Functional separability

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Abstract. A space X is functionally countable (FC) if for every continuous $f : X \to \mathbb{R}$, $|f(X)| \leq \omega$. The class of FC spaces includes ordinals, some trees, compact scattered spaces, Lindelöf P-spaces, σ-products in 2^κ, and some L-spaces. We consider the following three versions of functional separability: X is 1-FS if it has a dense FC subspace; X is 2-FS if there is a dense subspace $Y \subset X$ such that for every continuous $f : X \to \mathbb{R}$, $|f(Y)| \leq \omega$; X is 3-FS if for every continuous $f : X \to \mathbb{R}$, there is a dense subspace $Y \subset X$ such that $|f(Y)| \leq \omega$. We give examples distinguishing 1-FS, 2-FS, and 3-FS and discuss some properties of functionally separable spaces.

Keywords: functionally countable, pseudo-$\aleph_1$-compact, DCCC, P-space, τ-simple, scattered, 1-functionally separable, 2-functionally separable, 3-functionally separable, pseudocompact, dyadic compactum, σ-centered base, LOTS

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1. Introduction

By a space we mean a Tychonoff topological space. A space is called *functionally countable* (*FC*) ([14], [17]) iff for every continuous function $f : X \to \mathbb{R}$, $f(X)$ is at most countable. It is not difficult to see that X is FC iff every second countable continuous image of X is countable (A.V. Arhangel'skii calls spaces with this property ω-*simple*, see [1], [2]). Moreover, X is FC iff every metrizable image of X is countable. All functionally countable spaces are zero dimensional and DCCC. (DCCC is the *Discrete Countable Chain Condition*. This means that every discrete family of non empty open sets is at most countable. The other name for this property is pseudo-$\aleph_1$-compactness.) The class of FC spaces includes in particular:

- all scattered compact spaces;
- all ordinals;
- all Lindelöf P-spaces (moreover, a P-space, i.e. a space in which all G_δ-sets are open, is FC iff it is DCCC ([2, 2.7.7], or attributed to A.W. Hager in [9, Proposition 3.2]));
- all trees without uncountable antichains [18];
- σ-products in 2^κ;
- some L-spaces ([11], [12]).

Functional countability plays an important role in C_p-theory ([1], [2]) and in the theory of rings of continuous functions ([3], [4], [5], [8], [14], [15], [17]).

It is natural to ask which spaces have dense FC subspaces.

Definition 1. Say that X is:
1-FS if X has a dense FC subspace;
2-FS if there is a dense $Y \subset X$ such that for every continuous $f : X \to \mathbb{R}$, $|f(Y)| \leq \omega$;
3-FS if for every continuous $f : X \to \mathbb{R}$ there is a dense $Y \subset X$ such that $|f(Y)| \leq \omega$.

We use FS as an abbreviation for *Functionally Separable.* The following implications are obvious:

$$\begin{tikzcd}
\text{FC} \arrow[dr] & & & & \\
& \text{1-FS} \arrow[r] & \text{2-FS} \arrow[r] & \text{3-FS} \arrow[r] & \text{DCCC} \\
\text{separable} \arrow[ur] & & & &
\end{tikzcd}$$

The term "functional separability" is already used in Mathematical Physics (pointed out by the referee) and in DNA sequencing. However, we do not see a more natural name for the group of properties we consider here. We believe that using the same term in so distant areas will not lead to confusion.

In the next section we give examples distinguishing 1-FS, 2-FS and 3-FS. Then we discuss some properties of functionally separable spaces.

2. Distinguishing examples

All examples are constructed as subspaces in 2^κ for appropriate κ. The following remark was made by John Kulesza after the second author's talk on the topic of [10]. It is included here with his kind permission.

Proposition 2 (J. Kulesza)**.** *Let κ be any cardinal and X a dense pseudocompact subspace in 2^κ. Then X is 3-FS.*

Proof: Let $f : X \to \mathbb{R}$ be continuous. We must find a dense $D \subset X$ such that $f(D)$ is countable. Since X is a dense subspace in the product of second countable spaces, there is a countable $C \subset \kappa$ and a continuous function $f_C : \pi_C(X) \to \mathbb{R}$ such that $f = f_C \circ \pi_C$. (Here $\pi_C : 2^\kappa \to 2^C$ is the projection.) Pick a dense countable $D_C \subset \pi_C(X)$ and put $D = \pi_C^{-1}(D_C) \cap X$. Then $f(D) = f_C(D_C)$ is countable. It remains to show that D is dense in X. Let φ be a finite partial function from κ to 2. Put $C_\varphi = C \cup \mathrm{dom}(\varphi)$. D_C is dense in $\pi_C(X)$ and thus in 2^C there is $\tilde{\varphi} : C_\varphi \to 2$ such that $\tilde{\varphi}|_{\mathrm{dom}(\varphi)} = \varphi$ and $\tilde{\varphi}|_C \in D$. Since X is dense in 2^κ and pseudocompact, there is $x_\varphi \in X$ such that $x_\varphi|_{C_\varphi} = \tilde{\varphi}$; in particular, $x_\varphi|_{\mathrm{dom}(\varphi)} = \varphi$ which proves that D is dense in 2^κ and thus in X. $\square$

Remark. The result is true if 2^κ is replaced with a product of metrizable compacta.

It is worth mentioning here the following result from [10]: Every dense pseudocompact subspace of 2^{ω_1} is 1-FS.

Proposition 3. *There is a dense pseudocompact subspace X in $2^{\mathfrak{c}^+}$ such that $|X| = \mathfrak{c}^+$, and for every uncountable $Z \subset X$ there is a countable $C \subset \mathfrak{c}^+$ such that $\pi_C(Z)$ is uncountable.*

Proof: As in the well known Reznichenko construction [16], let $Q = \bigcup\{2^B : B \subset \mathfrak{c}^+ \text{ and } |B| \leq \omega\}$, and enumerate $Q = \{q_\alpha : \mathfrak{c} \leq \alpha < \mathfrak{c}^+\}$. Let $\mathfrak{c}^+ = \bigcup\{C_\gamma : \mathfrak{c} \leq \gamma < \mathfrak{c}^+\}$ be a partition such that each C_γ is countably infinite. For each γ with $\mathfrak{c} \leq \gamma < \mathfrak{c}^+$, well enumerate the points of 2^{C_γ} as $\{y_{\gamma,\alpha} : \alpha < \gamma\}$. For $\mathfrak{c} \leq \alpha < \mathfrak{c}^+$, define $x_\alpha \in 2^{\mathfrak{c}^+}$ by

$$x_\alpha(a) = \begin{cases} q_\alpha(a) & \text{if } \ a \in \operatorname{dom}(q_\alpha), \\ y_{\gamma,\alpha}(a) & \text{if } \ a \in C_\gamma \setminus \operatorname{dom}(q_\alpha) \ \text{ and } \ \alpha \leq \gamma, \\ 0 & \text{otherwise} \end{cases}$$

(where $a < \mathfrak{c}^+$). Put $X = \{x_\alpha : \alpha < \mathfrak{c}^+\}$. It follows from the first line in the definition of the x_α's that for every countable $B \subset \mathfrak{c}^+$, $\pi_B(X) = 2^B$. Therefore, X is dense in $2^{\mathfrak{c}^+}$ and pseudocompact.

Now let $Z \subset X$ be uncountable. Pick $Z_0 \subset Z$ with $|Z_0| = \omega_1$. Put $E = \bigcup\{\operatorname{dom}(q_\alpha) : x_\alpha \in Z_0\}$ and $\alpha^* = \sup\{\alpha : x_\alpha \in Z_0\}$. Then $E \subset \mathfrak{c}^+$, $|E| = \omega_1$, and $\alpha^* < \mathfrak{c}^+$. Since there are $\mathfrak{c}^+$ many C_γ's, there is $\gamma^* < \mathfrak{c}^+$ such that $\gamma^* \geq \alpha^*$, and E does not meet C_{γ^*}. Then for all $x_\alpha \in Z_0$ and all $a \in C_{\gamma^*}$, $x_\alpha(a)$ is calculated following the second line in the definition of the x_α's. It follows that the projection of Z_0 to $2^{C_{\gamma^*}}$ is one to one and thus $\pi_{C_{\gamma^*}}(Z)$ is uncountable. □

Example 4. There is a 3-FS space which is not 2-FS.

Indeed, X from Proposition 3 is 3-FS by Proposition 2. On the other hand, if D is a dense subspace of X then D is dense in $2^{\mathfrak{c}^+}$ and thus uncountable. By Proposition 3, there is a countable $C \subset \mathfrak{c}^+$ such that $\pi_C(D)$ is uncountable. This means that D continuously maps onto an uncountable subset of a second countable space 2^C. So X is not 2-FS. □

Example 5. There is a 2-FS space which is not 1-FS.

Let $X \subset 2^{\mathfrak{c}^+}$ be from Example 4 and let S be a σ-product in $2^{\mathfrak{c}^+}$ disjoint from X. Let $L(\omega_1)$ be $\omega_1 + 1$ with the one-point Lindelöfication topology (all points other than ω_1 are isolated; a basic neighborhood of ω_1 takes the form $L(\omega_1) \setminus C$ where C is arbitrary countable subset of ω_1). Put $Y = (X \times \{\omega_1\}) \cup (S \times \omega_1)$ (considered as a subspace of $2^{\mathfrak{c}^+} \times L(\omega_1)$.)

Then Y is not 1-FS. Indeed, Let D be a dense subspace of Y. Then $\tilde{D} = D \cap (X \times \{\omega_1\})$ is at most countable (because every uncountable subset of X has uncountable projection to some countable face in $2^{\mathfrak{c}^+}$). It follows that $\tilde{D}$ is nowhere dense in $2^{\mathfrak{c}^+} \times \{\omega_1\}$. Therefore, there is a basic open set K in $2^{\mathfrak{c}^+}$

such that $K \times \{\omega_1\}$ does not meet $\tilde{D}$. Put $D_K = D \cap (K \times L(\omega_1))$. If D were functionally countable, then so would be its clopen subspace D_K. However, since $D_K \cap (2^{\mathfrak{c}^+} \times \{\omega_1\}) = \emptyset$, D_K is the discrete union of uncountably many non empty subsets $D_K \cap (2^{\mathfrak{c}^+} \times \{\alpha\})$ where $0 \le \alpha < \omega_1$. So D_K cannot be functionally countable and neither is D.

Now we show that Y is 2-FS. The subspace $D = S \times \omega_1$ is dense in Y. Let $f : Y \to \mathbb{R}$ be continuous. We will show that $f(D)$ is at most countable. Put $\tilde{Y} = Y \cup (S \times \{\omega_1\}) = (X \times \{\omega_1\}) \cup (S \times A(\omega_1))$. Since X is dense in $2^{\mathfrak{c}^+}$ and pseudocompact, every continuous function from X to $\mathbb{R}$ continuously extends to $2^{\mathfrak{c}^+}$. It follows that $f|_{X \times \{\omega_1\}}$ extends to a continuous function $f' : (X \cup S) \times \{\omega_1\} \to \mathbb{R}$. Define $\tilde{f} : \tilde{Y} \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in Y, \\ f'(x) & \text{if } x \in (X \cup S) \times \{\omega_1\}. \end{cases}$$

We claim that $\tilde{f}$ is continuous. We have to show that (*) for every $x \in \tilde{Y}$ and every $\varepsilon > 0$ there is a neighborhood U of x in $\tilde{Y}$ such that for every $y \in U$, $|f(y) - f(x)| < \varepsilon$. For $x \in X \times \{\omega_1\}$, (*) follows from the fact that both f and f' are continuous and agree at x. For $x \in S \times \omega_1$, (*) follows from the fact that $S \times \omega_1$ is open in $\tilde{Y}$ and $\tilde{f}|_{S \times \omega_1} = f|_{S \times \omega_1}$. Now consider $x \in S \times \{\omega_1\}$. Since f' is continuous at x, there is a neighborhood W of x in $(X \cup S) \times \{\omega_1\}$ such that $|f'(y) - f'(x)| < \varepsilon/3$ for all $y \in W$. For every $y = \langle y_0, \omega_1\rangle \in W \cap (X \times \omega_1)$ there are a neighborhood V_y of y_0 in $2^{\mathfrak{c}^+}$ and an ordinal $\alpha_y < \omega_1$ such that $(V_y \cap (X \cup S)) \times \{\omega_1\} \subset W$ and for every $z \in (V_y \times (\alpha_y, \omega_1)) \cap Y$, $|f(z) - f(y)| < \varepsilon/3$. Since $X \cup S$ is CCC, there is a countable $T \subset W \cap (X \times \omega_1)$ such that $(\cup_{y \in T} V_y) \cap (X \times \omega_1)$ is dense in $W \cap (X \times \omega_1)$. Put $\alpha^* = \sup_{y \in T} \alpha_y$. Then $\alpha^* < \omega_1$. Put $U = \{\langle z, \alpha\rangle \in \tilde{Y} : \langle z, \omega_1\rangle \in W \text{ and } \alpha^* < \alpha \le \omega_1\}$. Then U is a neighborhood of x in $\tilde{Y}$. It is easy to see that U satisfies (*). So $\tilde{f}$ is continuous.

Being a σ-product in $2^{\mathfrak{c}^+}$, S is a countable union of scattered compact spaces. $L(\omega_1)$ is a scattered Lindelöf space. So $S \times L(\omega_1)$ is a countable union of scattered Lindelöf spaces, and hence functionally countable [9]. So $f(D) = \tilde{f}(D) \subset \tilde{f}(S \times L(\omega_1))$ is at most countable. $\square$

For completeness, we shall give one more simple example.

Example 6. For every $\kappa \ge \omega_1$, there is a dense subspace $X \subset 2^\kappa$ which is not 3-FS.

Let $Y = \{y_\alpha : \omega \le \alpha < \omega_1\}$ be a subspace of 2^ω such that all y_α's are distinct, and every basic open set in 2^ω contains uncountably many y_α's. For each α with $\omega \le \alpha < \omega_1$, put $X_\alpha = \{x \in 2^\kappa : x(\gamma) = y_\alpha(\gamma) \text{ for } \gamma < \omega \text{ and } x(\gamma) = 0 \text{ for } \alpha < \gamma < \omega_1\}$. Put $X = \bigcup\{X_\alpha : \omega \le \alpha < \omega_1\}$. It is easy to see that X is dense in 2^κ.

We claim that X is not 3-FS. Consider the projection $\pi_\omega : X \to 2^\omega$. Then for any countable $C \subset \pi_\omega(X)$, $\pi_\omega^{-1}(C)$ is not dense in 2^κ and thus not dense in X. Indeed, put $\alpha^* = \sup\{\alpha : y_\alpha \in C\} + 1$. Then $x(\alpha^*) = 0$ for all $x \in \pi_\omega^{-1}(C)$. $\square$

3. Which spaces are functionally separable?

Here we give only some partial answers to this many-faceted question. It is not difficult to see that a σ-product in any product of countable spaces can be represented as a countable union of scattered compact spaces. Since scattered compact spaces are FC, and FC is obviously preserved by countable unions and continuous images, we get the following:

Theorem 7. *Every product of separable spaces is* 1*-FS.*

Corollary 8. *Every dyadic compactum is* 1*-FS.*

In the case $\kappa \leq \mathfrak{c}$, Proposition 2 can be generalized in the following way. Recall that a space has a σ-centered base iff it is homeomorphic to a dense subspace of a separable space; moreover, X has a σ-centered base iff βX is separable [6].

Proposition 9. *A pseudocompact space with a σ-centered base is* 3*-FS.*

Proof: Let X be a pseudocompact space with a σ-centered base, and let $f : X \to \mathbb{R}$ be continuous. Since X is pseudocompact, f is bounded and thus extends to a continuous function $\beta f : \beta X \to \mathbb{R}$. Let C be a dense countable subspace of βX. Put $Y = (\beta f)^{-1}(\beta f(C)) \cap X$. Then $Y \subset X$, and $f(Y)$ is at most countable. That Y is dense in X follows from a simple lemma which may be considered folklore. $\square$

Lemma 10. *Let X be a dense pseudocompact subspace of Z and let $g : Z \to \mathbb{R}$ be continuous. Then for every $z \in Z$, $z \in \overline{g^{-1}(g(z)) \cap X}$.*

Pseudocompactness cannot be omitted in Proposition 9: see Example 6.

Recall that by a theorem of Noble and Ulmer [13], if in the product $X = \prod\{X_\alpha : \alpha \in \mathcal{A}\}$ all finite subproducts are DCCC, then X is DCCC, and every continuous function from X to $\mathbb{R}$ depends on countably many coordinates. It is easy to see that if all finite subproducts are 3-FS then so are all countable subproducts. So we get

Proposition 11. *If in the product $X = \prod\{X_\alpha : \alpha \in \mathcal{A}\}$ all finite subproducts are* 3*-FS, then X is* 3*-FS.*

Further, since the Lindelöf P-property is finitely productive and implies FC we get

Proposition 12. *Every product of Lindelöf P-spaces is* 3*-FS.*

In contrast with the generality of Theorem 7 and other results above, it is worth noting that functional separability is not preserved by finite products. Let X be ω_1 with the order topology and Y the one-point Lindelöfication of the discrete space of cardinality ω_1. Then both X and Y are FC while the product $X \times Y$ contains an uncountable discrete space as a clopen subset (hence $X \times Y$ is not even 3-FS). This example was mentioned in American Mathematical Monthly [7] as the answer to the problem whether FC is finitely productive.

We conclude the section with two special cases: LOTS, and pseudocompact spaces.

Proposition 13. *A CCC LOTS is* 3*-FS.*

Proof: Let X be a CCC LOTS and let $f : X \to \mathbb{R}$ be continuous. Put $T = \{r \in \mathbb{R} : \mathrm{Int}(f^{-1}(\{r\})) \neq \emptyset\}$. Then $|T| \leq \omega$. Put $Y = f^{-1}(T)$. Let $\mathcal{B}$ be a countable base for $\mathbb{R}$. For every $B \in \mathcal{B}$ fix a countable family $\mathcal{U}_B$ of closed intervals such that $f^{-1}(B) = \bigcup \mathcal{U}_B$ (this is possible because CCC LOTS are hereditarily Lindelöf). So if $B \in \mathcal{B}$ and $U \in \mathcal{U}_B$, then $U = [l_U, r_U]$ for some $l_U, r_U \in X$. Put $Z = \{l_U, r_U : U \in \mathcal{U}_B, B \in \mathcal{B}\}$ and $D = Y \cup Z$. Then $|f(D)| \leq \omega$.

We claim that D is dense in X. Let $O \subset X$ be a nonempty open set. If $O \cap Y \neq \emptyset$ then we are done, so assume $O \cap Y = \emptyset$. Pick $x \in O$. Let $\{B_n : n \in \omega\} \subset \mathcal{B}$ be a decreasing base of neighborhoods of $r = f(x)$. For each n pick U_n so that $x \in U_n \in \mathcal{U}_{B_n}$. Then $H = \cap\{U_n : n \in \omega\}$ is a nowhere dense convex set, so $1 \leq |H| \leq 2$. Then either $\{l_{U_n} : n \in \omega\}$ or $\{r_{U_n} : n \in \omega\}$ is a sequence of elements of D converging to x. $\square$

Proposition 14. *Let X be pseudocompact. Then the following conditions are equivalent:*

1. *X is* 3*-FS;*
2. *βX is* 3*-FS;*
3. *every compactification of X is* 3*-FS;*
4. *some compactification of X is* 3*-FS.*

Proof: (1)⇒(2)⇒(3)⇒(4) is obvious. (4)⇒(1) follows from Lemma 10. $\square$

Not all compact spaces are functionally separable: $\beta\omega \setminus \omega$ is not 3-FS. Indeed, let f be a continuous mapping from $\beta\omega \setminus \omega$ onto $[0,1]$. Since all G_δ-sets in $\beta\omega \setminus \omega$ have nonempty interior, any dense subset $Y \subset \beta\omega \setminus \omega$ must intersect all fibers of X and thus $f(Y) = [0,1]$ is uncountable. This fact can be generalized. Recall that X is a P$'$-space if all non empty G_δ-sets in X have non empty interior.

Proposition 15. *If a P$'$-space X is* 3*-FS, then X is FC.*

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