbitrary element in $\mathbb{P}$. We can take $\xi \in N \setminus \mathrm{dom}(q)$ because N is infinite. The function $p = q \cup \{(\xi, k)\}$ belongs to D^k_N and is less than q.

The partially ordered set $\mathbb{P}$ is ccc and $\mathcal{D} = \{D^k_N : k < \omega, N \in \mathcal{N}\}$ has cardinality strictly less than $\mathfrak{c}$. So, there exists a $\mathcal{D}$-generic filter G in $\mathbb{P}$. Take $f = \bigcup G$. Then $f : \kappa \to \omega$ is onto and $\kappa = \bigcup_{n<\omega} Y_n$ where $Y_n = f^{-1}[\{n\}]$.

Now, for each $n < \omega$, we consider the set $X_n = \bigcup_{\alpha \in Y_n} X_\alpha$. It is easy to prove that $\{X_n : n < \omega\}$ is a partition of $\bigcup_{n<\omega} X_n$. Moreover, each X_n is a dense subset of X. Indeed, let n_0 be a natural number. We are going to prove that X_{n_0} is dense. Let U be an open set of X. Because of the properties of $\mathcal{N}$, there is $N_0 \in \mathcal{N}$ such that $\{\alpha < \tau : X_\alpha \cap U \neq \emptyset\} \supseteq N_0$. We take $p \in D^{n_0}_{N_0} \cap G$. It happens that there is a $\xi \in N_0$ such that $p(\xi) = n_0$. Hence, $f(\xi) = n_0$. This means that $\xi \in f^{-1}[\{n_0\}] = Y_{n_0}$. By definition, X_ξ must have a non-empty intersection with U, and therefore $U \cap X_{n_0} = U \cap \bigcup_{\alpha \in Y_{n_0}} X_\alpha \neq \emptyset$. □

Assume that $\{x_\xi : \xi < \tau\}$ is a faithful enumeration of a space X. If X possesses a π-network $\mathcal{N}$ with infinite elements, the collection $\{M_N : N \in \mathcal{N}\}$ where $M_N = \{\xi < \tau : x_\xi \in N\}$, is a π-network of the partition $\{\{x_\xi\} : \xi < \tau\}$. So the following result is a corollary of Theorem 3.4.

3.5 Theorem. *Let X be a crowded topological space with a π-network $\mathcal{N}$ with cardinality $\kappa < \mathfrak{c}$ and such that each element in $\mathcal{N}$ is infinite. If we assume Martin's Axiom, then X is an ω-resolvable space.*

Recall that every biradial space is maximally resolvable. Moreover, every space with $\pi w(X) \leq \Delta(X)$ is maximally resolvable (see [4]). With respect to these ideas we have:

3.6 Corollary [MA]**.** *Every crowded space X with π-weight $< \mathfrak{c}$ is ω-resolvable. In particular, every space with weight $< \mathfrak{c}$ is hereditarily ω-resolvable.*

Proof: Let $\mathcal{N}$ be a π-base of X of cardinality $< \mathfrak{c}$. Since X is crowded and each element of $\mathcal{N}$ is open in X, then $|N| \geq \aleph_0$ for each $N \in \mathcal{N}$. On the other hand, $\mathcal{N}$ is a π-network in X, so the conclusion follows. □

It is easy to see that if X has π-character and density $\leq \kappa$, then X has a π-base of cardinality $\leq \kappa$.

3.7 Proposition [MA]**.** *If X is a space with density and π-character $< \mathfrak{c}$, then every dense subset of X is ω-resolvable.*

Proof: The space X has a π-base $\mathcal{B}$ of cardinality $< \mathfrak{c}$. Let H be an arbitrary dense subset of X. It happens now that $\mathcal{M} = \{N \cap H : N \in \mathcal{N}\}$ is a π-base of H and has cardinality $< \mathfrak{c}$. So, by Corollary 3.6, H is ω-resolvable. □

For every space X, $\max\{t(X), \pi\chi(X)\} \leq \chi(X)$, so, as a consequence of the last result, and related to Theorems 2.8(2) and 2.8(8), we have:

3.8 Theorem [MA]. *If X is a space such that $\chi(x, X) < \mathfrak{c}$ for each $x \in X$, then X is hereditarily ω-resolvable.*

PROOF: Let Y be a crowded subspace of X. The character of Y is strictly less than $\mathfrak{c}$; thus, the tightness of Y is $< \mathfrak{c}$. Hence, each point y in Y is contained in a crowded subspace Y_y of Y of cardinality $< \mathfrak{c}$ (Proposition 2.14). The density and character of each Y_y is strictly less than $\mathfrak{c}$. By Proposition 3.7, Y_y is ω-resolvable. Then Y is ω-resolvable (see Proposition 2.9(1)). $\square$

The following result is a generalization of Theorems 3.5 and 3.8, which answers, affirmatively, a question posed by the referee. A collection $\mathcal{N} \subseteq \mathcal{P}(X)$ is a *π-network of X at the point $x \in X$* if every open set of X containing x contains an element of $\mathcal{N}$. For each point $x \in X$, we define $\pi nw^*(x, X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a π-network of X at x and each element in $\mathcal{N}$ is infinite$\}$. Of course, for each $x \in X$, $\pi nw^*(x, X) \leq \chi(x, X)$. Since MA implies that $\mathfrak{c}$ is a regular cardinal, we have that, by Theorem 3.5, MA implies that every space X containing a dense subset Y of cardinality $\leq \kappa < \mathfrak{c}$ and such that for every $y \in Y$, $\pi nw^*(y, X) < \mathfrak{c}$, is ω-resolvable. This result can be ameliorated. Indeed, by using a similar proof to that of Proposition 2.14, if X is a space with $\pi nw^*(x, X) < \mathfrak{c}$ for each $x \in X$, then each point $x \in X$ is contained in a crowded subspace X_x of X of cardinality $< \mathfrak{c}$ and having, for each $y \in X_x$, $\pi nw^*(y, X_x) < \mathfrak{c}$. So:

3.9 Corollary [MA]. *Let X be a space such that for every $x \in X$, $\pi nw^*(x, X) < \mathfrak{c}$. Then X is ω-resolvable.*

We obtain another result with a slightly different mood of that of the previous corollary by defining for each point $x \in X$ the number $R(x, X) = \min\{|\Lambda| : \Lambda$ is a directed partially ordered set and there is a net $(x_\alpha)_{\alpha\in\Lambda}$ in $X \setminus \{x\}$ such that $(x_\alpha)_{\alpha\in\Lambda}$ converges to x in $X\}$. Indeed, following a similar argumentation to that given in the previous paragraph of Corollary 3.9, we obtain:

3.10 Corollary [MA]. *Let X be a space such that for every $x \in X$, $R(x, X) < \mathfrak{c}$. Then X is ω-resolvable.*

In Proposition 4.5 of [2] it was proved that every T_2 σ-space is almost-ω-resolvable. When X has a countable network, we can repeat that proof assuming only the weaker condition T_0. So every space with countable network is almost-ω-resolvable. With respect to σ-spaces, Proposition 4.5 in [2] and Martin's Axiom, Proposition 3.11 allows us to say something else which is, in some sense, stronger that Theorem 3.5:

3.11 Proposition [MA]. *Let κ be an infinite cardinal $< \mathfrak{c}$. Let X be a space with a network $\mathcal{N}$ such that for each finite subcollection $\mathcal{N}'$ of $\mathcal{N}$, $\bigcap \mathcal{N}'$ is infinite or empty, and for each $x \in X$, $|\{N \in \mathcal{N} : x \in N\}| \leq \kappa$. Then, X is hereditarily ω-resolvable.*

Proof: The space X is the condensation of a crowded space Y (Y is X with the topology generated by $\mathcal{N}$ as a base) which has character strictly less than $\mathfrak{c}$ (see Proposition 2.11). $\square$

Next, we obtain a result that we can locate between Theorem 3.5 which deals with π-networks and Corollary 3.6 which speaks of bases. First a definition and some remarks. A space X is called *σ-locally finite* if X can be written as $\bigcup_{n<\omega} X_n$ where, for each $n < \omega$, the collection $\{\{x\} : x \in X_n\}$ is locally finite in X. It can be proved that a σ-locally finite crowded space is hereditarily almost-ω-resolvable.

3.12 Theorem [MA]. *Let X be a crowded topological space with a network $\mathcal{N}$ with cardinality $\kappa < \mathfrak{c}$ and such that $\mathcal{N}_0 = \{N \in \mathcal{N} : |N| < \aleph_0\}$ is σ-locally finite in $\bigcup \mathcal{N}_0$. Then X can be written as $Y_0 \cup Y_1$ where Y_0 is a (possibly empty) regular closed ω-resolvable subspace and Y_1 is an open (possibly empty) almost-ω-resolvable, hereditarily ω-irresolvable space. Besides, if Y_1 is not void, it contains a non-empty open subset which is hereditarily almost-ω-resolvable. Moreover, if X is a T_1 Baire space, then X must be ω-resolvable.*

Proof: Let $\mathcal{M}$ be the collection of all subspaces of X which are ω-resolvable. Take $Y_0 = \mathrm{cl}_X \bigcup \mathcal{M}$ and $Y_1 = X \setminus Y_0$. Of course Y_0 is closed and ω-resolvable. Now, if Y_1 is empty, we have already finished; if the contrary happens, Y_1 is hereditarily ω-irresolvable and the collection $\mathcal{N}' = \{N \in \mathcal{N} : N \subseteq Y_1\}$ is a network in Y_1 with cardinality $< \mathfrak{c}$ and such that $\mathcal{N}'_0 = \{N \in \mathcal{N}' : |N| < \aleph_0\}$ is σ-locally finite in $\bigcup \mathcal{N}'_0$. Of course $\mathcal{N}'_0$ is not empty, because otherwise, by Theorem 3.5, Y_1 would be ω-resolvable, but this is not possible. Let Z be the subspace $\bigcup_{N \in \mathcal{N}'_0} N$ of X. The space Z is σ-locally finite. Since Y_1 is hereditarily ω-irresolvable, Z is a dense subset of Y_1. Then, Y_1 is almost-ω-resolvable. Furthermore, there must exist a non-empty open subset U of Y_1 such that each element of $\mathcal{N}'$ contained in U is finite because otherwise Y_1 would be ω-resolvable (again by Theorem 3.5). So, $\mathrm{int}\, Z$ is a non-empty open subset which is hereditarily almost-ω-resolvable.

Assume now that X is T_1 and satisfies all the conditions of our proposition including the Baire property. In this case Y_1 must be empty because if this is not the case, the subspace $\mathrm{int}\, Z$ of Y_1 would be a T_1 Baire hereditarily almost-ω-resolvable space. But this means, by Theorem 1.3, that $\mathrm{int}\, Z$ is ω-resolvable, which is not possible. $\square$

If we consider in the previous theorem π-networks instead of networks, we still get something interesting.

3.13 Proposition [MA]. *Let X be a crowded topological space with a π-network $\mathcal{N}$ with cardinality $\kappa < \mathfrak{c}$ and such that $\mathcal{N}_0 = \{N \in \mathcal{N} : |N| < \aleph_0\}$ is σ-locally finite. Then X is equal to $X_0 \cup X_1$ where $X_0 \cap X_1 = \emptyset$, X_0 is a regular closed (possibly empty) almost-ω-resolvable space and X_1 is an open (possibly empty) ω-resolvable subspace. In particular, X is, in this case, almost-ω-resolvable.*

Proof: Let Y be the subspace $\bigcup_{N\in\mathcal{N}_0} N$. The space Y is σ-locally finite. If Y is empty, we obtain our result by Theorem 3.5. If Y is crowded, then it is almost-ω-resolvable (see Theorem 3.5 in [26]). If Y is not empty and is not crowded, we can find an ordinal number $\alpha > 0$ and, for each $\beta < \alpha$, an ω-resolvable subspace M_β of X such that $X_0 = \mathrm{cl}_X(Y \cup \mathrm{cl}_X(\bigcup_{\beta<\alpha} M_\beta))$ is almost-ω-resolvable. In fact, let D_0 be the set of isolated points in $Y_0 = Y$. For each $x \in D_0$, there is an open set A_x in X such that $A_x \cap Y_0 = \{x\}$. Observe that $A_x \setminus \{x\}$ is a dense subset of A_x and it satisfies the conditions in Theorem 3.5, so it is ω-resolvable. Thus, $M_0 = \mathrm{cl}_X(\bigcup_{x\in D_0} A_x)$ is an ω-resolvable space. Assume that we have already constructed ω-resolvable subspaces M_β of X with $\beta < \gamma$. Put $Y_\gamma = Y \setminus \mathrm{cl}_X(\bigcup_{\beta<\gamma} M_\beta)$. If Y_γ is empty or crowded, we take $\alpha = \gamma$, and in this case $\mathrm{cl}_X(Y \cup \mathrm{cl}_X(\bigcup_{\beta<\gamma} M_\beta))$ is almost-ω-resolvable because Y_γ is empty or crowded and σ-locally finite. If Y_γ is not empty and is not crowded, let D_γ be the set of isolated points in Y_γ. For each $x \in D_\gamma$ there is an open set A_x in X such that $A_x \cap Y_\gamma = \{x\}$ and $A_x \cap \mathrm{cl}_X(\bigcup_{\beta<\gamma} M_\beta) = \emptyset$. Again $A_x \setminus \{x\}$ is a dense subset of A_x and it is ω-resolvable because of Theorem 3.5. Thus, $M_\gamma = \mathrm{cl}_X(\bigcup_{x\in D_\gamma} A_x)$ is an ω-resolvable space. Continuing with this process we have to find an ordinal number α for which $X_0 = \mathrm{cl}_X(Y \setminus \mathrm{cl}_X(\bigcup_{\beta<\alpha} M_\beta))$ is almost-ω-resolvable.

Now, if $X_1 = X \setminus X_0$ is not empty, then it is a crowded space and $\mathcal{N}_1 = \{N \in \mathcal{N} : N \subseteq X_1\}$ is a π-network in X_1 with infinite elements and $|\mathcal{N}_1| < \mathfrak{c}$. Then, again by Theorem 3.5, X_1 is ω-resolvable. Therefore, $X = X_0 \cup X_1$, and X_0, X_1 satisfy the conditions of our proposition. ☐

3.14 Questions. (1) *Let X be a crowded space with cardinality $< \mathfrak{c}$. Does MA+¬CH imply that X is almost-ω-resolvable?*

(2) *Is there a combinatorial axiom on ω_1 ensuring that every card-homogeneous topology in ω_1 is almost-ω-resolvable?*

(3) *Does $\diamondsuit$ imply that every card-homogeneous topology in ω_1 is almost-ω-resolvable?*

4. Martin's Axiom, cellularity and ω-resolvable Baire spaces

It is well known that MA(ω_1) implies that a Souslin line does not exist. That is, MA(ω_1) ⇒SH. We show that it is enough to assume SH in order to prove that every T_2 space with countable cellularity is almost-ω-resolvable.

4.1 Theorem [SH]. *Every crowded T_2 space with countable cellularity is almost-ω-resolvable.*

Proof: Let $a_0 \in X$ and $F_0 = \{a_0\}$. Let $\mathcal{C}_0$ be a maximal cellular family of open sets in $X \setminus F_0$ containing at least two elements. Let X_0 be equal to $\bigcup \mathcal{C}_0$. Assume that we have already constructed, by recursion, families $\{\mathcal{C}_\alpha : \alpha < \gamma\}$, $\{X_\alpha : \alpha < \gamma\}$ and $\{F_\alpha : \alpha < \gamma\}$, such that

(1) for all $\alpha < \gamma$, $\mathcal{C}_\alpha$ is a maximal cellular collection of open sets in X;

(2) if $\alpha < \xi < \gamma$, then $\mathcal{C}_\xi$ properly refines $\mathcal{C}_\alpha$;
(3) if $\alpha < \xi < \gamma$ and $C \in \mathcal{C}_\alpha$, then $\mathcal{C}_\xi$ contains a maximal cellular family of proper open sets of C having more than one element;
(4) $X_\alpha = \bigcup \mathcal{C}_\alpha$ for each $\alpha < \gamma$;
(5) the family $\{X_\alpha : \alpha < \gamma\}$ is a strictly decreasing γ-sequence of open sets in X;
(6) $F_\alpha \neq \emptyset$ for every $\alpha < \gamma$;
(7) $F_\alpha \subseteq (\bigcap_{\xi<\alpha} X_\xi) \setminus X_\alpha$ for all $\alpha < \gamma$;
(8) $\mathrm{int}(F_\alpha) = \emptyset$ for all $\alpha < \gamma$.

If γ is a successor ordinal, say $\gamma = \xi + 1$, take for each $C \in \mathcal{C}_\xi$ a point $a^\gamma_C \in C$. Now, take a maximal cellular family of open proper subsets in $C \setminus \{a^\gamma_C\}$ with more than one element, $\mathcal{C}^\xi_C$ (this is possible because C is T_2 and infinite). Put $\mathcal{C}_\gamma = \bigcup_{C\in\mathcal{C}_\gamma} \mathcal{C}^\xi_C$, $X_\gamma = \bigcup \mathcal{C}_\gamma$ and $F_\gamma = \{a^\gamma_C : C \in \mathcal{C}_\xi\}$.

If γ is a limit ordinal, analyse the set $\bigcap_{\xi<\gamma} X_\xi$: if $\mathrm{int}(\bigcap_{\xi<\gamma} X_\xi) = \emptyset$, declare our process finished; and if $\mathrm{int}(\bigcap_{\xi<\gamma} X_\xi)$ is not empty, take a point $a_\gamma \in \mathrm{int}(\bigcap_{\xi<\gamma} X_\xi)$ and take a maximal cellular family $\mathcal{C}_\gamma$ with cardinality bigger than one of open proper subsets in $\mathrm{int}(\bigcap_{\xi<\gamma} X_\xi) \setminus F_\gamma$ where $F_\gamma = \{a_\gamma\}$. Put $X_\gamma = \bigcup \mathcal{C}_\gamma$.

In this way we can find an ordinal number α_0 and families $\mathfrak{C} = \{\mathcal{C}_\alpha : \alpha < \alpha_0\}$, $\mathcal{X} = \{X_\alpha : \alpha < \alpha_0\}$ and $\mathcal{F} = \{F_\alpha : \alpha < \alpha_0\}$ satisfying properties from (1) to (8) above where α_0 is an ordinal number such that $\mathrm{int}(\bigcap_{\xi<\alpha_0} X_\xi) = \emptyset$ and for each $\alpha < \alpha_0$, $\mathrm{int}(\bigcap_{\xi<\alpha} X_\xi) \neq \emptyset$.

First, observe that α_0 must be a limit ordinal and every X_α is an open set of X. Now, consider the collection $\mathcal{Y} = \{Y_\alpha : \alpha < \alpha_0\}$ of subspaces of X where $Y_0 = X \setminus X_0$, and $Y_\alpha = (\bigcap_{\xi<\alpha} X_\alpha) \setminus X_\alpha$ if $\alpha > 0$. We have that $F_\alpha \subseteq Y_\alpha$ and $\mathrm{int}(Y_\alpha) = \emptyset$ for every $\alpha < \alpha_0$.

The set $\bigcup_{\alpha<\alpha_0} \mathcal{C}_\alpha$ with the order relation $\subseteq$ is a tree T and each element in it has at least two immediate successors.

Claim 1. The height of T, α_0, is at most $c(X)^+ = \omega_1$.

In fact, if $\alpha_0 > \omega_1$, then $\mathcal{C}_{\omega_1} \neq \emptyset$. Take $C_{\omega_1} \in \mathcal{C}_{\omega_1}$. Let $\mathcal{C} = \{C \in T : C \supseteq C_{\omega_1}$ and $C \neq C_{\omega_1}\}$. Since T is a tree, $\mathcal{C}$ is a well ordered set with order type ω_1. We can rename $\mathcal{C}$ as $\{C_\alpha : \alpha < \omega_1\}$ where C_α is the only element in $\mathcal{C}_\alpha$ which belongs to $\mathcal{C}$. For each $\alpha < \omega_1$, there is $A_{\alpha+1} \in \mathcal{C}_{\alpha+1}$ such that $A_{\alpha+1} \subseteq C_\alpha$ and $A_{\alpha+1} \cap C_{\alpha+1} = \emptyset$. The set $\mathcal{A} = \{A_{\alpha+1} : \alpha < \omega_1\}$ is an antichain in T. Indeed, let $A_{\alpha+1}$ and $A_{\xi+1}$ be two different elements of $\mathcal{A}$. Assume that $\alpha < \xi$. Hence, $A_{\xi+1} \subseteq C_\xi$ and $C_\xi \subseteq C_{\alpha+1}$. But $C_{\alpha+1} \cap A_{\alpha+1} = \emptyset$. Therefore, $A_{\alpha+1} \cap A_{\xi+1} = \emptyset$. This means that $c(X) > \aleph_0$, which is a contradiction. We get that every chain and every antichain of T has cardinality $\leq \aleph_0$. Since we are assuming the Souslin's Hypothesis, there are no Souslin trees. Therefore $\alpha_0 < \omega_1$.

It is not difficult to prove that the set $Z = X \setminus X_{\alpha_0}$ is equal to $\bigcup_{\alpha<\alpha_0} Y_\alpha$ and that the collection $\{Y_\lambda : \alpha < \alpha_0\}$ is a partition of Z.

Claim 2. The collection $\{Y_\alpha : \alpha < \alpha_0\} \cup \{X_{\alpha_0}\}$ is an almost-ω-resolution for X; that is, X is almost-ω-resolvable.

The collection $\mathcal{Y} = \{Y_\alpha : \alpha < \alpha_0\} \cup \{X_{\alpha_0}\}$ is a countable partition of X. Assume that A is a non-empty open set of X and $|\{\alpha < \alpha_0 : A \cap Y_\alpha \neq \emptyset\}| < \aleph_0$. Assume that $H = \{\alpha < \alpha_0 : A \cap Y_\alpha \neq \emptyset\}$ is equal to $\{\xi_1, \ldots, \xi_n\}$ with $\xi_1 < \xi_2 < \cdots < \xi_n$.

If $B = A \cap X_{\alpha_0} \neq \emptyset$, then $A \cap X_{\xi_n} = B$. But A and X_{ξ_n} are open sets in X, so B is a non-empty open set in X, contradicting the fact that $\operatorname{int}(X_{\alpha_0}) = \emptyset$. This means that $A \cap X_{\alpha_0}$ must be empty.

Now, let $B = A \cap Y_{\xi_n}$. B is not empty and $A \cap X_{\xi_{n-1}} = B$. Thus, B is a non-empty open set in X which does not intersect any member of $\mathcal{C}_{\xi_n}$. If $\xi_n = \alpha + 1$, $\mathcal{C}_{\xi_n}$ is a maximal cellular collection of open sets contained in $(\bigcup \mathcal{C}_\alpha) \setminus \{a^\alpha_C : C \in \mathcal{C}_\alpha\} = X_\alpha \setminus \{a^\alpha_C : C \in \mathcal{C}_\alpha\}$. Hence, $B \cap \{a^\alpha_C : C \in \mathcal{C}_\alpha\} \neq \emptyset$. Let $a^\gamma_C \in B$. We have that $M = (C \cap B) \setminus \{a^\gamma_C\}$ is an open set contained in $X_\alpha \setminus \{a^\alpha_C : C \in \mathcal{C}_\alpha\}$ and no element in $\mathcal{C}_\xi$ intersects M. By maximality of $\mathcal{C}_\xi$, we must have that M is empty; that is, $C \cap B = \{a^\gamma_C\}$, and this is not possible because X does not have isolated points.

Now assume that ξ_n is a limit ordinal. Since B is open and $B \subseteq \bigcap_{\xi < \xi_n} X_\xi$, B must be contained in $\operatorname{int}(\bigcap_{\xi < \xi_n} X_\xi)$. Since $\{a_{\xi_n}\}$ is closed and B does not intersects any element of $\mathcal{C}_{\xi_n}$ which is a maximal cellular family of open sets contained in the set $\operatorname{int}(\bigcap_{\xi < \xi_n} X_\xi) \setminus \{a_{\xi_n}\}$, B must be equal to $\{a_{\xi_n}\}$, which is again a contradiction.

Therefore, $|\{\xi < \alpha_0 : A \cap Y_\xi \neq \emptyset\}|$ must be equal to $\aleph_0$. □

Since the cellularity of a space is a monotone function when it is applied on dense subspaces, and using Theorem 1.3, we conclude:

4.2 Corollary [SH]. *Every T_2 Baire space with $c(X) \leq \aleph_0$ is ω-resolvable.*

Example 4.3 in [26] (see Example 2.3 above) gives us a space which is Baire, T_1 with countable cellularity but it is not almost-ω-resolvable. This example is constructed assuming the existence of measurable cardinals. Moreover, there is a model M in which SH holds and there are measurable cardinals. So we cannot get anything stronger than our results of this section by assuming only T_1. Furthermore, we cannot erase the Baire condition in Corollary 4.2 because there is in ZFC a Tychonoff, countable irresolvable space (see Examples 2.5). Finally, in 2.2 we list an example of a space with cellularity $\leq \aleph_1$ which is Baire and is not almost-ω-resolvable. This last example is given by assuming the existence of an ω_1-complete ideal over ω_1 which has a dense set of cardinality ω_1. Hence, it is natural to ask:

4.3 Question. *Does MA imply that every crowded T_2 space of cellularity $< \mathfrak{c}$ is almost-ω-resolvable?*

In this question, we cannot change "almost-ω-resolvable" for "resolvable" since there is in ZFC an irresolvable countable space.

5. Almost-ω-irresolvable spaces

A space is *almost-ω-irresolvable* if it is not almost-ω-resolvable. In a similar way we define almost irresolvable spaces.

5.1 Proposition. *If X is almost-ω-irresolvable, then there is a non-empty open subset U of X which is hereditarily almost-ω-irresolvable.*

Proof: Let $\mathcal{U}$ be the collection of all almost-ω-resolvable subspaces Y of X. The set $Z = \mathrm{cl}_X(\bigcup \mathcal{U})$ is almost-ω-resolvable and $U = X \setminus Z$ is not empty and satisfies the requirements. $\square$

5.2 Proposition. *If X is open hereditarily almost-ω-irresolvable, then X is a Baire space.*

Proof: Let $\{U_n : n < \omega\}$ be a sequence of open and dense subsets of X. We can choose this sequence to be $\subseteq$-decreasing. Denote by F the set $\bigcap_{n<\omega} U_n$. We claim that F is dense in X. In fact, if for a $k < \omega$, $\mathrm{cl}_X F \supseteq U_k$, then $\mathrm{cl}_X F \supseteq \mathrm{cl}_X U_k = X$ and F is dense. Now, assume that for each $n < \omega$, $U_n \setminus \mathrm{cl}_X F$ is not empty. In this case, the collection $T = \{i < \omega : (U_i \setminus U_{i+1}) \cap (X \setminus \mathrm{cl}_x F) \neq \emptyset\}$ is infinite. For each $i \in T$, we put $T_i = (U_i \setminus U_{i+1}) \cap (X \setminus \mathrm{cl}_x F)$. The collection $\{T_i : i < \omega\}$ forms an almost-ω-resolution of $X \setminus \mathrm{cl}_X F$. But this is not possible. $\square$

5.3 Corollary. *If there is an almost resolvable space X which is almost-ω-irresolvable, then there is a resolvable Baire open subspace U of X which is hereditarily almost-ω-irresolvable.*

Proof: Let X be an almost-resolvable almost-ω-irresolvable space. The space X contains a non-empty open subspace U which is hereditarily almost-ω-irresolvable. By Proposition 5.2, U is a Baire space; so, it is resolvable being almost resolvable. $\square$

5.4 Corollary. *There is an almost resolvable space X which is almost-ω-irresolvable if and only if there is an almost resolvable Baire space which is hereditarily almost-ω-irresolvable.*

As a consequence of the previous result, we have that almost resolvability and almost-ω-resolvability coincide in the class of spaces X in which every open subset is not a Baire space. Even more was obtained in [2, Corollary 5.21]: every space which does not contain a Baire open subspace is almost-ω-resolvable.

5.5 Proposition. *Let X be a T_1 space. Then X is hereditarily resolvable if and only if X is hereditarily ω-resolvable.*

Proof: Let Y be a crowded subspace of X and assume that Y is not ω-resolvable. Then, there is $k \in \omega$ with $k > 1$ such that X is k-resolvable but X is not $(k+1)$-resolvable [15]. So there are $D_0, \dots, D_{k-1}$ dense and pairwise disjoint subspaces of Y. But, then, each D_i is crowded and irresolvable, a contradiction. □

5.6 Proposition. *Let X have the property that every of its crowded subspaces is Baire. Then X is hereditarily ω-resolvable iff X is hereditarily resolvable iff X is hereditarily almost-ω-resolvable iff X is hereditarily almost resolvable.*

Several results established in [2, Section 5] and [26, Section 4] relate Baire irresolvable spaces with the property of almost-ω-resolvability (see also [1, Section 3]). In the following theorem we obtain the most general possible result in the mood of these propositions.

5.7 Theorem. *For crowded T_1 spaces and for a crowded-hereditarily topological property P, the following assertions are equivalent:*

(1) *every Baire space with P is ω-resolvable,*
(2) *every Baire space with P is resolvable,*
(3) *every space with P is almost-ω-resolvable,*
(4) *every space with P is almost resolvable.*

Proof: The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are evident.

(2) $\Rightarrow$ (3): Assume that X is not almost-ω-resolvable and satisfies P. The space X contains an open and non-empty subset U which is hereditarily almost-ω-irresolvable. By Proposition 5.5, U is not hereditarily resolvable, so there is a crowded subspace Y which is not resolvable. Observe that Y is hereditarily almost-ω-irresolvable, then Y is an irresolvable Baire space because of Proposition 5.2. Since P is a crowded-hereditarily topological property, Y satisfies P too.

(4) $\Rightarrow$ (2): Assume that X is a Baire space with P. By hypothesis, X is almost resolvable and every Baire almost resolvable space is resolvable (see [2, Corollary 5.4]).

(3) $\Rightarrow$ (1): Assume that X is a Baire space with P. By hypothesis, every crowded subspace Y of X has P and so it is almost-ω-resolvable; hence X is ω-resolvable because of Theorem 1.3. □

Taking P equal to "X is a crowded topological space", we have:

5.8 Corollary. *For crowded T_1 spaces, the following assertions are equivalent:*

(1) *every Baire space is ω-resolvable,*
(2) *every Baire space is resolvable,*
(3) *every space is almost-ω-resolvable,*
(4) *every space is almost resolvable.*

A space is *locally homogeneous* if each of its points has a homogeneous neighborhood. For a cardinal number $\kappa \geq 1$, we will say that X is *exactly κ-resolvable*, in symbols $\mathrm{E}_\kappa\mathrm{R}$, if X is κ-resolvable but is not κ^+-resolvable. The space X is said to be $\mathrm{OE}_\kappa\mathrm{R}$ if every non-empty open set in X is $\mathrm{E}_\kappa\mathrm{R}$. The concept and examples of $\mathrm{E}_n\mathrm{R}$ spaces for $n \in \omega$ have existed in the literature for some time (see, for example, [10] and [8]). It is clear that the $\mathrm{OE}_\kappa\mathrm{R}$ spaces are $\mathrm{E}_\kappa\mathrm{R}$. The above definitions can be viewed as natural generalizations of the concepts of irresolvable and open-hereditarily irresolvable spaces since $\mathrm{E}_1\mathrm{R}$ and irresolvability are the same concept and $\mathrm{OE}_1\mathrm{R}$ and open-hereditarily irresolvability coincide.

It was proved in [1, Theorem 3.13] that every locally homogeneous irresolvable space such that its cardinality is not a measurable cardinal is of the first category. Also, Li Feng and O. Masaveu [13] proved that every crowded topological space X can be written as

$$X = \Omega \cup \mathrm{cl}_X\left(\bigcup_{n=1}^{\infty} O_n\right),$$

where

(1) for each n, O_n is an open, possibly empty, subset of X;
(2) for each n, if $O_n \neq \emptyset$, then it is $\mathrm{OE}_n\mathrm{R}$;
(3) for $n \neq m$, $O_n \cap O_m = \emptyset$; and
(4) Ω is an open, possibly empty, ω-resolvable subset of X.

Thus we obtain the following:

5.9 Proposition. *Every locally homogeneous Baire space of cardinality strictly less than the first measurable cardinal is resolvable.*

Proof: Let X be a locally homogeneous Baire space. Write X as Feng and Masaveu say: $X = \Omega \cup \mathrm{cl}_X(\bigcup_{n=1}^{\infty} O_n)$. Assume that O_1 is not empty and take $x \in O_1$. There is a homogeneous neighborhood W of x. (Observe that W has to be contained in $X \setminus \mathrm{cl}_X(\Omega \cup \bigcup_{n>1} O_n) \subseteq \mathrm{int}_X\, \mathrm{cl}_X\, O_1$). On the other hand, O_1 is open hereditarily irresolvable, so $\mathrm{int}_X\, W \cap O_1$ is irresolvable. Since $\mathrm{int}_X\, W \cap O_1$ is a non-empty open subset of W, W is irresolvable. By Theorem 3.13 in [1], W is of first category. In particular the open and non-empty subset $O_1 \cap \mathrm{int}_X\, W$ of X is of first category in itself, but this is not possible because X is a Baire space. Hence, $O_1 = \emptyset$ and X is resolvable. $\square$

5.10 Questions. (1) *Is every pseudocompact (resp., Čech-complete) Tychonoff space almost-ω-resolvable in ZFC?*

(2) *Is every Baire locally homogeneous space (resp., homogeneous space, topological group) ω-resolvable?*

(3) *For each $n > 1$, is there a Baire OE_nR space?*

6. The infinite π-netweight and $\mathrm{Seq}(u_t)$ spaces

We define the *infinite π-networkweight* of a crowded space X, $\pi nw^*(X)$, as the minimum infinite cardinal of a π-network with infinite elements. And $\pi nw(X)$ is the minimum infinite cardinal of a π-network in X. It is easy to prove that $\pi nw(X) = d(X)$ for every topological space X. Moreover, for a crowded space X, we have $d(X) \leq \pi nw^*(X) \leq \min\{d(X) \cdot \sup\{\pi nw^*(x, X) : x \in X\}, d(X) \cdot R(X), \pi w(X)\}$, where $nw^*(x, X)$ and $R(X)$ were defined before Corollaries 3.9 and 3.10. Besides, for every metrizable space X we have $d(X) = w(X)$. So, for a crowded metrizable space X, the equality $\pi nw^*(X) = \pi nw(X)$ always holds. We have the same phenomenon for spaces of the form $C_p(X)$, the space of real continuous function defined on X with the pointwise convergent topology (here, X is not necessarily crowded). Indeed, for $f \in C_p(X)$, the sequence $(f_n)_{n<\omega}$ where $f_n = f + 1/n$, converges to f. So, if D is a dense subset of $C_p(X)$ with cardinality equal to $d(C_p(X))$, the collection $\{\{f\} \cup \{f_i : i \geq n\} : f \in D, n < \omega\}$ is a π-network of cardinality $d(C_p(X))$ constituted by infinite elements. So, $\pi nw^*(C_p(X)) = \pi nw(C_p(X))$. In particular, for every cardinal number κ, $\pi nw^*(\mathbb{R}^\kappa) = d(\mathbb{R}^\kappa)$. The same can be said for spaces of the form $C_p(X, 2)$ where X is an infinite zero-dimensional T_2 space. In fact, we can take an infinite discrete subspace $Y = \{x_n : n < \omega\}$ of X, and clopen subsets $\{V_n : n < \omega\}$ such that, for each $n < \omega$, $Y \cap V_n = \{x_n\}$. The characteristic functions χ_{V_n} constitute a sequence which converge to the constant function 0. So, in this case too, $\pi nw^*(C_p(X, 2)) = d(C_p(X, 2))$.

We have already mentioned that in [1] a dense countable subset Y of $2^{\mathfrak{c}}$ which is irresolvable was constructed in ZFC. This space has $\pi nw(Y) = \aleph_0$, but every of its countable π-networks has to have finite elements, because otherwise Y would be maximally resolvable (see Theorem 2.8(1)). The $\mathrm{Seq}(u_t)$ spaces considered below are also examples of spaces of this kind.

We recall that for a $p \in \omega^*$, $\chi(p) = \min\{|b| : b \text{ is a base for } p\}$. Of course we can also define: $\pi\chi(p) = \min\{|b| : b \text{ is a } \pi\text{-base for } p\}$ where a family of infinite sets $\mathcal{G}$ in ω is a π-base for p if every member of p contains an element of $\mathcal{G}$. It is not difficult to prove that for every $p \in \omega^*$, $\pi\chi(p) \leq \chi(p)$ and $\pi\chi(p) > \aleph_0$. In fact, assume that $N_0, \dots, N_k, \dots$ are infinite subsets of ω. By recursion, we can construct two sequences $A = \{a_0, \dots, a_n, \dots\}$ and $B = \{b_0, \dots, b_n, \dots\}$ such that the elements in $A \cup B$ are pairwise different, and for each $n < \omega$, $a_n, b_n \in N_n$. If $A \in p$ then A is an element of p which does not contain any N_k. If $A \notin p$, then $\omega \setminus A$ belongs to p and does not contain any N_k.

By Seq we mean the set of all finite sequences of natural numbers. More precisely, for each natural number $n \in \omega$, let ${}^n\omega = \{t : t \text{ is a function and } t : n \to \omega\}$. Then $\mathrm{Seq} = \bigcup_{n\in\omega} {}^n\omega$. If $t \in \mathrm{Seq}$, with domain $k = \{0, 1, \dots, (k-1)\}$, and $n \in \omega$, let $t^\frown n$ denote the function $t \cup \{(k, n)\}$. For every $t \in \mathrm{Seq}$ let u_t be a non-principal ultrafilter on ω. By $\mathrm{Seq}(\{u_t : t \in \mathrm{Seq}\})$ we denote the space with underlying set Seq and topology defined by declaring a set $U \subseteq \mathrm{Seq}$ to be open if and only if

$$(\forall t \in U)\,\{n \in \omega : t^\frown n \in U\} \in u_t.$$

For short, we write $\mathrm{Seq}(u_t)$ instead of $\mathrm{Seq}(\{u_t : t \in \mathrm{Seq}\})$. We also consider the case where there is a single non-principal ultrafilter p in ω such that $u_t = p$ for all $t \in \mathrm{Seq}$, and in this case we write $\mathrm{Seq}(p)$ instead of $\mathrm{Seq}(u_t)$.

We use the following notation of W. Lindgren and A. Szymanski [20]; put $L_n = \{s \in \mathrm{Seq} : \mathrm{dom}(s) = n\}$, and for any $s \in \mathrm{Seq}$ the *cone over* s is defined by $C(s) = \{t \in \mathrm{Seq} : s \subseteq t\}$. In particular, $L_0 = \{\emptyset\}$. We add some other notations: For each $s \in L_n$, $T(s) = \{t \in L_{n+1} : s \subseteq t\}$. Observe that for every $s \in \mathrm{Seq}$, $C(s)$ is a clopen subset of $\mathrm{Seq}(u_t)$.

It is well-known that for any choice of $\{u_t : t \in \mathrm{Seq}\} \subseteq \omega^*$, the space $\mathrm{Seq}(u_t)$ is a zero-dimensional, extremally disconnected, Hausdorff space with no isolated points. By the way, $\mathrm{Seq}(p)$ is homogeneous and if p is Ramsey, there is a binary group operation $+$ such that $(\mathrm{Seq}(p), +)$ is a topological group (see [27]).

6.1 Proposition. *Every* $\mathrm{Seq}(u_t)$ *space is ω-resolvable.*

Proof: In fact, let $\{E_n : n < \omega\}$ be a partition of ω where each E_n is infinite. Set $D_n = \bigcup_{i \in E_n} L_i$. Each D_n is dense in $\mathrm{Seq}(u_t)$ and $D_n \cap D_m = \emptyset$ if $n \neq m$. □

6.2 Proposition. *Let* $\{u_t : t \in \mathrm{Seq}\} \subseteq \omega^*$. *Then, the infinite π-netweight of* $\mathrm{Seq}(u_t)$ *is not countable.*

Proof: For each $n < \omega$, each $s \in L_n$, and each sequence S of subcollections of the form

$$\begin{gathered}\{B(s)\}, \{B(s, i_{n+1}) : i_{n+1} \in B(s)\}, \{B(s, i_{n+1}, i_{n+2}) : i_{n+1} \in B(s),\\ i_{n+2} \in B(s, i_{n+1})\}, \dots, \{B(s, i_{n+1}, \dots, i_{n+k+1}) : i_{n+1} \in B(s),\\ i_{n+1} \in B(s, i_{n+1}), \dots, i_{n+k+1} \in B(s, i_{n+1}, \dots, i_{n+k})\}, \dots\end{gathered}$$

where $B(s) \in u_s$ and, if $i_{n+1} \in B(s), i_{n+2} \in B(s, i_{n+1}), \dots, i_{n+k} \in B(s, i_{n+1}, \dots, i_{n+k-1})$, $B(s, i_{n+1}, \dots, i_{n+k}) \in u_t$ with $t = s^\frown i_{n+1}^\frown \dots ^\frown i_{n+k}$, we define a set $V(s, S)$ as follows:

$$\begin{aligned}V(s,S) = \{s\} \cup \{t \in \mathrm{Seq}(p) :\ & m \in \omega, t \in L_{n+m+1}, s \subseteq t, t(n+1) \in B(s),\\ & t(n+2) \in B(s, t(n+1)), \dots,\\ & t(n+m+1) \in B(s, t(n+1), t(n+2), \dots, t(n+m))\}.\end{aligned}$$

We call this set $V(s,S)$ *cascade of* $\mathrm{Seq}(p)$ *defined by* (s,S). Moreover, we will called each sequence S, described as above, *fan on* $(s, (u_t))$.

Of course, the collection of cascades forms a base of clopen sets for $\mathrm{Seq}(u_t)$.

Claim 1. If $\mathcal{N} = \{N_0, \ldots, N_k, \ldots\}$ is a countable set of infinite subsets of $\mathrm{Seq}(u_t)$, then $\mathcal{N}$ is not a π-network of $\mathrm{Seq}(u_t)$.

We are going to prove Claim 1 in several lemmas.

Claim 1.1. If $\mathcal{M}$ is a finite collection of subsets of Seq, then there is a non-empty open set A of $\mathrm{Seq}(u_t)$ such that $M \setminus A \neq \emptyset$ for all $M \in \mathcal{M}$.

Proof: Take $s_0, \ldots, s_n$ elements in Seq such that each M in $\mathcal{M}$ contains one of this points. There is $k < \omega$ such that $s_i \in L_m$ implies $m < k$ for all $i \in \{0, \ldots, n\}$. Take $s \in L_k$. The cone $C(s)$ is open and contains no element in $\mathcal{M}$. □

Claim 1.2. Assume that $F \subseteq \mathrm{Seq}(u_t)$ is such that $|F \cap T(s)| \leq 1$ for every $s \in \mathrm{Seq}$. Then, F is a proper closed subset of $\mathrm{Seq}(u_t)$.

Proof: Let P be the set $\{s < \mathrm{Seq} : F \cap T(s) \neq \emptyset\}$. Let z_s be the only point belonging to $F \cap T(s)$ for each $s \in P$. Let $x \in \mathrm{Seq}(u_t) \setminus F$. Assume that $x = (n_0, \ldots, n_k)$ (the argument is similar if $x = \emptyset$). Let

$$S = \{\{B(x)\}, \{B(x, i_0) : i_0 \in B(x)\}, \{B(x, i_0, i_1) : i_0 \in B(x), i_1 \in B(x, i_0)\}, \ldots,$$
$$\{B(x, i_0, \ldots, i_{k+1}) : i_0 \in B(x), i_1 \in B(x, i_1), \ldots, i_{k+1} \in B(x, i_0, \ldots, i_k)\}, \ldots\}$$

be a fan on $(x, (u_t))$. We claim that the set $V(x, S) \setminus F$ is an open set. Indeed, if $y \in V(x, S) \setminus F$, y is of the form $(n_0, \ldots, n_k, i_0, \ldots, i_{m+1})$ where $m < \omega$, $i_0 \in B(x), i_1 \in B(x, i_0), \ldots, i_{m+1} \in B(x, i_0, i_1, \ldots, i_m)$.

The set $\{l < \omega : (n_0, \ldots, n_k, i_0, \ldots, i_{m+1}, l) \in V(x, s) \setminus F\}$ is equal to

$$B\left(x, i_0, i_1, \ldots, i_{m+1}\right) \setminus F.$$

Moreover, the set $B(x, i_0, i_1, \ldots, i_{m+1}) \cap F = G$ is either empty if $F \cap T(x, i_0, i_1, \ldots, i_{m+1}) = \emptyset$, or $G = \{z_{(x, i_0, i_1, \ldots, i_{m+1})}\}$ if $F \cap T(x, i_0, i_1, \ldots, i_{m+1}) \neq \emptyset$. Of course, in both cases, $B(x, i_0, i_1, \ldots, i_{m+1}) \setminus F$ belongs to u_t where $t = x^\frown i_0^\frown \ldots ^\frown i_{m+1}$. This means that $V(x, s) \setminus F$ is open. □

Claim 1.3. Let $\mathcal{M} = \{N \in \mathcal{N} : \forall s \in \mathrm{Seq}(|N \cap T(s)| < \aleph_0)\}$. Then, there is a non-empty open set A of $\mathrm{Seq}(u_t)$ such that $N \setminus A \neq \emptyset$ for all $N \in \mathcal{M}$.

Proof: First, we define in Seq a well order $\sqsubseteq$ as follows: $\emptyset$ is the $\sqsubseteq$-first element, and for two elements s and t different to $\emptyset$, we define $s \sqsubset t$ if either $s \in L_{n+1}$, $t \in L_{m+1}$ and $n < m$, or $n = m$ and $s(n) < t(n)$.

Because of Claim 1.1, we can assume that $\mathcal{M}$ is infinite. We faithfully enumerate $\mathcal{M}$ as $\{M_0, M_1, \ldots, M_k, \ldots\}$. Consider the set $J = \{s \in \mathrm{Seq} : \exists M \in \mathcal{M}$ such that $T(s) \cap M \neq \emptyset\}$. Because of the definition of $\mathcal{M}$, we must have $|J| = \aleph_0$. Hence, we can enumerate J as $\{s_m : m < \omega\}$ in such a way that $s_0 \sqsubset s_1 \sqsubset \cdots \sqsubset s_n \sqsubset s_{n+1} \sqsubset \ldots$.

Let k_0 be the first natural number m such that $M_m \cap T(s_0) \neq \emptyset$. We take $z_0 \in M_{k_0} \cap T(s_0)$. Assume that we have already defined two finite sequences $k_0, \dots, k_l$ and $z_0, \dots, z_l$ such that

(1) for each $i \in \{0, \dots, l-1\}$, k_{i+1} is the first natural number $m \in \omega \setminus \{k_0, \dots, k_i\}$ such that $M_m \cap T(s_{i+1}) \neq \emptyset$, and
(2) $z_{i+1} \in M_{k_{i+1}} \cap T(s_{i+1})$ for each $i \in \{0, \dots, l-1\}$.

We define now k_{l+1} as the first natural number $m \in \omega \setminus \{k_0, \dots, k_l\}$ such that $M_m \cap T(s_{l+1}) \neq \emptyset$. Take $z_{l+1} \in M_{k_{l+1}} \cap T(s_{l+1})$.

Observe that $\{k_i : i < \omega\} = \omega$. Indeed, assume that $\{0, \dots, m\} \subseteq \{k_i : i < \omega\}$ and $\{k_{i_0}, \dots, k_{i_m}\} = \{0, \dots, m\}$. Let j be a natural number greater than k_{i_l} for all $l \in \{0, \dots, m\}$ and such that $M_{m+1} \cap T(s_j) \neq \emptyset$. Then we must have $m + 1 \in \{k_0, \dots, k_j\}$.

We put $F = \{z_i : i < \omega\}$. The set F satisfies the conditions required in Claim 1.2; so, F is a proper closed subset of $\mathrm{Seq}(u_t)$. Therefore, $A = \mathrm{Seq}(u_t) \setminus F$ is a non-empty open set which does not contain any of the sets $M \in \mathcal{M}$. □

Claim 1.4. Let $\mathcal{O} = \mathcal{N} \setminus \mathcal{M} = \{N \in \mathcal{N} : \exists s \in \mathrm{Seq}(|N \cap T(s)| \geq \aleph_0)\}$. Then, there is an open set B of $\mathrm{Seq}(u_t)$ such that $N \setminus B \neq \emptyset$ for all $N \in \mathcal{O}$.

Proof: Let $T = \{n < \omega : N_n \in \mathcal{O}\}$. The open set B will be an open cascade $V(s, S)$ defined by (s, S) where $s = \emptyset$ and the fan

$$S = \{\{B(s)\}, \{B(s, i_1) : i_1 \in B(s)\}, \{B(s, i_1, i_2) : i_1 \in B(s), i_2 \in B(s, i_1)\}, \dots, \{B(s, i_1, \dots, i_{k+1}) : i_1 \in B(s), i_1 \in B(s, i_1), \dots, i_{k+1} \in B(s, i_1, \dots, i_k)\}, \dots\}$$

will be constructed by recursion.

Assume that we have already selected

$$\{\{B(s)\}, \{B(s, i_1) : i_1 \in B(s)\}, \{B(s, i_1, i_2) : i_1 \in B(s), i_2 \in B(s, i_1)\}, \dots, \{B(s, i_1, \dots, i_k) : i_1 \in B(s), i_2 \in B(s, i_1), \dots, i_k \in B(s, i_1, \dots, i_{k-1})\}\}.$$

For each sequence $i_1 \in B(s), i_2 \in B(s, i_1), \dots, i_{k+1} \in B(s, i_1, i_2, \dots, i_k)$, consider the ultrafilter u_t where $t = s^\frown i_1^\frown \dots ^\frown i_k$, and consider the set $P(s, i_1, \dots, i_{k+1}) = \{n \in T : |N_n \cap T(s, i_1, \dots, i_k)| \geq \aleph_0\}$. If $P(s, i_1, \dots, i_{k+1})$ is empty, we choose $B(s, i_1, \dots, i_{k+1})$ to be an arbitrary element of u_t. If $P(s, i_1, \dots, i_{k+1})$ is not empty, there is $B(s, i_1, \dots, i_{k+1}) \in u_t$ such that $N_n \setminus B(s, i_1, \dots, i_{k+1}) \neq \emptyset$ for every $n \in P(s, i_1, \dots, i_{k+1})$ because $\pi\chi(u_t) > \aleph_0$.

We have already finished the description of the recursive process that define the fan S. The set $B = V(s, S)$ is the required open set.

We finished the proof of Claim 1 by saying that the open set $A \cap B$, where A was defined in the proof of Claim 1.3 and B in that of Claim 1.4, is not empty and does not contain any of the elements in $\mathcal{N}$. $\square$

References

[1] Alas O., Sanchis M., Tkačhenko M.G., Tkachuk V.V., Wilson R.G., *Irresolvable and submaximal spaces: Homogeneity versus σ discreteness and new ZFC examples*, Topology Appl. **107** (2000), 259–273.

[2] Angoa J., Ibarra M., Tamariz-Mascarúa Á., *On ω-resolvable and almost-ω-resolvable spaces*, Comment. Math. Univ. Carolin. **49** (2008), 485–508.

[3] Bell M., Kunen K., *On the π-character of ultrafilters*, C.T. Math. Rep. Acad. Sci. Canada **3** (1981), 351–356.

[4] Biernias J., Terepeta M., *A sufficient condition for maximal resolvability of topological spaces*, Comment. Math. Univ. Carolin. **41** (2004), 139–144.

[5] Bolstein R., *Sets of points of discontinuity*, Proc. Amer. Math. Soc. **38** (1973), 193–197.

[6] Comfort W.W., Feng L., *The union of resolvable spaces is resolvable*, Math. Japonica **38** (1993), 413–114.

[7] Comfort W.W., García-Ferreira S., *Resolvability: a selective survey and some new results*, Topology Appl. **74** (1996), 149–167.

[8] van Douwen E.K., *Applications of maximal topologies*, Topology Appl. **51** (1993), 125–139.

[9] El'kin A.G., *On the maximal resolvability of products of topological spaces*, Soviet Math. Dokl. **10** (1969), 659–662.

[10] El'kin A.G., *Resolvable spaces which are not maximally resolvable*, Moscow Univ. Math. Bull. **24** (1969), 116–118.

[11] Feng L., *Strongly exactly n-resolvable spaces of arbitrary large dispersion character*, Topology Appl. **105** (2000), 31–36.

[12] Foran J., Liebnits P., *A characterization of almost resolvable spaces*, Rend. Circ. Mat. Palermo (2) **40** (1991), 136–141.

[13] Feng L., Masaveu O., *Exactly n-resolvable spaces and ω-resolvability*, Math. Japonica **50** (1999), 333–339.

[14] Hewitt E., *A problem of set-theoretic topology*, Duke Math. J. **10** (1943), 306–333.

[15] Illanes A., *Finite and ω-resolvability*, Proc. Amer. Math. Soc. **124** (1996), 1243–1246.

[16] Katětov M., *On topological spaces containing no disjoint dense sets*, Mat. Sb. **21** (1947), 3–12.

[17] Kunen K., *Set Theory. An Introduction to Independence Proofs*, Studies in Logic and the Foundations of Mathematics, 102, North Holland, sixth impression, Amsterdam, London, New York, Tokyo, 1995.

[18] Kunen K., Tall F., *On the consistency of the non-existence of Baire irresolvable spaces*, Topology Atlas, `http://at.yorku.ca/v/a/a/a/27.htm` (1998).

[19] Kunen K., Szymansky A., Tall F., *Baire irresolvable spaces and ideal theory*, Ann. Math. Silesiana **2 (14)** (1986), 98-107.

[20] Lindgren W.F., Szymanski A.A., *A non-pseudocompact product of countably compact spaces via Seq*, Proc. Amer. Math. Soc. **125** (1997), 3741–3746.

[21] Malykhin V.I., *On extremally disconnected topological groups*, Soviet Math. Dokl. **16** (1975), 21–25.

[22] Malykhin V.I., *On the resolvability of the product of two spaces and a problem of Katětov*, Dokl. Akad. Nauk SSSR **222** (1975), 765–729.

[23] Malykhin V.I., *Irresolvable countable spaces of weight less than $\mathfrak{c}$*, Comment. Math. Univ. Carolin. **40** (1999), no. 1, 181–185.

[24] Pavlov O., *On resolvability of topological spaces*, Topology Appl. **126** (2002), 37-47.
[25] Pytke'ev E.G., *On maximally resolvable spaces*, Proc. Steklov. Inst. Math. **154** (1984), 225–230.
[26] Tamariz-Mascarúa Á., Villegas-Rodríguez H., *Spaces of continuous functions, box products and almost-ω-resolvable spaces*, Comment. Math. Univ. Carolin. **43** (2002), no. 4, 687–705.
[27] Vaughan J.E., *Two spaces homeomorphic to $Seq(p)$*, manuscript.
[28] Villegas L.M., *On resolvable spaces and groups*, Comment. Math. Univ. Carolin. **36** (1995), 579–584.
[29] Villegas L.M., *Maximal resolvability of some topological spaces*, Bol. Soc. Mat. Mexicana **5** (1999), 123–136.

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