# Sharp generalized Trudinger inequalities via truncation for embedding into multiple exponential spaces

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*Abstract.* We prove that the generalized Trudinger inequality for Orlicz-Sobolev spaces embedded into multiple exponential spaces implies a version of an inequality due to Brézis and Wainger.

*Keywords:* Orlicz spaces, Sobolev inequalities

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## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. The classical Sobolev embedding theorem asserts that $W_0^{1,p}(\Omega)$ is continuously embedded into $L^{p^*}(\Omega)$ if $1 \leq p < n$ and $p^* = \frac{pn}{n-p}$. Further $W_0^{1,p}(\Omega)$, $p > n$, is embedded to $L^\infty(\Omega)$. Even though $p^*$ tends to infinity as $p \to n-$, there are unbounded functions in $W_0^{1,n}(\Omega)$.

A famous result by Trudinger [25] (see also [12], [22], [24] and [26]) states that the space $W_0^{1,n}(\Omega)$ is continuously embedded in the Orlicz space $\exp L^{\frac{n}{n-1}}(\Omega)$ (see Preliminaries for the definition of Orlicz spaces), i.e. there exist $C_1 = C_1(n)$ and $C_2 = C_2(n)$ such that

$$\int_\Omega \exp\Big(\Big(\frac{|u(x)|}{C_1\|\nabla u\|_{L^n(\Omega)}}\Big)^{\frac{n}{n-1}}\Big)\,dx \leq C_2\mathcal{L}_n(\Omega) \tag{1.1}$$

for every non-trivial function $u \in W_0^{1,n}(\Omega)$.

It is known (see [13], [7] and [3]) that $\exp L^{\frac{n}{n-1}}(\Omega)$ is the smallest Orlicz space with this property. However, even sharper inequalities exist in other scales. By a result of Brézis and Wainger [1] and independently Hansson [11] (see also [19] for a simple proof) we know that

$$\int_0^{\mathcal{L}_n(\Omega)} \frac{(u^*(t))^n}{\log^n(\frac{e\mathcal{L}_n(\Omega)}{t})}\frac{dt}{t} \leq C\|\nabla u\|^n_{L^n(\Omega)} \tag{1.2}$$

for every $u \in W_0^{1,n}(\Omega)$. This inequality can be also derived from capacitary estimates by Maz'ya [17]. The results in [8] and [4] tell us that this inequality gives us the smallest rearrangement invariant Banach function space $Y(\Omega)$ such that $W_0^{1,n}(\Omega)$ is continuously embedded into $Y(\Omega)$. From [1, Proof of Theorem 3(b)] one can easily see that equality (1.2) is stronger than (1.1).

Next we would like to have a version of (1.2) which is suitable for Orlicz-Sobolev spaces embedded into multiple exponential Orlicz spaces. Recall that for $s > 0$, a measure $\mu$ on $\Omega$, $f : \Omega \mapsto \mathbb{R}$ $\mu$-measurable and for $\psi : [0, \mathcal{L}_n(\Omega)] \mapsto [0,\infty)$ non-decreasing and continuous on $[0, \mathcal{L}_n(\Omega)]$, differentiable on $(0, \mathcal{L}_n(\Omega))$ and satisfying $\psi(0) = 0$, we have the following well-known identity

$$\int_0^{\mathcal{L}_n(\Omega)} (f_\mu^*(t))^s \psi'(t)\,dt = \int_0^\infty \psi\big(\mu(\{x \in \Omega : |f(x)| > r\})\big) s r^{s-1}\,dr \tag{1.3}$$

($f_\mu^*$ denotes the non-increasing rearrangement of $f$ with respect to the measure $\mu$). Using (1.3) and some easy estimates we obtain that (1.2) is equivalent to

$$\int_0^\infty \frac{t^{n-1}}{\log^{n-1}\Big(\frac{e\mathcal{L}_n(\Omega)}{\mathcal{L}_n(\{x\in\Omega: |u(x)|\geq t\})}\Big)}\,dt \leq C\|\nabla u\|^n_{L^n(\Omega)} \tag{1.4}$$

with the convention that we integrate only over $t \in (0,\infty)$ such that $\mathcal{L}_n(\{|u| \geq t\}) > 0$ (we define $\psi(t) = \log^{1-n}(\frac{e\mathcal{L}_n(\Omega)}{t})$ for $t \in (0, \mathcal{L}_n(\Omega)]$ and $\psi(0) = 0$). We use this convention throughout the paper.

When $\Omega$ is sufficiently nice, (1.1) turns to the following inequality for functions that do not have a zero trace on the boundary: there are $C_1 = C_1(n)$ and $C_2 = C_2(n)$ so that for every non-trivial $u \in W^{1,n}(\Omega)$ we have

$$\inf_{c\in\mathbb{R}} \int_\Omega \exp\Big(\Big(\frac{|u(x)-c|}{C_1\|\nabla u\|_{L^n(\Omega)}}\Big)^{\frac{n}{n-1}}\Big)\,dx \leq C_2\mathcal{L}_n(\Omega) \tag{1.5}$$

and (1.4) turns to

$$\inf_{c\in\mathbb{R}} \int_0^\infty \frac{t^{n-1}}{\log^{n-1}\Big(\frac{e\mathcal{L}_n(\Omega)}{\mathcal{L}_n(\{x\in\Omega: |u(x)-c|\geq t\})}\Big)}\,dt \leq C\|\nabla u\|^n_{L^n(\Omega)} \tag{1.6}$$

for every $u \in W^{1,n}(\Omega)$.

It is a surprising result by Koskela and Onninen [16] that if $\Omega$ is such that (1.5) is valid for every $u \in W^{1,n}(\Omega)$, then (1.6) is also valid for every $u \in W^{1,n}(\Omega)$. That is, with no additional requirement on $\Omega$ we have that the validity of the embedding (1.5) implies the validity of the sharper embedding (1.6). It is also proved in [16] that the Sobolev inequality for $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, $1 \leq p < n$, improves the same way into an inequality by O'Neil [20] and Peetre [21].

In recent paper [15], Hencl proves a version of the result from [16] for Orlicz-Sobolev spaces embedded into single and double exponential spaces.

The aim of this note is to show that the same phenomenon occurs in all Orlicz-Sobolev spaces embedded into multiple exponential Orlicz spaces.

Let us give some information concerning the spaces we are interested in. The space $W_0L^n\log^\alpha L(\Omega)$, $\alpha < n-1$, of the (first order) Sobolev type, modeled on the Zygmund space $L^n \log^\alpha L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^{\frac{n}{n-1-\alpha}})$ for large $t$. These results are due to Fusco, Lions, Sbordone [9] for $\alpha < 0$ and Edmunds, Gurka, Opic [5] in general. Moreover it is shown in [5] (see also [3] and [7]) that in the limiting case $\alpha = n-1$ we have the embedding into a double exponential space, i.e. the space $W_0 L^n \log^{n-1} L \log^\alpha \log L(\Omega)$, $\alpha < n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^{\frac{n}{n-1-\alpha}}))$ for large $t$. Further in the limiting case $\alpha = n-1$ we have the embedding into triple exponential space and so on. The borderline case is always $\alpha = n-1$ and for $\alpha > n-1$ we have the embedding into $L^\infty(\Omega)$. It is well-known that the Zygmund space $L^n \log^\alpha L(\Omega)$ coincides with the Orlicz space $L^\Phi(\Omega)$, where

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^\alpha(t)} = 1,$$

the space $L^n \log^{n-1} L \log^\alpha \log L(\Omega)$ coincides with $L^\Phi(\Omega)$ where

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^{n-1}(t) \log^\alpha(\log(t))} = 1,$$

and so on. For a further discussion about the limiting cases $\alpha = n-1$ see [6].

To simplify our notation when working with the multiple exponential spaces, let us write for $\ell \in \mathbb{N}$, $\ell \geq 2$

$$\log_{[\ell]}(t) = \log(\log_{[\ell-1]}(t))\,, \quad \text{where} \quad \log_{[1]}(t) = \log(t)$$

and

$$\exp_{[\ell]}(t) = \exp(\exp_{[\ell-1]}(t)), \quad \text{where} \quad \exp_{[1]}(t) = \exp(t).$$

Next, let us recall the version of (1.1) for embedding into multiple exponential spaces. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, let $\ell \in \mathbb{N}$, $\ell \geq 2$, let $\alpha < n-1$ and let $\Phi$ be a Young function satisfying

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \Big(\Pi_{i=1}^{\ell-1} \log_{[i]}^{n-1}(t)\Big) \log_{[\ell]}^{\alpha}(t)} = 1.$$

Then it is shown in [5] and [9] (see also [3], [14] and [2]) that there are constants $C_1$ and $C_2$ such that

$$\int_\Omega \exp_{[\ell]}\Big(\Big(\frac{|u(x)|}{C_1\|\nabla u\|_{L^\Phi(\Omega)}}\Big)^{\frac{n}{n-1-\alpha}}\Big)\,dx \leq C_2$$

for every non-trivial $u \in W_0 L^\Phi(\Omega)$.

Following [16] and [15] we state our results in the generality which can be applied in the context of analysis on metric measure spaces. In what follows $X$ is always a metric space equipped with a Borel measure $\mu$ and $\Omega$ is a measurable subset of $X$.

In the sequel we consider differentiable Young functions $\Phi$ such that

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^s\left(\Pi_{i=1}^{\ell-1}\log_{[i]}^{s-1}(t)\right)\log_{[\ell]}^{\alpha}(t)} = 1 \tag{1.7}$$

with $\ell \in \mathbb{N}$, $\ell \geq 2$, $s > 1$ and $\alpha < s-1$. We further suppose that there are $C, \delta > 0$ satisfying

$$\frac{1}{C}t^s \leq \Phi(t) \leq Ct^s \qquad \text{for } t \in [0,\delta). \tag{1.8}$$

**Theorem 1.1.** *Let $\Omega \subset X$ be a domain with $\mu(\Omega) < \infty$ and let $u, g : \Omega \to \mathbb{R}$. Fix $\ell \in \mathbb{N}$, $\ell \geq 2$, $s \in (1,\infty)$ and $\alpha \in \mathbb{R}$, $\alpha < s-1$. Set $E = \exp_{[\ell]}(1)$. Suppose that $\Phi$ is a Young function satisfying* (1.7) *and* (1.8). *Assume that the inequality*

$$\inf_{c\in\mathbb{R}} \int_\Omega \exp_{[\ell]}\Big(\Big(\frac{|u(y)-c|}{C_1\|g\|_{L^\Phi(\Omega)}}\Big)^{\frac{s}{s-1-\alpha}}\Big)\, d\mu(y) \leq C_2 \tag{1.9}$$

*is stable under truncation. Then*

$$\inf_{c\in\mathbb{R}} \int_0^\infty \frac{t^{s-1}}{\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(\{x\in\Omega: |u(x)-c|\geq t\})}\right)}\, dt < \infty\,. \tag{1.10}$$

The requirement that the inequality (1.9) is stable under truncation means that for every $d \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $z \in \{-1,1\}$ the pairs $v_{t_1}^{t_2}$, $g_{t_1}^{t_2} = g\chi_{\{t_1 < v \leq t_2\}}$, where $v = z(u-d)$ and $v_{t_1}^{t_2} = \min\{\max\{0, v-t_1\}, t_2 - t_1\}$, also satisfy (1.9):

$$\inf_{c\in\mathbb{R}} \int_\Omega \exp_{[\ell]}\Big(\Big(\frac{|v_{t_1}^{t_2}(y)-c|}{C_1\|g_{t_1}^{t_2}\|_{L^\Phi(\Omega)}}\Big)^{\frac{s}{s-1-\alpha}}\Big)\, d\mu(y) \leq C_2.$$

Notice that the function $u$ clearly satisfies the truncation property if $\Omega \subset \mathbb{R}^n$, $s = n$, $\mu = \mathcal{L}_n$ and $g = |\nabla u|$. For further applications of the powerful truncation technique which was first used in [18] we refer the reader to [17], [10] and references given there.

The validity of (1.10) is known in the Euclidean setting if we deal only with functions with zero traces (see [5], [8] and [4]). Again these spaces serve as the best rearrangement invariant target space of the embedding of $W_0L^\Phi(\Omega)$. Our approach gives a new proof of these embeddings and we have additional information if we deal with functions that do not have a zero trace on the boundary.

The paper is organized the following way. In the third section we study some properties of the functions $\exp_{[j]}$ and $\log_{[j]}$, $j \in \mathbb{N}$. The fourth section is devoted to the proof of Theorem 1.1.

## 2. Preliminaries

We denote by $\mathcal{L}_n$ the $n$-dimensional Lebesgue measure. For two functions $h, g : I \mapsto \mathbb{R}$ we write $h \sim g$ on $I$ if there is a constant $C > 1$ such that $\frac{1}{C}h(t) \leq g(t) \leq Ch(t)$ for every $t \in I$. When $I = [0, \infty)$ we simply write $h \sim g$.

A function $\Phi : [0, \infty) \mapsto [0, \infty)$ is a Young function if $\Phi(0) = 0$, $\Phi$ is increasing, convex and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. For a fixed measure $\mu$, we denote by $L^\Phi(\Omega)$ the Orlicz space corresponding to a Young function $\Phi$ on a set $\Omega$ with a measure $\mu$. This space is equipped with the Luxemburg norm

$$\|f\|_{L^\Phi(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu(x) \leq 1\Big\}.$$

For an introduction to Orlicz spaces see [23]. By $WL^\Phi(\Omega)$ we denote the set of functions $f$ such that $f, |\nabla f| \in L^\Phi(\Omega)$ and by $W_0L^\Phi(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $WL^\Phi(\Omega)$.

Let $\ell \in \mathbb{N}$, $\ell \geq 2$, $s > 1$ and $\alpha < s - 1$. Suppose that the Young function $\Phi$ satisfies (1.7) and (1.8). Let us define auxiliary functions $\varphi_1, \Phi_1 : [0, \infty) \mapsto [0, \infty)$ by

$$\varphi_1(t) = \Big(\prod_{j=1}^{\ell} \log_{[j]}^{s-1}(E + t)\Big) \log_{[j]}^{\alpha}(E + t), \quad \Phi_1(t) = t^s\varphi_1(t), \qquad t \geq 0.$$

From conditions (1.7), (1.8) we see that for any fixed $t_0 > 0$ we have

$$\Phi_1(t) \geq \frac{1}{C}t^s, \quad \Phi \sim \Phi_1, \quad \varphi_1 \sim 1 \text{ on } [0, t_0] \text{ and } \Phi_1(t) \sim t^s \text{ on } [0, t_0]. \tag{2.1}$$

We say that a function $\Phi$ satisfies the $\Delta_2$-condition if there is $C_\Delta > 0$ such that $\Phi(2t) \leq C_\Delta\Phi(t)$ for every $t \geq 0$. If $\Phi$ satisfies the $\Delta_2$-condition then (see [23, Proposition 6, p. 77])

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\Big)\, d\mu(x) = 1 \qquad \text{provided } \|f\|_{L^\Phi(\Omega)} > 0. \tag{2.2}$$

Notice that our function $\Phi$ satisfies $\Delta_2$-condition thanks to (1.7) and (1.8). And so do $\varphi_1$ and $\Phi_1$.

Let $\Psi : [0, \infty) \mapsto [0, \infty)$ be an increasing convex function and let $h : S \to \mathbb{R}$ be a non-negative function. Then we can use the following version of Jensen's inequality:

$$\frac{1}{\mu(S)} \int_S h(x)\, dx \leq \Psi^{-1}\Big(\frac{1}{\mu(S)} \int_S \Psi(h(x))\, dx\Big). \tag{2.3}$$

We also use a simple lemma from [16].

**Lemma 2.1.** *Let $\nu$ be a finite measure on a set $Y$. If $w : Y \mapsto [0,\infty)$ is a $\nu$-measurable function such that $\nu(\{y \in Y : w(y) = 0\}) \geq \frac{\nu(Y)}{2}$, then, for every $t > 0$ we have*

$$\nu(\{y \in Y : w(y) \geq t\}) \leq 2 \inf_{c\in\mathbb{R}} \nu\Big(\Big\{y \in Y : |w(y) - c| \geq \frac{t}{2}\Big\}\Big).$$

By $C$ we denote a generic positive constant that may depend on $\ell$, $s$, $\alpha$, $C_1$, $K$, $\|g\|_{L^\Phi(\Omega)}$ and $\|f\|_{L^\Phi(\Omega)}$. This constant may vary from expression to expression as usual.

## 3. Some properties of the functions $\exp_{[j]}$ and $\log_{[j]}$

**Lemma 3.1.** *Let $a, b, d \geq 1$. Then for every $j \in \mathbb{N}$, $j \leq \ell$ we have*

$$\log_{[j]}(E + ab) \leq 2\log(E + b)\log_{[j]}(E + a) \tag{3.1}$$

*and*

$$\log_{[j]}(E + a^d) \leq C \log_{[j]}(E + a). \tag{3.2}$$

Proof: Let us prove (3.1). Using the fact that for $x, y \geq 1$ we have $x + y \leq 2xy$ we obtain

$$\begin{aligned}\log(E + ab) &\leq \log(Eb + ab) = \log(b) + \log(E + a)\\ &\leq \log(E + b) + \log(E + a) \leq 2\log(E + b)\log(E + a).\end{aligned}$$

Similarly we use the inequality $2\log(E+b) \leq E+b$ and above estimate to obtain

$$\begin{aligned}\log_{[2]}(E + ab) &\leq \log\Big(2\log(E + b)\log(E + a)\Big) \leq \log\Big((E + b)\log(E + a)\Big)\\ &= \log(E + b) + \log_{[2]}(E + a) \leq 2\log(E + b)\log_{[2]}(E + a)\end{aligned}$$

and we continue by induction.

Now, let us prove (3.2). We have

$$\log(E + a^d) \leq \log((E + a)^d) = d\log(E + a)$$

and thus

$$\log_{[2]}(E + a^d) \leq \log(C\log(E + a)) = \log(C) + \log_{[2]}(E + a) \leq C\log_{[2]}(E + a).$$

We continue by induction. □

**Lemma 3.2.** *If $t \geq 0$, then*

$$t^{k_\ell} \leq \frac{\prod_{i=1}^{\ell} k_i!}{\prod_{i=1}^{\ell-1} k_i^{k_{i+1}}} \exp_{[\ell]}(t)$$

*whenever $k_i \in \mathbb{N}$, $i = 1, \dots, \ell$.*

Proof: We have $\exp(t)=\sum_{k_1=0}^{\infty}\frac{t^{k_1}}{k_1!}$,

$$\exp_{[2]}(t)=\sum_{k_1=0}^{\infty}\frac{\exp^{k_1}(t)}{k_1!}=\sum_{k_1=0}^{\infty}\frac{\exp(k_1t)}{k_1!}=\sum_{k_1,k_2=0}^{\infty}\frac{k_1^{k_2}t^{k_2}}{k_1!k_2!}$$

and by induction

$$\exp_{[\ell]}(t)=\sum_{k_1,\dots,k_\ell=0}^{\infty}\frac{\prod_{i=1}^{\ell-1}k_i^{k_{i+1}}}{\prod_{i=1}^{\ell}k_i!}t^{k_\ell}.$$

Each summand on the right hand side is estimated by $\exp_{[\ell]}(t)$ and we are done. $\square$

**Lemma 3.3.** *Suppose that $\xi,\psi>0$ satisfy*

$$\xi^{\frac{1}{k_\ell}}\le C\frac{\prod_{i=1}^{\ell}k_i^{\frac{k_i}{k_\ell}}}{\prod_{i=1}^{\ell-1}k_i^{\frac{k_{i+1}}{k_\ell}}}\psi \qquad \text{where } k_i\in\mathbb{N},\ k_i\le k_\ell\ ,\ i=1,\dots,\ell\,.$$

*Then*

$$\xi^{\frac{1}{a_\ell}}\le C\frac{\prod_{i=1}^{\ell}a_i^{\frac{a_i}{a_\ell}}}{\prod_{i=1}^{\ell-1}a_i^{\frac{a_{i+1}}{a_\ell}}}\psi \qquad \text{for every } a_i\in[1,\infty),\ a_i\le a_\ell\,,\ i=1,\dots,\ell\,.$$

Proof: First let us show that we have

$$\xi^{\frac{1}{b}}\le C\frac{\prod_{i=1}^{\ell}k_i^{\frac{k_i}{b}}}{\prod_{i=1}^{\ell-1}k_i^{\frac{k_{i+1}}{b}}}\psi \qquad \text{for every}\ \ b\in[1,\infty)\ ,\ k_i\le b+1\,,\ i=1,\dots,\ell\,. \tag{3.3}$$

Let $m\in\mathbb{N}$ be the integer part of $b$. Then by assumption we have

$$\begin{aligned}\xi^{\frac{1}{b}}&\le\max(\xi^{\frac{1}{m+1}},\xi^{\frac{1}{m}})\le C\psi\max\left(\frac{\prod_{i=1}^{\ell}k_i^{\frac{k_i}{m}}}{\prod_{i=1}^{\ell-1}k_i^{\frac{k_{i+1}}{m}}}\,,\ \frac{\prod_{i=1}^{\ell}k_i^{\frac{k_i}{m+1}}}{\prod_{i=1}^{\ell-1}k_i^{\frac{k_{i+1}}{m+1}}}\right)\\&\le C\psi\frac{\prod_{i=1}^{\ell}k_i^{\frac{k_i}{m}}}{\prod_{i=1}^{\ell-1}k_i^{\frac{k_{i+1}}{m+1}}}\,.\end{aligned} \tag{3.4}$$

Next let us prove

$$k_i^{\frac{k_i}{m}}\le Ck_i^{\frac{k_i}{b}}\ ,\ i=1,\dots,\ell \qquad \text{and} \qquad k_i^{\frac{k_{i+1}}{b}}\le Ck_i^{\frac{k_{i+1}}{m+1}}\ ,\ i=1,\dots,\ell-1. \tag{3.5}$$

The first inequality in (3.5) follows from

$$k_i^{\frac{k_i}{m}-\frac{k_i}{b}}=k_i^{\frac{k_i(b-m)}{bm}}\le k_i^{\frac{k_i}{bm}}\le(b+1)^{\frac{b+1}{bm}}\le(3m)^{\frac{3m}{m^2}}=(3m)^{\frac{3}{m}}\le C.$$

The second inequality in (3.5) is proved by

$$k_i^{\frac{k_{i+1}}{b}-\frac{k_{i+1}}{m+1}} = k_i^{\frac{k_{i+1}(m+1-b)}{b(m+1)}} \le k_i^{\frac{k_{i+1}}{bm}} \le (b+1)^{\frac{b+1}{bm}} \le (3m)^{\frac{3m}{m^2}} = (3m)^{\frac{3}{m}} \le C.$$

Now, (3.3) follows from (3.4) and (3.5).

Next, we are going to prove assertion of the lemma applying inequality (3.3) with $k_i$ being the integer parts of $a_i$, $i = 1, \dots, \ell$. For $i = 1, \dots, \ell - 1$ we observe that

$$a_i^{\frac{a_{i+1}}{b}} = \Big(\frac{a_i}{k_i}\Big)^{\frac{a_{i+1}}{b}} k_i^{\frac{a_{i+1}}{b}-\frac{k_{i+1}}{b}} k_i^{\frac{k_{i+1}}{b}} \le 2^2 k_i^{\frac{1}{b}} k_i^{\frac{k_{i+1}}{b}} \le 2^2 (2b)^{\frac{1}{b}} k_i^{\frac{k_{i+1}}{b}} \le C k_i^{\frac{k_{i+1}}{b}}.$$

Therefore

$$\text{(3.6)} \quad \frac{\prod_{i=1}^{\ell} k_i^{\frac{k_i}{b}}}{\prod_{i=1}^{\ell-1} k_i^{\frac{k_{i+1}}{b}}} \le C \frac{\prod_{i=1}^{\ell} a_i^{\frac{a_i}{b}}}{\prod_{i=1}^{\ell-1} a_i^{\frac{a_{i+1}}{b}}} \quad \text{for every } \ b \in [1, \infty),\ a_i \le b+1,\ i = 1, \dots, \ell\,.$$

Now, we set $a_\ell = b$ and (3.3) together with (3.6) conclude the proof. □

**Lemma 3.4.** *Let $\Psi$ be a non-negative increasing function satisfying $\Psi(t) \sim t\varphi_1(t)$ for $t \ge 0$. Then there is $C_\Psi > 0$ such that the inverse function $\Psi^{-1}$ satisfies on $[0, \infty)$*

$$\Psi^{-1}(t) \le C_\Psi t \Big(\prod_{j=1}^{\ell-1} \log_{[j]}^{1-s}(E+t)\Big) \log_{[\ell]}^{-\alpha}(E+t) = C_\Psi \frac{t}{\varphi_1(t)} =: \tilde{\Psi}(t).$$

Proof: First, let us prove that there is $t_1 > 0$ such that

$$\text{(3.7)} \qquad \log_{[j]}(E + t^{\frac{1}{2}}) \ge \frac{1}{2} \log_{[j]}(E+t) \quad \text{for } \ t \ge t_1,\ j \in \mathbb{N},\ j \le \ell\,.$$

For $j = 1$ it is obvious. For $j = 2$ we have

$$\log_{[2]}(E + t^{\frac{1}{2}}) \ge \log\Big(\frac{1}{2}\log(E+t)\Big) = \log_{[2]}(E+t) - \log(2) \ge \frac{1}{2}\log_{[2]}(E+t)$$

provided $t$ is large enough. And we continue by induction.

Further, we see that for $\alpha \ge 0$ there is $t_2 \ge t_1$ such that for $t \ge t_2$ we have from (3.7)

$$\text{(3.8)} \qquad \log_{[\ell]}^{\alpha}(E + \tilde{\Psi}(t)) \ge \log_{[\ell]}^{\alpha}(E + t^{\frac{1}{2}}) \ge \frac{1}{2^\alpha} \log_{[\ell]}^{\alpha}(E+t)$$

while for $\alpha < 0$ we find $t_2 \ge t_1$ so that for every $t \ge t_2$ we obtain

$$\text{(3.9)} \qquad \log_{[\ell]}^{\alpha}(E + \tilde{\Psi}(t)) \ge \log_{[\ell]}^{\alpha}(E+t) > \frac{1}{2^{|\alpha|}} \log_{[\ell]}^{\alpha}(E+t).$$

Therefore by (3.7), (3.8) and (3.9) we have for $t \geq t_2$

$$\begin{aligned}
\Psi(\tilde{\Psi}(t)) &\geq \frac{1}{C}\tilde{\Psi}(t)\varphi_1(\tilde{\Psi}(t)) \\
&= \frac{C_\Psi}{C} t\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{1-s}(E+t)\Big)\log_{[\ell]}^{-\alpha}(E+t)\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{s-1}(E+\tilde{\Psi}(t))\Big) \\
&\qquad\qquad \times \log_{[\ell]}^{\alpha}(E+\tilde{\Psi}(t)) \\
&\geq \frac{C_\Psi}{C} t\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{1-s}(E+t)\Big)\log_{[\ell]}^{-\alpha}(E+t) \\
&\qquad\qquad \times \frac{1}{2^{(s-1)(\ell-1)}}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{s-1}(E+t)\Big)\frac{1}{2^{|\alpha|}}\log_{[\ell]}^{\alpha}(E+t) \\
&\geq \frac{C_\Psi}{C} t.
\end{aligned}$$

Thus $\Psi^{-1}(t) \leq \tilde{\Psi}(t)$ on $[t_2, \infty)$ provided $C_\Psi$ is large enough. On the other hand we have $\Psi(t) \sim t$ on every bounded interval by (2.1) and thus $\Psi^{-1}(t) \sim t$ on every bounded interval. As $\frac{1}{\varphi_1}$ is bounded away from zero on any bounded interval, we have $\tilde{\Psi}(t) \sim t$ there and we are done. $\square$

## 4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Our proof is very similar to the proofs from [15] (thanks to our auxiliary lemmata from the previous section).

**Lemma 4.1.** *Suppose that the functions $f_k : \Omega \to \mathbb{R}$ have pairwise disjoint supports and that $f = \sum_{k=1}^\infty f_k \in L^\Phi(\Omega)$. We further assume that for every $k \in \mathbb{N}$ such that $\|f_k\|_{L^\Phi(\Omega)} > 0$ we have*

$$(s+2)\log\Big(\frac{1}{\|f_k\|_{L^\Phi(\Omega)}}\Big) < \log\Big(\frac{E\mu(\Omega)}{\mu(\{f_k \neq 0\})}\Big) + C. \tag{4.1}$$

*Then*

$$\sum_{k=1}^\infty \|f_k\|^s_{L^\Phi(\Omega)} < \infty\,.$$

Proof: Denote $\lambda_k = \|f_k\|_{L^\Phi(\Omega)}$. Without loss of generality we can suppose that $\lambda_k > 0$ for every $k \in \mathbb{N}$. We can further suppose that $\|f\|_{L^\Phi(\Omega)} = 1$. Indeed, otherwise we replace $f_k$ with $\frac{f_k}{\|f\|_{L^\Phi(\Omega)}}$, $k \in \mathbb{N}$, which are functions satisfying the following version of (4.1)

$$
\begin{aligned}
(s+2)\log\Big(\frac{1}{\|\frac{f_k}{\|f\|_{L^\Phi(\Omega)}}\|_{L^\Phi(\Omega)}}\Big)
&= (s+2)\log\Big(\frac{1}{\|f_k\|_{L^\Phi(\Omega)}}\Big) + (s+2)\log(\|f\|_{L^\Phi(\Omega)})\\
&\le \log\Big(\frac{E\mu(\Omega)}{\mu(\{f_k\neq 0\})}\Big) + C + (s+2)\max(0,\log(\|f\|_{L^\Phi(\Omega)}))\\
&= \log\Big(\frac{E\mu(\Omega)}{\mu(\{\frac{f_k}{\|f\|_{L^\Phi(\Omega)}}\neq 0\})}\Big) + C.
\end{aligned}
$$

Hence we have $\lambda_k \in (0,1]$, for every $k\in\mathbb{N}$. Notice that (4.1) implies

$$
(s+2)\log\Big(E+\frac{1}{\lambda_k}\Big) < \log\Big(\frac{E\mu(\Omega)}{\mu(\{f_k\neq 0\})}\Big) + C. \tag{4.2}
$$

Let $k_0\in\mathbb{N}$ be fixed (value of $k_0$ is given bellow, we need (4.8) to be satisfied). The function $\varphi_1$ is increasing for $t$ large and satisfies the $\Delta_2$-condition. Hence by (3.2) from Lemma 3.1 and the inequality $ab\le a^2+b^2$, $a,b\in\mathbb{R}$, we have

$$
\begin{aligned}
\varphi_1\Big(\frac{|f_k|}{\lambda_k}\Big) &\le C+\varphi_1\Big(|f_k|^2+\frac{1}{\lambda_k^2}\Big) \le C + C\varphi_1\Big(|f_k|^2\Big) + C\varphi_1\Big(\frac{1}{\lambda_k^2}\Big)\\
&\le C + C\varphi_1(|f_k|) + C\varphi_1\Big(\frac{1}{\lambda_k}\Big).
\end{aligned}
$$

Therefore (2.1) and (2.2) give

$$
\begin{aligned}
\sum_{k=1}^{\infty}\lambda_k^s &= \sum_{k=1}^{k_0}\lambda_k^s + \sum_{k=k_0+1}^{\infty}\int_\Omega \lambda_k^s\Phi\Big(\frac{|f_k|}{\lambda_k}\Big)\,d\mu\\
&\le \sum_{k=1}^{k_0}\|f\|^s_{L^\Phi(\Omega)} + C\sum_{k=k_0+1}^{\infty}\int_\Omega \lambda_k^s\Phi_1\Big(\frac{|f_k|}{\lambda_k}\Big)\,d\mu\\
&= C + C\sum_{k=k_0+1}^{\infty}\int_\Omega |f_k|^s\varphi_1\Big(\frac{|f_k|}{\lambda_k}\Big)\,d\mu\\
&\le C + C\Big(\sum_{k=k_0+1}^{\infty}\int_\Omega |f_k|^s\,d\mu + \sum_{k=k_0+1}^{\infty}\int_\Omega |f_k|^s\varphi_1(|f_k|)\,d\mu\\
&\qquad\qquad + \sum_{k=k_0+1}^{\infty}\int_\Omega |f_k|^s\varphi_1\Big(\frac{1}{\lambda_k}\Big)\,d\mu\Big)\\
&= C + C(S_1+S_2+S_3).
\end{aligned} \tag{4.3}
$$

Notice that we have by (2.1) and (2.2)

$$\sum_{k=1}^{\infty} \int_{\Omega} \Phi_1(|f_k|)\, d\mu = \int_{\Omega} \Phi_1(|f|)\, d\mu \le C \int_{\Omega} \Phi(|f|)\, d\mu = C \tag{4.4}$$

and

$$\sum_{k=1}^{\infty} \int_{\Omega} |f_k|^s\, d\mu \le C \sum_{k=1}^{\infty} \int_{\Omega} \Phi_1(|f_k|)\, d\mu \le C. \tag{4.5}$$

From (4.5) we obtain

$$S_1 = \sum_{k=k_0+1}^{\infty} \int_{\Omega} |f_k|^s\, d\mu \le C \tag{4.6}$$

and (4.4) implies

$$S_2 = \sum_{k=k_0+1}^{\infty} \int_{\Omega} \Phi_1(|f_k|)\, d\mu \le C. \tag{4.7}$$

It remains to estimate $S_3$. First, we claim that there is $k_0 \in \mathbb{N}$ such that

$$\log\Big(E + \frac{1}{\lambda_k}\Big) \le C \log\Big(E + \frac{1}{\mu(\{f_k \neq 0\})} \int_{\Omega} \Phi(|f_k|) d\mu\Big) \tag{4.8}$$

for every $k \ge k_0$. Let us prove this claim. From (2.2), $\lambda_k \le 1$ and inequality (3.1) from Lemma 3.1 we obtain

$$\begin{aligned}
\lambda_k^s &= \int_{\Omega} \lambda_k^s \Phi\Big(\frac{|f_k|}{\lambda_k}\Big)\, d\mu \le C \int_{\Omega} \lambda_k^s \Phi_1\Big(\frac{|f_k|}{\lambda_k}\Big)\, d\mu = C \int_{\Omega} |f_k|^s \varphi_1\Big(\frac{|f_k|}{\lambda_k}\Big)\, d\mu \\
&= \int_{\Omega} |f_k|^s \Big(\prod_{j=1}^{\ell-1} \log_{[j]}^{s-1}\Big(E + \frac{|f_k|}{\lambda_k}\Big)\Big) \log_{[\ell]}^{\alpha}\Big(E + \frac{|f_k|}{\lambda_k}\Big)\, d\mu \\
&\le C \log^{(\ell-1)(s-1)+|\alpha|}\Big(E + \frac{1}{\lambda_k}\Big) \int_{\Omega} |f_k|^s \Big(\prod_{j=1}^{\ell-1} \log_{[j]}^{s-1}(E + |f_k|\Big) \log_{[\ell]}^{\alpha}(E + |f_k|)\, d\mu \\
&\le C \frac{1}{\lambda_k} \int_{\Omega} |f_k|^s \Big(\prod_{j=1}^{\ell-1} \log_{[j]}^{s-1}(E + |f_k|\Big) \log_{[\ell]}^{\alpha}(E + |f_k|)\, d\mu \\
&= C \frac{1}{\lambda_k} \int_{\Omega} \Phi_1(|f_k|)\, d\mu \le C \frac{1}{\lambda_k} \int_{\Omega} \Phi(|f_k|)\, d\mu.
\end{aligned}$$

This implies

$$-(s+1) \log\Big(E + \frac{1}{\lambda_k}\Big) \le C + \log\Big(\int_{\Omega} \Phi(|f_k|)\, d\mu\Big).$$

Summing up this inequality and (4.2) we obtain

$$\log\Big(E+\frac{1}{\lambda_k}\Big)\le\log\Big(E+\frac{1}{\mu(\{f_k\neq 0\})}\int_\Omega\Phi(|f_k|)\,d\mu\Big)+C.$$

Therefore, since $\lambda_k\to 0$ we easily find $k_0\in\mathbb{N}$ large enough so that (4.8) is satisfied for every $k\ge k_0$.

Now, we can start estimating $S_3$. From the definition of $\varphi_1$, the fact that $\varphi_1(t)$ is increasing for large $t$ and from (4.8) we obtain

$$\varphi_1\Big(\frac{1}{\lambda_k}\Big)\le C+C\varphi_1\Big(\frac{1}{\mu(\{f_k\neq 0\})}\int_\Omega\Phi(|f_k|)\,d\mu\Big).$$

Hence

(4.9)

$$\begin{aligned}S_3&=\sum_{k=k_0+1}^{\infty}\varphi_1\Big(\frac{1}{\lambda_k}\Big)\int_\Omega|f_k|^s\,d\mu\\&\le C\sum_{k=k_0+1}^{\infty}\int_\Omega|f_k|^s\,d\mu+C\sum_{k=k_0+1}^{\infty}\varphi_1\Big(\frac{1}{\mu(\{f_k\neq 0\})}\int_\Omega\Phi(|f_k|)\,d\mu\Big)\int_\Omega|f_k|^s\,d\mu.\end{aligned}$$

Thus we need a suitable estimate of $\int_\Omega|f_k|^s\,d\mu$.

Fix an increasing convex function $\Psi:[0,\infty)\mapsto[0,\infty)$ such that $\Psi(t)\sim t\varphi_1(t)$. Therefore $\Psi$ and $\Psi^{-1}$ satisfy the $\Delta_2$-condition and $\Psi^{-1}$ can be estimated by $\tilde\Psi$ from Lemma 3.4. Thus from Jensen's inequality (2.3) for the function $h=|f_k|^s$ and $S=\{f_k\neq 0\}$ we obtain

$$\begin{aligned}\frac{1}{\mu(\{f_k\neq 0\})}\int_{\{f_k\neq 0\}}|f_k|^s\,d\mu&\le\Psi^{-1}\Big(\frac{1}{\mu(\{f_k\neq 0\})}\int_{\{f_k\neq 0\}}\Psi(|f_k|^s)\,d\mu\Big)\\&\le\Psi^{-1}\Big(\frac{1}{\mu(\{f_k\neq 0\})}\int_{\{f_k\neq 0\}}C|f_k|^s\varphi_1(|f_k|^s)\,d\mu\Big).\end{aligned}$$

Next we use the fact that $\varphi_1(t^s)\le C\varphi_1(t)$ (see (3.2)), (2.1) and Lemma 3.4

$$\begin{aligned}\frac{1}{\mu(\{f_k\neq 0\})}\int_{\{f_k\neq 0\}}|f_k|^s\,d\mu&\le\Psi^{-1}\Big(\frac{1}{\mu(\{f_k\neq 0\})}\int_{\{f_k\neq 0\}}C|f_k|^s\varphi_1(|f_k|)\,d\mu\Big)\\&=\Psi^{-1}\Big(\frac{1}{\mu(\{f_k\neq 0\})}\int_{\{f_k\neq 0\}}C\Phi_1(|f_k|)\,d\mu\Big)\\&\le\Psi^{-1}\Big(\frac{1}{\mu(\{f_k\neq 0\})}\int_{\{f_k\neq 0\}}C\Phi(|f_k|)\,d\mu\Big)\\&\le\tilde\Psi\Big(\frac{1}{\mu(\{f_k\neq 0\})}\int_{\{f_k\neq 0\}}C\Phi(|f_k|)\,d\mu\Big).\end{aligned}$$

Now, we can plainly suppose that the constant $C$ on the last line satisfies $C\ge 1$. Therefore, as $\varphi_1(t)$ is non-decreasing for large $t$ and bounded away from zero on

$[0, \infty)$, we have $\frac{1}{\varphi_1(Ct)} \leq \frac{C}{\varphi_1(t)}$ and thus $\tilde{\Psi}(Ct) \leq C\tilde{\Psi}(t)$ on $[0, \infty)$. Hence we obtain

$$\frac{1}{\mu(\{f_k \neq 0\})} \int_{\{f_k \neq 0\}} |f_k|^s \, d\mu \leq C\tilde{\Psi}\Big(\frac{1}{\mu(\{f_k \neq 0\})} \int_{\{f_k \neq 0\}} \Phi(|f_k|) \, d\mu\Big).$$

Therefore we have

$$\int_{\{f_k \neq 0\}} |f_k|^s \, d\mu \leq C \int_{\{f_k \neq 0\}} \Phi(|f_k|) \, d\mu \, \frac{1}{\varphi_1(\frac{1}{\mu(\{f_k \neq 0\})} \int_{\{f_k \neq 0\}} \Phi(|f_k|) \, d\mu)} \, .$$

This estimate, (2.1), (4.4), (4.5) and (4.9) imply

$$S_3 \leq C. \tag{4.10}$$

Now (4.3), (4.6), (4.7) and (4.10) conclude the proof. □

Proof of Theorem 1.1: Let us choose $d \in \mathbb{R}$ such that

$$\mu(\{u \geq d\}) \geq \frac{\mu(\Omega)}{2} \qquad \text{and} \qquad \mu(\{u \leq d\}) \geq \frac{\mu(\Omega)}{2} \, .$$

Set $v_+ = \max\{u - d, 0\}$ and $v_- = -\min\{u - d, 0\}$. In the sequel $v$ stands for $v_+$ and $v_-$, respectively. Our aim is to prove

$$\int_0^\infty \frac{t^{s-1}}{\log_{[\ell]}^{s-1-\alpha}\big(\frac{E\mu(\Omega)}{\mu(\{v \geq t\})}\big)} \, dt < \infty \qquad \text{for } v = v_+ \, , \; v = v_- \, . \tag{4.11}$$

First, let us show how (4.11) concludes the proof. Since $\{|u - d| \geq t\} = \{v_+ \geq t\} \cup \{v_- \geq t\}$, we have

$$\mu(\{|u - d| \geq t\}) \leq 2 \max\{\mu(\{v_+ \geq t\}), \mu(\{v_- \geq t\})\}.$$

Moreover we have for all $s \in [1, \infty)$

$$\frac{1}{\log_{[\ell]]}(Es)} \leq C \frac{1}{\log_{[\ell]]}(2Es)} \, .$$

From this estimate and (4.11) we obtain

$$\begin{aligned} &\inf_{c \in \mathbb{R}} \int_0^\infty \frac{t^{s-1}}{\log_{[\ell]}^{s-1-\alpha}\big(\frac{E\mu(\Omega)}{\mu(\{x \in \Omega : |u(x) - c| \geq t\})}\big)} \, dt \\ &\qquad \leq \int_0^\infty \frac{t^{s-1}}{\log_{[\ell]}^{s-1-\alpha}\big(\frac{E\mu(\Omega)}{\mu(\{x \in \Omega : |u(x) - d| \geq t\})}\big)} \, dt \\ &\qquad \leq C \left( \int_0^\infty \frac{t^{s-1}}{\log_{[\ell]}^{s-1-\alpha}\big(\frac{E\mu(\Omega)}{\mu(\{v_+ \geq t\})}\big)} \, dt + \int_0^\infty \frac{t^{s-1}}{\log_{[\ell]}^{s-1-\alpha}\big(\frac{E\mu(\Omega)}{\mu(\{v_- \geq t\})}\big)} \, dt \right) < \infty \end{aligned} \tag{4.12}$$

which is the assertion of the theorem.

In the rest of the proof we establish (4.11). We distinguish two cases.

If $v \in L^\infty(\Omega)$, then inequality (4.11) is obviously satisfied (recall the convention that we integrate over $t \in (0,\infty)$ such that $\mu(\{v \geq t\}) > 0$ only) and thus we are done.

Hence we can suppose that $v \notin L^\infty(\Omega)$ in the rest of the proof.

STEP 1.
Fix $0 < t_1 < t_2 < \infty$. From (1.9), the truncation property and Lemma 3.2 we have

$$\inf_{c\in\mathbb{R}}\Big(\int_\Omega |v_{t_1}^{t_2} - c|^{\frac{sk_\ell}{s-1-\alpha}}\, d\mu\Big)^{\frac{s-1-\alpha}{sk_\ell}} \leq C\Big(\frac{\prod_{i=1}^{\ell} k_i!}{\prod_{i=1}^{\ell-1} k_i^{k_{i+1}}}\Big)^{\frac{s-1-\alpha}{sk_\ell}} \|g_{t_1}^{t_2}\|_{L^\Phi(\Omega)} \tag{4.13}$$

whenever $k_i \in \mathbb{N}$, $i = 1,\dots,\ell$. From Lemma 2.1 and the weak form of (4.13) we obtain

$$\begin{aligned} t[\mu(\{v_{t_1}^{t_2} \geq t\})]^{\frac{s-1-\alpha}{sk_\ell}} &\leq C\inf_{c\in\mathbb{R}} \frac{t}{2}\Big[\mu\Big(\Big\{|v_{t_1}^{t_2} - c| \geq \frac{t}{2}\Big\}\Big)\Big]^{\frac{s-1-\alpha}{sk_\ell}} \\ &\leq C\big(\mu(\Omega)\big)^{\frac{s-1-\alpha}{sk_\ell}}\Big(\frac{\prod_{i=1}^{\ell} k_i!}{\prod_{i=1}^{\ell-1} k_i^{k_{i+1}}}\Big)^{\frac{s-1-\alpha}{sk_\ell}} \|g_{t_1}^{t_2}\|_{L^\Phi(\Omega)} \end{aligned}$$

for $k_i \in \mathbb{N}$, $i = 1,\dots,\ell$ and every $t > 0$. Since $(k!)^{\frac{1}{l}} \sim k^{\frac{k}{l}}$ if $k \leq l$, from above and from Lemma 3.3 we see that

$$t\Big(\frac{\mu(\{v_{t_1}^{t_2} \geq t\})}{E\mu(\Omega)}\Big)^{\frac{s-1-\alpha}{sa_\ell}} \leq C\left(\frac{\prod_{i=1}^{\ell} a_i^{\frac{a_i}{a_\ell}}}{\prod_{i=1}^{\ell-1} a_i^{\frac{a_{i+1}}{a_\ell}}}\right)^{\frac{s-1-\alpha}{s}} \|g_{t_1}^{t_2}\|_{L^\Phi(\Omega)} \tag{4.14}$$

for $a_i \in [1,\infty)$, $a_i \leq a_\ell$, $i = 1,\dots,\ell$ and $t > 0$.

STEP 2.
Our next step is to prove

$$\frac{2^i}{\log_{[\ell]}^{\frac{s-1-\alpha}{s}}\Big(\frac{E\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\Big)} \leq C\|g_{2^i}^{2^{i+1}}\|_{L^\Phi(\Omega)} \quad \text{whenever} \quad i \in \mathbb{N}\,. \tag{4.15}$$

Let us define $b = \frac{E\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}$. We set

$$a_i = \frac{\log(b)}{\log_{[i+1]}(b)} \quad \text{for } i = 1,\dots,\ell-1 \quad \text{and} \quad a_\ell = \log(b).$$

Hence as $t^{\frac{1}{\log(t)}} = e$, $(\frac{1}{t})^{\frac{1}{\log(t)}} = e^{-1}$, $b \geq E$ and $\lim_{t\to\infty}(\frac{1}{t})^{\frac{1}{t}} = 1$, we obtain

$$
\begin{aligned}
\frac{\prod_{i=1}^{\ell} a_i^{\frac{a_i}{a_\ell}}}{\prod_{i=1}^{\ell-1} a_i^{\frac{a_{i+1}}{a_\ell}}} &= \frac{\Big(\prod_{i=1}^{\ell-1}\big(\frac{\log(b)}{\log_{[i+1]}(b)}\big)^{\frac{1}{\log_{[i+1]}(b)}}\Big)\log(b)}{\Big(\prod_{i=1}^{\ell-2}\big(\frac{\log(b)}{\log_{[i+1]}(b)}\big)^{\frac{1}{\log_{[i+2]}(b)}}\Big)\frac{\log(b)}{\log_{[\ell]}(b)}} \\
&= \frac{\log_{[\ell]}(b)\log^{\frac{1}{\log_{[2]}(b)}}(b)\Big(\prod_{i=1}^{\ell-1}\big(\frac{1}{\log_{[i+1]}(b)}\big)^{\frac{1}{\log_{[i+1]}(b)}}\Big)}{\prod_{i=1}^{\ell-2}\big(\frac{1}{\log_{[i+1]}(b)}\big)^{\frac{1}{\log_{[i+2]}(b)}}} \\
&\sim \log_{[\ell]}(b).
\end{aligned}
\tag{4.16}
$$

Next we observe that $(\frac{1}{b})^{\frac{s-1-\alpha}{s\log(b)}} = e^{-\frac{s-1-\alpha}{s}} = C$ and $\{v_{2^i}^{2^{i+1}} \geq 2^i\} = \{v \geq 2^{i+1}\}$. Hence from (4.14) with $t = 2^i$, $t_1 = 2^i$, $t_2 = 2^{i+1}$ and (4.16) we obtain (4.15).

STEP 3.
Set $S_i = \{v \geq 2^i\}$,

$$
G = \Big\{ i \in \mathbb{N}_0 : \log_{[\ell]}\Big(\frac{E\mu(\Omega)}{\mu(S_{i+1})}\Big) < K4^{\frac{s}{s-1-\alpha}} \log_{[\ell]}\Big(\frac{E\mu(\Omega)}{\mu(S_i)}\Big)\Big\}
$$

and $B = \mathbb{N}_0 \setminus G$, where $K \geq 1$ is large enough so that $0 \in G$. Notice that $G$ and $B$ are well-defined, because $v \notin L^\infty(\Omega)$.

Lemma 2.1 implies

$$
\mu(\{v \geq 2^{i+1}\}) = \mu(\{v_{2^i}^{2^{i+1}} \geq 2^i\}) \leq 2 \inf_{c\in\mathbb{R}} \mu(\{|v_{2^i}^{2^{i+1}} - c| \geq 2^{i-1}\}).
$$

Hence we can use (1.9) and the truncation property for $t_1 = 2^i$ and $t_2 = 2^{i+1}$ to obtain

$$
\mu(\{v \geq 2^{i+1}\}) \exp_{[\ell]}\Big(\Big(\frac{2^{i-1}}{C\|g_{2^i}^{2^{i+1}}\|_{L^\Phi(\Omega)}}\Big)^{\frac{s}{s-1-\alpha}}\Big) \leq C_2.
$$

Further we observe that

$$
\{g_{2^i}^{2^{i+1}} \neq 0\} = \{g\chi_{2^i < v \leq 2^{i+1}} \neq 0\} \subset \{2^i < v\} \subset \{2^i \leq v\} = S_i.
$$

Thus for $i \in G$ we have

$$
\begin{aligned}
\frac{1}{\|g_{2^i}^{2^{i+1}}\|_{L^\Phi(\Omega)}} &\leq C \log_{[\ell]}^{\frac{s-1-\alpha}{s}}\Big(E + \frac{C}{\mu(S_{i+1})}\Big) \\
&\leq C \log_{[\ell]}^{\frac{s-1-\alpha}{s}}\Big(E + \frac{C}{\mu(S_i)}\Big) \\
&\leq C \log_{[\ell]}^{\frac{s-1-\alpha}{s}}\Big(E + \frac{C}{\mu(\{g_{2^i}^{2^{i+1}} \neq 0\})}\Big).
\end{aligned}
$$

This verifies assumption (4.1) and therefore Lemma 4.1 and (4.15) give us

$$\sum_{i\in G}\frac{2^{si}}{\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\right)}\leq C\sum_{i\in G}\|g_{2^i}^{2^{i+1}}\|^s_{L^\Phi(\Omega)}<\infty\,. \tag{4.17}$$

Next, let us suitably decompose $B$. For each $i\in G$ we define

$$B_i=\big\{j\in B: j>i \ \text{ and } \ \{i+1,i+2,\dots,j\}\subset B\big\}.$$

From the definition of $B$, simple induction and (4.17) we have

$$\begin{aligned}\sum_{j\in B}\frac{2^{sj}}{\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(\{v\geq 2^{j+1}\})}\right)}&=\sum_{i\in G}\sum_{j\in B_i}\frac{2^{sj}}{\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(S_{j+1})}\right)}\\&\leq C\sum_{i\in G}\sum_{j=i+1}^{\infty}\frac{2^{sj}}{4^{s(j-i)}\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(S_{i+1})}\right)}\\&\leq C\sum_{i\in G}\frac{2^{si}}{\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\right)}\sum_{j=i+1}^{\infty}\frac{1}{2^{s(j-i)}}<\infty.\end{aligned} \tag{4.18}$$

From (4.17) and (4.18) we obtain

$$\sum_{i=0}^{\infty}\frac{2^{si}}{\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\right)}<\infty\,. \tag{4.19}$$

STEP 4.

We raise estimate (4.15) to the power $s$ and sum over $i\in\mathbb{N}$ and we infer from (4.19)

$$\int_2^{\infty}\frac{t^{s-1}}{\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(\{v\geq t\})}\right)}\,dt\leq C\sum_{i=0}^{\infty}\frac{2^{si}}{\log_{[\ell]}^{s-1-\alpha}\left(\frac{E\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\right)}<\infty\,. \tag{4.20}$$

From (4.20) for $v=v_+$ and $v=v_-$, respectively, we obtain (4.11). Since (4.11) implies (4.12), we are done. $\square$

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