Comment.Math.Univ.Carolin. 51,3 (2010) 453–458

A note on propagation of singularities of semiconcave functions of two variables

Luděk Zajíček

Abstract. P. Albano and P. Cannarsa proved in 1999 that, under some applicable conditions, singularities of semiconcave functions in $\mathbb{R}^n$ propagate along Lipschitz arcs. Further regularity properties of these arcs were proved by P. Cannarsa and Y. Yu in 2009. We prove that, for $n = 2$, these arcs are very regular: they can be found in the form (in a suitable Cartesian coordinate system) $\psi(x) = (x, y_1(x) - y_2(x))$, $x \in [0, \alpha]$, where y_1, y_2 are convex and Lipschitz on $[0, \alpha]$. In other words: singularities propagate along arcs with finite turn.

Keywords: semiconcave functions, singularities

Classification: Primary 26B25; Secondary 35A21

1. Introduction

Let u be a function defined on an open set $\Omega \subset \mathbb{R}^n$ which is locally (linearly) semiconcave; i.e., u is locally representable in the form $u(x) = g(x) + K\|x\|^2$, where g is concave (cf. [3]).

Let $\Sigma(u)$ be the singular set of u, i.e.

$$\Sigma(u) = \{x \in \Omega : \ u \ \text{ is not differentiable at } \ x\}.$$

It is clear that in many questions concerning $\Sigma(u)$ we can suppose that u is concave (or convex), since the results for semiconcave functions then easily follow. But it is reasonable to formulate theorems for semiconcave functions, since these functions are important in a number of applications (see [3]).

It is well-known that $\Sigma(u)$ is a rather small set: it can be covered by countably many Lipschitz DC hypersurfaces ([12]). (Note that for $A \subset \mathbb{R}^n$ there exists a convex (resp. semiconcave) function u on $\mathbb{R}^n$ such that $A = \Sigma(u)$, if and only if A is an F_σ set which can be covered by countably many Lipschitz DC hypersurfaces, see [8].)

The set $\Sigma(u)$ can have isolated points, but P. Albano and P. Cannarsa [1] found applicable conditions ensuring that $\Sigma(u)$ is in a sense big in each neighbourhood of a given $x_0 \in \Sigma(u)$. (The results of [1] can be found also in the book [3].) In particular, they proved that if $\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset$ (see Preliminaries for the

The research was supported by the grant MSM 0021620839 from the Czech Ministry of Education and by the grant GAČR 201/09/0067.

definitions), then a Lipschitz arc $\xi : [0,\tau] \to \Omega$ emanating from x_0 is a subset of the singular set $\Sigma(u)$. The results of [1] were refined in [5]; in particular it is proved in [5, Corollary 4.3] that ξ has nonzero (right continuous) right derivative at all points.

The purpose of the present note is to show that in $\mathbb{R}^2$ the results of [5] and methods from [12] and [10] easily imply that the restriction of ξ to an interval $[0,\tau']$ has an equivalent parametrization of the form (in a suitable Cartesian coordinate system) $\psi(x) = (x, y_1(x) - y_2(x))$, $x \in [0,\alpha]$, where y_1, y_2 are convex and Lipschitz on $[0,\alpha]$. (This result is equivalent to the assertion that the restriction of ξ to an interval $[0,\tau^*]$ has finite turn, cf. Remark 3.3). In particular, ξ has (left continuous) left halftangents at all points.

The question whether the results can be generalized to the case $n > 2$ remains open.

2. Preliminaries

By $B(x,r)$ we denote the open ball with center x and radius r. The scalar product of $v, w \in \mathbb{R}^n$ is denoted by $\langle v, w\rangle$. If $A \subset \mathbb{R}^n$, $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then we define the sets $A + v$ and cA by the usual way and similarly set $\langle v, A\rangle := \{\langle v, a\rangle : a \in A\}$. The boundary and the convex hull of a set $A \subset \mathbb{R}^n$ are denoted by ∂A and $\operatorname{conv} A$, respectively. The (Fréchet) derivative $Df(a)$ of a function f on $\mathbb{R}^n$ at $a \in \mathbb{R}^n$ is considered as an element of $\mathbb{R}^n$. The one-sided derivatives of a real or vector function ξ of one variable at $x \in \mathbb{R}$ are denoted by $\xi'_+(x)$ and $\xi'_-(x)$.

If f is a function defined on a subset of $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then we define the one-sided directional derivative as

$$f'_+(x,v) := \lim_{h\to 0+} \frac{f(x+hv) - f(x)}{h}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set and u a locally semiconcave function on Ω (see Introduction). Then u is locally Lipschitz and so differentiable a.e. in Ω. For $x \in \Omega$, we define (see [1] or [3, p. 54]) the set

$$D^*u(x) = \{p \in \mathbb{R}^n : \ \Omega \ni x_i \to x,\ Du(x_i) \to p\}$$

of all *reachable gradients* of u at x (note that $D^*u(x)$ is also called limiting subdifferential, cf. [1, p. 725]).

The *superdifferential* $D^+u(x)$ of u at x can be defined as the convex hull of $D^*u(x)$ (see [1, p. 723], cf. [3, Theorem 3.3.6]).

Always $D^*u(x) \subset \partial D^+u(x)$ (see [3, Proposition 3.3.4]). Note that the superdifferential $D^+u(x) = \operatorname{conv} D^*u(x)$ coincides with the Clarke's subdifferential $\partial^C u(x)$ (since $\partial^C u(x) = \operatorname{conv} D^*u(x)$, see, e.g., [4]).

Let $u(x) = g(x) + K\|x\|^2$, where g is concave, on a ball $B(x_0,\delta) \subset \Omega$. Set $f := -g$. Since $D(K\|x\|^2) = 2Kx$, we easily obtain that $D^*u(x_0) = -D^*f(x_0) + 2Kx_0$, and therefore

$$D^+u(x_0) = -\partial f(x_0) + 2Kx_0, \tag{2.1}$$

where ∂f is the classical subdifferential of the convex function f.

Recall that a function defined on an open convex subset of $\mathbb{R}^n$ is a *DC function* if it is a difference of two convex functions. We will need the following simple lemma which is a special case of the "mixing lemma" [10, Lemma 4.8].

Lemma 2.1. *Let $\varphi_1, \dots, \varphi_p$ be DC functions on $\mathbb{R}$, and let h be a continuous function on $\mathbb{R}$ such that*

$$h(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\} \quad \text{for each} \quad x \in \mathbb{R}.$$

Then h is DC on $\mathbb{R}$.

We will need also the well-known fact that convex functions are semismooth (see [7, Proposition 3], cf. also [9, Proposition 2.3]). In other words:

Lemma 2.2. *Let f be a convex function on an open convex set $C \subset \mathbb{R}^n$ and $x_0 \in C$. Let $0 \neq q \in \mathbb{R}^n$, $q_n \to q$, $t_n \searrow 0$, and $z_n \in \partial f(x_n)$, where $x_n := x_0 + t_n q_n$, be given. Then $\langle q, z_n \rangle \to f'_+(x_0, q)$. In particular,*

$$\operatorname{diam}\langle q, \partial f(x_n)\rangle \to 0. \tag{2.2}$$

3. The result and its proof

The following result is an immediate consequence of [5, Corollary 4.3].

Theorem CY. *Let u be a semiconcave function on an open set $\Omega \subset \mathbb{R}^n$, $x_0 \in \Sigma(u)$ be a singular point of u and*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset.$$

Then there exist $q \in \mathbb{R}^n$ with $\|q\| = 1$, $\tau > 0$, and a Lipschitz curve $\xi : [0, \tau] \to \Sigma(u)$ such that

(i) $\xi'_+(0) = q$,

(ii) $\lim_{s \to 0+} \xi'_+(s) = q$, *and*

(iii) $\inf_{s \in [0,\tau]} \operatorname{diam} D^+u(\xi(s)) > 0$.

Note that it is proved in [5] also that $\xi'_+(s)$ exists for each $s \in [0, \tau)$ and ξ'_+ is right continuous on $[0, \tau)$. Further note that the result without (ii) was proved already in [1].

Using Theorem CY and the method of the proof of the implicit function theorem for DC functions [10, Theorem 4.4], we easily prove the following result.

Theorem 3.1. *Let u be a semiconcave function on an open set $\Omega \subset \mathbb{R}^2$, $x_0 \in \Sigma(u)$ be a singular point of u and*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset.$$

Then there exist a Cartesian coordinate system in $\mathbb{R}^2$ given by an isometry $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A(x_0) = (0, 0)$, and convex Lipschitz functions y_1, y_2 on some $[0, \alpha]$ ($\alpha > 0$) such that, denoting $\psi(x) := (x, y_1(x) - y_2(x))$, $x \in [0, \alpha]$, we have $\psi(0) = (0, 0)$ and $A^{-1}(\psi([0, \alpha])) \subset \Sigma(u)$.

Proof: Let $\xi : [0, \tau] \to \Sigma(u)$ and $q \in \mathbb{R}^2$ have properties from Theorem CY. We will proceed in four steps. In Steps 1–3 we will suppose that

$$x_0 = (0,0) \text{ and } q = (1,0). \tag{3.1}$$

Step 1. Set $e_2 := (0,1)$. Let $u(x) = g(x) + K\|x\|^2$ for $x \in B(x_0, \delta) \subset \Omega$, where g is concave and Lipschitz with a constant $L > 0$ on $B(x_0, \delta)$. Set $f := -g$. Applying (2.1) to any point $x \in B(x_0, \delta)$, we obtain $D^+u(x) = -\partial f(x) + 2Kx$, $x \in B(x_0, \delta)$. So (iii) (of Theorem CY) easily implies that, for some $0 < \tau_1 < \tau$, we have that $f(\xi(s)) \in B(x_0, \delta)$ and $\partial f(\xi(s)) \subset B(0, L)$ for each $s \in [0, \tau_1]$, and

$$\inf_{s \in [0, \tau_1]} \operatorname{diam} \partial f(\xi(s)) > 0. \tag{3.2}$$

We will show that there exists $0 < \tau_2 < \tau_1$ such that

$$\delta := \inf_{s \in (0, \tau_2]} \operatorname{diam} \langle e_2, \partial f(\xi(s)) \rangle > 0. \tag{3.3}$$

Suppose on the contrary that there exits a sequence (t_n) such that $t_n \searrow 0$ and

$$\lim_{n \to \infty} \operatorname{diam} \langle e_2, \partial f(\xi(t_n)) \rangle = 0. \tag{3.4}$$

Set $q_n := \xi(t_n)/t_n$ and $x_n := \xi(t_n) = t_n q_n$. Since $q_n \to q$ by (i), Lemma 2.2 gives that

$$\lim_{n \to \infty} \operatorname{diam} \langle q, \partial f(\xi(t_n)) \rangle = 0. \tag{3.5}$$

Since (3.4) and (3.5) clearly imply $\lim_{n\to\infty} \operatorname{diam} \partial f(\xi(t_n)) = 0$, we obtain a contradiction with (3.2).

Step 2. Let $\xi = (\xi_1, \xi_2)$. By (ii), we have $\lim_{s \to 0+} (\xi_1)'_+(s) = 1$ and therefore there exits $0 < \tau_3 < \tau_2$ such that $1/2 \le (\xi_1)'(s)$ for a.e. $s \in (0, \tau_3)$. So ξ_1 is Lipschitz strictly increasing on $[0, \tau_3]$ and $(\xi_1)^{-1}$ is Lipschitz on $[0, \alpha]$, where $\alpha := \xi_1(\tau_3)$. Set $d(x) := \xi_2 \circ (\xi_1)^{-1}(x)$, $x \in [0, \alpha]$. Then d is Lipschitz and $\psi(x) := (x, d(x))$, $x \in [0, \alpha]$, is an equivalent parametrization of $\xi|_{[0, \tau_3]}$.

Step 3. Choose a partition $\{-L = y_0 < y_1 < \cdots < y_p = L\}$ of the interval $[-L, L]$ such that $\max\{y_i - y_{i-1},\ i = 1, \ldots, p\} < \delta/2$. For each $x \in (0, \alpha)$, the set $\langle e_2, \partial f(\psi(x)) \rangle \subset [-L, L]$ is a closed interval of length at least δ and so we can choose $i_x \in \{1, \ldots, p\}$ such that

$$y_{i_x} \in \langle e_2, \partial f(\psi(x)) \rangle \quad \text{and} \quad y_{i_x - 1} \in \langle e_2, \partial f(\psi(x)) \rangle. \tag{3.6}$$

For $i \in \{1, \ldots, p\}$, set $A_i := \{x \in (0, \alpha) : i_x = i\}$. We will show that, for each $i \in \{1, \ldots, p\}$ with $A_i \neq \emptyset$, the function $d|_{A_i}$ can be extended to a Lipschitz DC function φ_i on $\mathbb{R}$.

To this end, fix a such i and set

$$\omega_1(x) := f(x, d(x)) - y_i d(x) \quad \text{and} \quad \omega_2(x) := f(x, d(x)) - y_{i-1} d(x) \quad \text{for} \quad x \in A_i.$$

Since $\omega_1(x) - \omega_2(x) = (y_{i-1} - y_i)d(x)$, $x \in A_i$, it is sufficient to prove that ω_j $(j = 1, 2)$ can be extended to a Lipschitz convex function c_j defined on $\mathbb{R}$.

For each $x \in A_i$, choose $p_x \in \mathbb{R}$ such that $(p_x, y_i) \in \partial f(x, d(x))$ and consider the affine function

$$a_x(t) := \omega_1(x) + p_x(t - x), \quad t \in \mathbb{R}.$$

Set

$$c_1(t) := \sup\{a_x(t) : \ x \in A_i\}, \quad t \in \mathbb{R}.$$

Since ω_1 is clearly bounded on A_i and $|p_x| \leq L$ for $x \in A_i$, it is easy to see that c_1 is a Lipschitz convex function on $\mathbb{R}$.

Now consider arbitrary $x, t \in A_i$, $x \neq t$. Since $(p_x, y_i) \in \partial f(x, d(x))$, we have

$$f(t, d(t)) - f(x, d(x)) \geq p_x(t - x) + y_i(d(t) - d(x)),$$

and therefore

$$\omega_1(t) = f(t, d(t)) - y_i d(t) \geq f(x, d(x)) - y_i d(x) + p_x(t - x) = a_x(t).$$

Since $a_t(t) = \omega_1(t)$, $t \in A_i$, we obtain that c_1 extends ω_1. Quite similarly we can find a convex Lipschitz extension c_2 of ω_2.

Since $d(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\}$ for each $x \in (0, \alpha)$, and d, $\varphi_1, \dots, \varphi_p$ are continuous on $[0, \alpha]$, we can clearly find $i_0, i_\alpha \in \{1, \dots, p\}$ such that $d(0) = \varphi_{i_0}(0)$ and $d(\alpha) = \varphi_{i_\alpha}(\alpha)$.

Let h be the extension of d with $h(x) = \varphi_{i_0}(x)$, $x < 0$ and $h(x) = \varphi_{i_\alpha}(x), x > \alpha$. Then h is continuous on $\mathbb{R}$ and $h(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\}$ for each $x \in \mathbb{R}$. Thus Lemma 2.1 implies that h is DC on $\mathbb{R}$, i.e., $h = \gamma_1 - \gamma_2$, where γ_1 and γ_2 are convex on $\mathbb{R}$. Then $y_j := \gamma_j|_{[0,\alpha]}$, $j = 1, 2$, are clearly convex Lipschitz functions, and $\psi(x) = (x, y_1(x) - y_2(x))$, $x \in [0, \alpha]$.

Step 4. If (3.1) does not hold, we can choose a Cartesian system of coordinates given by an isometry $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A(x_0) = (0, 0)$ and $A(q) = (1, 0)$. Applying steps 1-3 to $u^* := u \circ A^{-1}$ and $\xi^* := A \circ \xi$, we obtain ψ of the demanded form with $\psi([0, \alpha]) \subset \Sigma(u^*) = A(\Sigma(u))$. $\square$

Remark 3.2. Well-known elementary properties of convex functions on $\mathbb{R}$ easily imply that the one-sided derivative ψ'_+ (ψ'_-) exists and is right (left) continuous on $[0, \alpha)$ $((0, \alpha])$ and has finite variation on this interval. In other words, ψ has *bounded convexity* (see [11, Theorem 3.1] or [6, Lemma 5.5]). Further, since clearly $|\psi'_+| \geq 1$, $|\psi'_-| \geq 1$ we obtain that the curve ψ has *finite turn* (see [2, Theorem 5.4.2] or [6, Theorem 5.11]). So the curve $\psi^* := A^{-1} \circ \psi$, for which $\psi^*([0, \alpha]) \subset \Sigma(u)$, has also bounded convexity and finite turn.

Remark 3.3. The proof of Theorem 3.1 and Remark 3.2 show that, for the curve $\xi : [0, \tau] \to \Sigma(u)$ from Theorem CY, there exists $0 < \tau^* < \tau$ such that $\xi|_{[0,\tau^*]}$ has finite turn. In fact, this assertion "is not weaker" than Theorem 3.1, since it implies quickly Theorem 3.1 by standard methods.

Remark 3.4. We *did not show* that the curve ξ from Theorem CY has near 0 (left-continuous) left derivative ξ'_- at all points. However, the proof of Theorem 3.1 clearly implies that ξ has (left-continuous) left half-tangent on $(0, \tau^*]$ for some $0 < \tau^* < \tau$.

We will not give detailed proofs of facts from Remarks 3.2–3.4, since they would be inadequately long, and these facts are not essential for the present short note.

References

[1] Albano P., Cannarsa P., *Structural properties of singularities of semiconcave functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 719–740.

[2] Alexandrov A.D., Reshetnyak Yu.G., *General theory of irregular curves*, Mathematics and its Applications (Soviet Series), Vol. 29., Kluwer Academic Publishers, Dordrecht, 1989.

[3] Cannarsa P., Sinestrari C., *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and their Applications, 58, Birkhäuser, Boston, 2004.

[4] Clarke F.H., *Optimization and nonsmooth analysis*, 2nd edition, Classics in Applied Mathematics, 5, SIAM, Philadelphia, 1990.

[5] Cannarsa P., Yu Y., *Singular dynamics for semiconcave functions*, J. Eur. Math. Soc. **11** (2009), 999–1024.

[6] Duda J., *Curves with finite turn*, Czechoslovak Math. J. **58** (133) (2008), 23–49.

[7] Mifflin R., *Semismooth and semiconvex functions in constrained optimization*, SIAM J. Control Optimization **15** (1977), 959–972.

[8] Pavlica D., *On the points of non-differentiability of convex functions*, Comment. Math. Univ. Carolin. **45** (2004), 727–734.

[9] Spingarn J.E., *Submonotone subdifferentials of Lipschitz functions*, Trans. Amer. Math. Soc. **264** (1981), 77–89.

[10] Veselý L., Zajíček L., *Delta-convex mappings between Banach spaces and applications*, Dissertationes Math. (Rozprawy Mat.) **289** (1989).

[11] Veselý L., Zajíček L., *On vector functions of bounded convexity*, Math. Bohemica **133** (2008), 321–335.

[12] Zajíček L., *On the differentiation of convex functions in finite and infinite dimensional spaces*, Czechoslovak Math. J. **29** (1979) 340–348.

Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail: zajicek@karlin.mff.cuni.cz

(*Received* February 12, 2010, *revised* April 12, 2010)