

Berezin-Weyl quantization for Cartan motion groups

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Abstract. We construct adapted Weyl correspondences for the unitary irreducible representations of the Cartan motion group of a noncompact semisimple Lie group by using the method introduced in [B. Cahen, *Weyl quantization for semidirect products*, Differential Geom. Appl. **25** (2007), 177–190].

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1. Introduction

In [3] and [4], we introduced the notion of adapted Weyl correspondence as a generalization of the usual quantization rules [1], [15]. The present paper is part of a larger program to study adapted Weyl correspondences for semisimple Lie groups and for semidirect products.

Let G be a connected Lie group, \mathfrak{g} the Lie algebra of G and \mathfrak{g}^* the dual space of \mathfrak{g} . Let π be a unitary irreducible representation of G on a Hilbert space \mathcal{H} . Suppose that π is associated with an orbit \mathcal{O} for the coadjoint action of G on \mathfrak{g}^* by the Kostant-Kirillov method of orbits [18], [20]. In [6], we gave the following definition for the notion of adapted Weyl correspondence.

Definition 1.1. An adapted Weyl correspondence is an isomorphism W from a vector space \mathcal{A} of complex-valued smooth functions on the orbit \mathcal{O} (called symbols) onto a vector space \mathcal{B} of (not necessarily bounded) linear operators on \mathcal{H} satisfying the following properties:

- (1) the elements of \mathcal{B} preserve a fixed dense domain \mathcal{D} of \mathcal{H} ;
- (2) the constant function 1 belongs to \mathcal{A} , the identity operator I belongs to \mathcal{B} and $W(1) = I$;
- (3) $A \in \mathcal{B}$ and $B \in \mathcal{B}$ implies $AB \in \mathcal{B}$;
- (4) for each f in \mathcal{A} the complex conjugate \bar{f} of f belongs to \mathcal{A} and the adjoint of $W(f)$ is an extension of $W(\bar{f})$;
- (5) the elements of \mathcal{D} are C^∞ -vectors for the representation π , the functions \tilde{X} ($X \in \mathfrak{g}$) defined on \mathcal{O} by $\tilde{X}(\xi) = \langle \xi, X \rangle$ are in \mathcal{A} and $W(i\tilde{X})v = d\pi(X)v$ for each $X \in \mathfrak{g}$ and each $v \in \mathcal{D}$.

Adapted Weyl correspondences were obtained in various situations, see the introduction of [6]. In particular, we constructed adapted Weyl correspondences for the principal series representations of a noncompact semisimple Lie group in [3] and [7]. We also obtained adapted Weyl correspondences for the unitary irreducible representations of the semidirect product $V \rtimes K$ of the real vector space V by a Lie group K acting linearly on V in the following situations:

- (1) K is a connected noncompact semisimple Lie group and the little group associated with the representation of $V \times K$ is a maximal compact subgroup of K [6];
- (2) K is a connected compact semisimple Lie group and the little group is the centralizer of a torus of K [10].

Let us mention that adapted Weyl correspondences have various applications in Harmonic Analysis and Deformation Theory as the construction of covariant star-products on coadjoint orbits [3] and the study of contractions of Lie group unitary representations [13], [5], [8], [9]. Recently, in [11], we have studied a contraction of the principal series of a semisimple Lie group to the unitary irreducible representations of its Cartan motion group by using the deformed Weyl calculus introduced in [3].

The present paper can be considered as a sequel of [6] and [10]. Let G_0 be a connected noncompact semisimple Lie group with Lie algebra \mathfrak{g}_0 and let K be a maximal compact subgroup of G_0 . Then we have the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus V$ where \mathfrak{k} is the Lie algebra of K and V is an $\text{Ad}(K)$ -invariant subspace of \mathfrak{g}_0 . The Cartan motion group associated with the pair (G_0, K) is the semidirect product $V \rtimes K$ formed with respect to the adjoint action of K on V .

It is known for a long time that the unitary irreducible representations of $V \rtimes K$ are similar to the principal series representations of G_0 [21]. This has been illustrated by means of contractions of representations in [14] (see also [11]). Here, we exploit this similarity to construct adapted Weyl correspondences for unitary irreducible representations of $V \rtimes K$ as it was done for principal series representations of G_0 in [7]. The method is essentially the same as in [6] and the explicit computations are partially based on those of [7].

More precisely, let \mathcal{O} be a coadjoint orbit of $V \rtimes K$ which is associated with a generic unitary irreducible representation π of $V \rtimes K$. We realize π on a Hilbert space of functions on \mathbb{R}^n where $n = (1/2) \dim \mathcal{O}$ and we compute the corresponding derived representation $d\pi$ (Section 3). We dequantize $d\pi$ by using a combination of the usual Weyl calculus on \mathbb{R}^{2n} and of the Berezin calculus on the little group orbit \mathcal{O}' (Section 4). Then we obtain an explicit symplectomorphism from $\mathbb{R}^{2n} \times \mathcal{O}'$ onto a dense open subset of \mathcal{O} and an adapted Weyl correspondence on \mathcal{O} (Section 5). In the case when G_0 is a complex Lie group, we verify that the adapted Weyl correspondence coincide with that of [10].

2. Preliminaries

In this section, we introduce some general facts on noncompact semisimple Lie groups and Cartan motion groups. Our main references are [16, Chapter VI], [19, Chapter V] and [22].

Let G_0 be a connected noncompact semisimple real Lie group with finite center. Let \mathfrak{g}_0 be the Lie algebra of G_0 . We denote by β the Killing form of \mathfrak{g}_0 defined by $\beta(X, Y) = \text{Tr}(\text{ad } X \text{ ad } Y)$ for X and Y in \mathfrak{g}_0 . Let θ be a Cartan involution of \mathfrak{g}_0 and let $\mathfrak{g}_0 = \mathfrak{k} \oplus V$ be the corresponding Cartan decomposition of \mathfrak{g}_0 . Let K be the connected compact (maximal) subgroup of G_0 with Lie algebra \mathfrak{k} . Let \mathfrak{a} be a maximal abelian subalgebra of V and let M be the centralizer of \mathfrak{a} in K . Let \mathfrak{m} denote the Lie algebra of M . Let $\Delta := \Delta(\mathfrak{a}, \mathfrak{g}_0)$ be the set of restricted roots and let

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda$$

be the root space decomposition of \mathfrak{g}_0 . We fix a Weyl chamber in \mathfrak{a} and we denote by Δ^+ the corresponding set of positive roots. We set $\mathfrak{n} = \sum_{\lambda \in \Delta^+} \mathfrak{g}_\lambda$ and $\bar{\mathfrak{n}} = \sum_{\lambda \in \Delta^+} \mathfrak{g}_{-\lambda}$. Then $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$. Let A, N and \bar{N} denote the analytic subgroups of G with algebras $\mathfrak{a}, \mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively.

Recall that $\bar{N}MAN$ is a open dense subset of G . We denote by $g = \bar{n}(g)m(g)a(g)n(g)$ the decomposition of $g \in \bar{N}MAN$. Also, recall that we have the Iwasawa decomposition $G = KAN$. We denote by $g = \tilde{k}(g)\tilde{a}(g)\tilde{n}(g)$ the decomposition of $g \in G$.

The Cartan motion group associated with the pair (G_0, K) is the semidirect product $G := V \rtimes K$. The group law of G is given by

$$(v, k) \cdot (v', k') = (v + \text{Ad}(k)v', kk')$$

for v, v' in V and $k, k' \in K$. The Lie algebra \mathfrak{g} of G is the space $V \times \mathfrak{k}$ endowed with the Lie bracket

$$[(w, U), (w', U')] = ([U, w']_0 - [U', w]_0, [U, U']_0)$$

where $[\cdot, \cdot]_0$ denotes the Lie bracket of \mathfrak{g}_0 .

Recall that β is positive defined on V and negative defined on \mathfrak{k} [16, p. 184]. Then, by using β , we can identify V^* with V and \mathfrak{k}^* with \mathfrak{k} , hence $\mathfrak{g}^* \simeq V^* \times \mathfrak{k}^*$ with $V \times \mathfrak{k}$. Under this identification, the coadjoint action of G on $\mathfrak{g}^* \simeq V \times \mathfrak{k}$ is given by

$$(v, k) \cdot (w, U) = (\text{Ad}(k)w, \text{Ad}(k)U + [v, \text{Ad}(k)w]_0)$$

for v, w in V, k in K and U in \mathfrak{k} . This is a particular case of the general formula for the coadjoint action of a semidirect product, see for instance [22].

Let $p_{\mathfrak{k}}^c$ and p_V^c be the projections of \mathfrak{g}_0 on \mathfrak{k} and V associated with the decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus V$. Recall that an element ξ_1 of \mathfrak{a} is said to be regular if $\lambda(\xi_1) \neq 0$ for each $\lambda \in \Delta$. We shall need the following lemma.

Lemma 2.1. *For each regular element ξ_1 of \mathfrak{a} , the space $\text{ad } \xi_1(V)$ is the orthogonal complement of \mathfrak{m} in \mathfrak{k} .*

PROOF: For each $\lambda \in \Delta^+$, let $E_\lambda \neq 0$ be in \mathfrak{g}_λ . Note that the space $p_{\mathfrak{k}}^c(\mathfrak{n}) = p_{\mathfrak{k}}^c(\bar{\mathfrak{n}})$ is generated by the elements $E_\lambda + \theta(E_\lambda)$ and hence orthogonal to \mathfrak{m} . Now, by applying successively $p_{\mathfrak{k}}^c$ and p_V^c to the decomposition $\mathfrak{g}_0 = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} + \bar{\mathfrak{n}}$ we get the decompositions $\mathfrak{k} = \mathfrak{m} + p_{\mathfrak{k}}^c(\mathfrak{n})$ and $V = \mathfrak{a} + p_V^c(\mathfrak{n})$. This shows that $p_{\mathfrak{k}}^c(\mathfrak{n})$ is the orthogonal complement of \mathfrak{m} in \mathfrak{k} . On the other hand, since $p_V^c(\mathfrak{n})$ is generated by the elements $E_\lambda - \theta(E_\lambda)$, we see that the space $\text{ad } \xi_1(V)$ is generated by the elements

$$\text{ad } \xi_1(E_\lambda - \theta(E_\lambda)) = \lambda(\xi_1)(E_\lambda + \theta(E_\lambda))$$

where $\lambda(\xi_1) \neq 0$ for $\lambda \in \Delta$. Hence $\text{ad } \xi_1(V) = p_{\mathfrak{k}}^c(\mathfrak{n})$ is the orthogonal complement of \mathfrak{m} in \mathfrak{k} . □

The coadjoint orbits of the semidirect product of a Lie group by a vector space were described in [22]. For each $(w, U) \in \mathfrak{g}^* \simeq \mathfrak{g}$, we denote by $O(w, U)$ the orbit of (w, U) under the coadjoint action of G . The following lemma shows that, for almost all (w, U) , the orbit $O(w, U)$ is of the form $O(\xi_1, \xi_2)$ with $\xi_1 \in \mathfrak{a}$ and $\xi_2 \in \mathfrak{m}$.

Lemma 2.2. (1) *Let \mathcal{O} be a coadjoint orbit for the coadjoint action of G on $\mathfrak{g}^* \simeq \mathfrak{g}$. Then there exists an element of \mathcal{O} of the form (ξ_1, U) with $\xi_1 \in \mathfrak{a}$. Moreover, if ξ_1 is regular then there exists $\xi_2 \in \mathfrak{m}$ such that $(\xi_1, \xi_2) \in \mathcal{O}$.*
 (2) *Let ξ_1 be a regular element of \mathfrak{a} . Then M is the stabilizer of ξ_1 in K .*

PROOF: (1) Let $(w, U) \in \mathcal{O}$. For each $k \in K$ we have

$$(0, k) \cdot (w, U) = (\text{Ad}(k)w, \text{Ad}(k)U).$$

By [19, p. 120], we have $\text{Ad}(K)\mathfrak{a} = V$ and then one can choose $k \in K$ so that $\text{Ad}(k)w \in \mathfrak{a}$. We set $\xi_1 := \text{Ad}(k)w$. If we assume that ξ_1 is regular then, by Lemma 2.1, we can write $U = \xi_2 + [\xi_1, v]$ where $\xi_2 \in \mathfrak{m}$ and $v \in V$. Then $(\xi_1, U) = (v, e) \cdot (\xi_1, \xi_2)$. Hence $\mathcal{O} = O(\xi_1, \xi_2)$.

(2) By [7, Lemma 4.2], the stabilizer of ξ_1 in \mathfrak{g}_0 is MA . Then, the stabilizer of ξ_1 in K is $MA \cap K = M$. □

Let $\xi_1 \in \mathfrak{a}$ be a regular element. Denote by $O_V(\xi_1)$ the orbit of ξ_1 in V under the action of K . In the next section, we shall need the chart on $O_V(\xi_1) \simeq K/M$ which is given by the following lemma.

Lemma 2.3 ([26, Lemma 7.6.8]). *The map $\tau : y \rightarrow \text{Ad}(\tilde{k}(y))\xi_1$ is a diffeomorphism from \bar{N} onto a dense open subset of $O_V(\xi_1)$. Let us consider the action of $k \in K$ on $y \in \bar{N}$ defined by $\tau(k \cdot y) = \text{Ad}(k)\tau(y)$ or, equivalently, by $k \cdot y = \bar{n}(ky)$. Then the K -invariant measure on \bar{N} is given by $e^{-2\rho(\log \bar{a}(y))} dy$ where dy is a Haar measure on \bar{N} .*

In the rest of the paper, we fix the normalization of dy as follows. Let $(E_i)_{1 \leq i \leq n}$ be an orthonormal basis of $\bar{\mathfrak{n}}$ with respect to the scalar product defined by

$(Y, Z) := -\beta(Y, \theta(Z))$. Denote by (y_1, y_2, \dots, y_n) the coordinates of $Y \in \bar{\mathfrak{n}}$ in this basis and let $dY = dy_1 dy_2 \dots dy_n$ be the Euclidean measure on $\bar{\mathfrak{n}}$. The exponential map \exp is a diffeomorphism from $\bar{\mathfrak{n}}$ onto \bar{N} and we set $dy := (\exp^{-1})^*(dY)$.

We shall also denote by $k \cdot Y$ the action of $k \in K$ on $Y \in \bar{\mathfrak{n}}$ defined by $\exp(k \cdot Y) = k \cdot \exp(Y)$.

3. Representations

We retain the notation of Section 2. Let $\xi_1 \in \mathfrak{a}$ be a regular element and let $\xi_2 \in \mathfrak{m}$. We denote by $o(\xi_2)$ the orbit of ξ_2 under the adjoint action of M on \mathfrak{m} . Let σ be a unitary irreducible representation of M on a complex (finite-dimensional) vector space E . In the rest of the paper, we assume that σ is associated with the orbit $o(\xi_2)$ in the following sense, see [27, Section 4]. Given a maximal torus T of M with Lie algebra \mathfrak{t} and a set of positive roots in $\Delta(\mathfrak{t}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}})$, the element $i\beta(\xi_2, \cdot)$ of $i\mathfrak{t}^*$ is the highest weight of σ . Under these assumptions, the orbit $O(\xi_1, \xi_2)$ is associated with the unitarily induced representation

$$\hat{\pi} = \text{Ind}_{V \times M}^G \left(e^{i\beta(\xi_1, \cdot)} \otimes \sigma \right)$$

(see [20], [22] and [23]). By a result of Mackey, $\hat{\pi}$ is irreducible since σ is irreducible [24, p. 149]. Moreover, in the terminology of [22] and [23], the group M is called the little group and the orbit $o(\xi_2)$ the little orbit.

The representation $\hat{\pi}$ is usually realized on the space of square-integrable sections of a Hermitian vector bundle over $O_V(\xi_1)$, see [10], [22]. Following [23] and using Lemma 2.3 and the section $y \rightarrow \tilde{k}(y)$ from \bar{N} to K , we immediately obtain the realization π_0 of $\hat{\pi}$ defined by

$$(\pi_0(v, k)\psi)(y) = e^{i\beta(\text{Ad}(\tilde{k}(y))\xi_1, v)} \sigma(\tilde{k}(y)^{-1}k\tilde{k}(k^{-1} \cdot y))\psi(k^{-1} \cdot y)$$

on the Hilbert space \mathcal{H}_0 which is the completion of the space of compactly supported smooth functions $\psi : \bar{N} \rightarrow E$ with respect to the norm

$$\|\psi\|_0^2 = \int_{\bar{N}} \langle \psi(y), \psi(y) \rangle_E e^{-2\rho(\log \tilde{a}(y))} dy.$$

For the Weyl calculus, it is more convenient to realize $\hat{\pi}$ on the Hilbert space $\mathcal{H} := L^2(\bar{\mathfrak{n}}, E)$ which is the completion of the space $C_0(\bar{\mathfrak{n}}, E)$ of compactly supported smooth functions $\phi : \bar{\mathfrak{n}} \rightarrow E$ with respect to the norm

$$\|\phi\|^2 = \int_{\bar{\mathfrak{n}}} \langle \phi(Y), \phi(Y) \rangle_E dY.$$

To this end, we use the unitary isomorphism B from \mathcal{H} onto \mathcal{H}_0 defined by $B(\phi)(\exp Y) = e^{\rho(\log \tilde{a}(y))}\phi(Y)$ and we set $\pi(g) := B^{-1}\pi_0(g)B$ for $g \in G$. We immediately obtain, for $(v, k) \in G$,

$$(\pi(v, k)\phi)(Y) = e^{i\beta(\text{Ad}(\tilde{k}(y))\xi_1, v) + \rho(\log \tilde{a}(k^{-1} \cdot y) - \log \tilde{a}(y))} \sigma(m(k, y))\phi(k^{-1} \cdot Y)$$

where we have set $y = \exp Y$ and $m(k, y) := \tilde{k}(y)^{-1} k \tilde{k}(k^{-1} \cdot y) \in M$. This formula can be simplified as follows. Let $k \in K$ and $y \in \bar{N}$. Write $k^{-1}y = \bar{n}(k^{-1}y)m(k^{-1}y)a(k^{-1}y)n(k^{-1}y)$. Then $k^{-1}\tilde{k}(y) = \tilde{k}(\bar{n}(k^{-1}y))m(k^{-1}y)$. Thus $m(k, y) = m(k^{-1}y)^{-1}$. Also, we have that

$$\tilde{a}(y) = \tilde{a}(k^{-1}y) = \tilde{a}(\bar{n}(k^{-1}y))a(k^{-1}y) = \tilde{a}(k^{-1} \cdot y)a(k^{-1}y).$$

This gives

$$\begin{aligned} (\pi(v, k)\phi)(Y) &= e^{-\rho(\log a(k^{-1} \exp Y)) + i\beta(\text{Ad}(\tilde{k}(\exp Y))\xi_1, v)} \sigma(m(k^{-1} \exp Y))^{-1} \\ &\quad \times \phi(\log \bar{n}(k^{-1} \exp Y)). \end{aligned}$$

Now, we compute the derived representation $d\pi$. Let $p_{\mathfrak{a}}$, $p_{\mathfrak{m}}$ and $p_{\bar{\mathfrak{n}}}$ be the projections of \mathfrak{g}_0 onto \mathfrak{a} , \mathfrak{m} and $\bar{\mathfrak{n}}$ associated with the direct decomposition $\mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m} + \mathfrak{n} + \bar{\mathfrak{n}}$. For $X \in \bar{\mathfrak{n}}$ we denote by X^+ the right invariant vector field on \bar{N} generated by X , that is, $X^+(y) = \left. \frac{d}{dt}(\exp tX)y \right|_{t=0}$ for $y \in \bar{N}$.

Lemma 3.1 ([7]). (1) For each $X \in \bar{\mathfrak{n}}$ and each $Y \in \bar{\mathfrak{n}}$, we have

$$d \log(\exp Y) (X^+(\exp Y)) = \frac{\text{ad } Y}{e^{\text{ad } Y} - 1} (X).$$

(2) For each $X \in \mathfrak{g}_0$ and each $y \in \bar{N}$, we have

$$\begin{aligned} \left. \frac{d}{dt} a(\exp(tX)y) \right|_{t=0} &= p_{\mathfrak{a}}(\text{Ad}(y^{-1})X) \\ \left. \frac{d}{dt} m(\exp(tX)y) \right|_{t=0} &= p_{\mathfrak{m}}(\text{Ad}(y^{-1})X) \\ \left. \frac{d}{dt} \bar{n}(\exp(tX)y) \right|_{t=0} &= (\text{Ad}(y) p_{\bar{\mathfrak{n}}}(\text{Ad}(y^{-1})X))^+(y). \end{aligned}$$

From this lemma, we immediately deduce the following proposition.

Proposition 3.2. For each $(w, U) \in \mathfrak{g}$, $\phi \in C_0(\bar{\mathfrak{n}}, E)$ and $Y \in \bar{\mathfrak{n}}$, we have

$$\begin{aligned} (d\pi(w, U)\phi)(Y) &= i\beta(\text{Ad}(\tilde{k}(\exp Y))\xi_1, w)\phi(Y) \\ &\quad + \rho(p_{\mathfrak{a}}(\text{Ad}(\exp(-Y))U))\phi(Y) + d\sigma(p_{\mathfrak{m}}(\text{Ad}(\exp(-Y))U))\phi(Y) \\ &\quad - d\phi(Y) \left(\frac{\text{ad } Y}{1 - e^{-\text{ad } Y}} p_{\bar{\mathfrak{n}}}(\text{Ad}(\exp(-Y))U) \right). \end{aligned}$$

4. Dequantization

In this section, we first introduce the Berezin-Weyl calculus on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$. Recall that the Berezin calculus on $o(\xi_2)$ associates with each operator B on the finite-dimensional complex vector space E a complex-valued function $s(B)$ on the orbit $o(\xi_2)$ called the symbol of the operator B (see [2]). The following properties of the Berezin calculus are well-known, see [12], [3], [10].

Proposition 4.1. (1) *The map $B \rightarrow s(B)$ is injective.*

(2) *For each operator B on E , we have $s(B^*) = \overline{s(B)}$.*

(3) *For each operator B on E , each $m \in M$ and each $\varphi \in o(\xi_2)$, we have*

$$s(B)(\text{Ad}(m)\varphi) = s(\sigma(m)^{-1}B\sigma(m))(\varphi).$$

(4) *For $X \in \mathfrak{m}$ and $\varphi \in o(\xi_2)$, we have $s(d\sigma(X))(\varphi) = i\beta(\varphi, X)$.*

In particular, we see that the map s^{-1} is an adapted Weyl transform on $o(\xi_2)$ in the sense of Definition 1.1.

We say that a complex-valued smooth function $f : (Y, Z, \varphi) \rightarrow f(Y, Z, \varphi)$ is a symbol on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ if for each $(Y, Z) \in \bar{\mathfrak{n}}^2$ the function $\varphi \rightarrow f(Y, Z, \varphi)$ is the symbol in the Berezin calculus on $o(\xi_2)$ of an operator on E denoted by $\hat{f}(Y, Z)$. A symbol f on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ is called an \mathcal{S} -symbol if the function \hat{f} belongs to the Schwartz space of rapidly decreasing smooth functions on $\bar{\mathfrak{n}}^2$ with values in $\text{End}(E)$. The Weyl calculus for $\text{End}(E)$ -valued functions is a slight refinement of the usual Weyl calculus for complex-valued functions [17], [15]. For each \mathcal{S} -symbol on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$, we define the operator $\mathcal{W}(f)$ on the Hilbert space $L^2(\bar{\mathfrak{n}}, E)$ by

$$(4.1) \quad (\mathcal{W}(f)\phi)(Y) = (2\pi)^{-n} \int_{\bar{\mathfrak{n}}^2} e^{i\langle T, Z \rangle} \hat{f}\left(Y + \frac{1}{2}T, Z\right) \phi(Y + T) dT dZ$$

for $\phi \in C_0(\bar{\mathfrak{n}}, E)$.

It is well-known that the Weyl calculus can be extended to much larger classes of symbols (see for instance [17]). Here we only consider a class of polynomial symbols. We say that a symbol f on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ is a P -symbol if the function $\hat{f}(Y, Z)$ is polynomial in the variable Z . Let f be the P -symbol defined by $f(Y, Z, \varphi) = u(Y)Z^\alpha$ where $u \in C^\infty(\bar{\mathfrak{n}})$ and $Z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ for each multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Then, by [25], we have

$$(4.2) \quad (\mathcal{W}(f)\phi)(Y) = \left(i \frac{\partial}{\partial Z}\right)^\alpha \left(u\left(Y + \frac{1}{2}Z\right) \phi(Y + Z)\right) \Big|_{Z=0}.$$

In particular, if $f(Y, Z, \varphi) = u(Y)$ then $(\mathcal{W}(f)\phi)(Y) = u(Y)\phi(Y)$ and if $f(Y, Z, \varphi) = u(Y)z_k$ then

$$(4.3) \quad (\mathcal{W}(f)\phi)(Y) = i \left(\frac{1}{2} \partial_k u(Y) \phi(Y) + u(Y) \partial_k \phi(Y)\right)$$

where ∂_k denotes partial derivative with respect to the variable y_k .

We need the following lemma. The trace of an endomorphism u of $\bar{\mathfrak{n}}$ is denoted by $\text{Tr}_{\bar{\mathfrak{n}}} u$.

Lemma 4.2. *For $U \in \mathfrak{k}$ let $c_U : \bar{\mathfrak{n}} \rightarrow \bar{\mathfrak{n}}$ be the map defined by*

$$c_U(Y) = s(\text{ad } Y)p_{\bar{\mathfrak{n}}}(\text{Ad}(\exp(-Y))U)$$

where s is the function defined by $s(z) = \frac{e^z}{1-e^{-z}}$ for $z \neq 0$ and $s(0) = 1$. Then we have

$$\mathrm{Tr}_{\bar{n}} dc_U(Y) = -2\rho(p_{\mathfrak{a}}(\mathrm{Ad}(\exp(-Y))X)).$$

PROOF: This is a particular case of [7, Lemma 3.3]. □

Then we get the following proposition.

Proposition 4.3. *For each $(w, U) \in \mathfrak{g}$, the Berezin-Weyl symbol of the operator $-id\pi(w, U)$ is the P-symbol $f_{(w,U)}$ on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ defined by*

$$f_{(w,U)}(Y, Z, \varphi) = \beta(\mathrm{Ad}(\tilde{k}(\exp Y))\xi_1, w) + \beta(p_{\mathfrak{m}}(\mathrm{Ad}(\exp(-Y))U), \varphi) + (c_U(Y), Z).$$

PROOF: Set $c_U^k(Y) = (c_U(Y), E_k)$ for each $k = 1, 2, \dots, n$. By using (4) of Proposition 4.1 and Formula (4.3), we immediately see that the symbol of $-id\pi(w, U)$ is

$$f_{(w,U)}(Y, Z, \varphi) = -i\rho(p_{\mathfrak{a}}(\mathrm{Ad}(\exp(-Y))U)) + \beta(\mathrm{Ad}(\tilde{k}(\exp Y))\xi_1, w) + \beta(p_{\mathfrak{m}}(\mathrm{Ad}(\exp(-Y))U), \varphi) + \sum_{k=1}^n c_U^k(Y)z_k - \frac{i}{2} \sum_{k=1}^n \partial_k c_U^k(Y).$$

But by Lemma 4.2, we have

$$-\frac{i}{2} \sum_{k=1}^n \partial_k c_U^k(Y) = -\frac{i}{2} \mathrm{Tr}_{\bar{n}}(dc_U(Y)) = i\rho(p_{\mathfrak{a}}(\mathrm{Ad}(\exp(-Y))U)).$$

The result follows. □

5. Adapted Weyl correspondence

In this section, we use the dequantization procedure of Section 4 in order to obtain an explicit diffeomorphism from $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ onto the dense open subset $\tilde{O}(\xi_1, \xi_2)$ of $O(\xi_1, \xi_2)$ defined by

$$\tilde{O}(\xi_1, \xi_2) = \{(v, k) \cdot (\xi_1, \xi_2) : v \in V, k \in K \cap \bar{N}MAN\}$$

and then to construct an adapted Weyl correspondence on $O(\xi_1, \xi_2)$.

Proposition 5.1. *Let Ψ be the map from $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ to \mathfrak{g} defined by*

$$\Psi(Y, Z, \varphi) = \left(\mathrm{Ad}(\tilde{k}(\exp Y))\xi_1, p_{\mathfrak{t}}^c \left(\mathrm{Ad}(\exp Y) \left(\varphi + p_{\mathfrak{n}} \left(\frac{\mathrm{ad} Y}{e^{\mathrm{ad} Y} - 1} \theta(Z) \right) \right) \right) \right).$$

Then, for each $(w, U) \in \mathfrak{g}$, we have

$$f_{(w,U)}(Y, Z, \phi) = \langle \Psi(Y, Z, \varphi), (w, U) \rangle.$$

PROOF: We use Proposition 4.3. Note that we have $\beta(\mathfrak{a} + \mathfrak{m}, \mathfrak{n} + \bar{\mathfrak{n}}) = (0)$, $\beta(\mathfrak{n}, \mathfrak{n}) = (0)$ and $\beta(\bar{\mathfrak{n}}, \bar{\mathfrak{n}}) = (0)$. Then for $(Y, Z, \varphi) \in \bar{\mathfrak{n}}^2 \times o(\xi_2)$ and $(w, U) \in \mathfrak{g}$, we can write

$$\begin{aligned} (c_U(Y), Z) &= -\beta(c_U(Y), Z) \\ &= -\beta\left(\frac{\text{ad } Y}{1 - e^{-\text{ad } Y}} p_{\bar{\mathfrak{n}}}(\text{Ad}(\exp(-Y)) U), \theta(Z)\right) \\ &= \beta\left(p_{\bar{\mathfrak{n}}}(\text{Ad}(\exp(-Y)) U), \frac{\text{ad } Y}{e^{\text{ad } Y} - 1} \theta(Z)\right) \\ &= \beta\left(\text{Ad}(\exp(-Y)) U, p_{\mathfrak{n}}\left(\frac{\text{ad } Y}{e^{\text{ad } Y} - 1} \theta(Z)\right)\right) \\ &= \beta\left(U, p_{\mathfrak{t}}^c\left(\text{Ad}(\exp Y) p_{\mathfrak{n}}\left(\frac{\text{ad } Y}{e^{\text{ad } Y} - 1} \theta(Z)\right)\right)\right). \end{aligned}$$

Similarly, we have

$$\beta(\varphi, p_{\mathfrak{m}}(\text{Ad}(\exp(-Y))U)) = \beta(\varphi, \text{Ad}(\exp(-Y))U) = \beta(\text{Ad}(\exp Y)\varphi, U).$$

The result then follows from Proposition 4.3. □

Let ω and ω_0 be the Kirillov 2-forms on $O(\xi_1, \xi_2)$ and $o(\xi_2)$, respectively. We endow $\bar{\mathfrak{n}}^2$ with the symplectic form $dY \wedge dZ := \sum_{k=1}^n dy_k \wedge dz_k$.

Proposition 5.2. *The map Ψ is a symplectomorphism from the symplectic product $(\bar{\mathfrak{n}}^2 \times o(\xi_2), (dY \wedge dZ) \otimes \omega_0)$ onto $(\hat{O}(\xi_1, \xi_2), \omega|_{\hat{O}(\xi_1, \xi_2)})$.*

PROOF: The proof is similar to that of Proposition 6.2 in [10]. □

Now, we obtain an adapted Weyl transform on $O(\xi_1, \xi_2)$ by transferring to $O(\xi_1, \xi_2)$ the Berezin-Weyl calculus on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$. We say that a smooth function f on $O(\xi_1, \xi_2)$ is a symbol on $O(\xi_1, \xi_2)$ (respectively a P -symbol, an \mathcal{S} -symbol) if $f \circ \Psi$ is a symbol (respectively a P -symbol, an \mathcal{S} -symbol) for the Berezin-Weyl calculus on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$.

Proposition 5.3. *Let \mathcal{A} be the space of P -symbols on $O(\xi_1, \xi_2)$ and let \mathcal{B} be the space of differential operators on $\bar{\mathfrak{n}}$ with coefficients in $C^\infty(\bar{\mathfrak{n}}, E)$. Then the map $W : \mathcal{A} \rightarrow \mathcal{B}$ that assigns to each $f \in \mathcal{A}$ the operator $\mathcal{W}(f \circ \Psi)$ on $L^2(\bar{\mathfrak{n}}, E)$ is an adapted Weyl correspondence in the sense of Definition 1.1.*

PROOF: Properties (1), (2) and (3) of the definition of an adapted Weyl correspondence are clearly satisfied with $\mathcal{D} = C_0(\bar{\mathfrak{n}}, E)$. Property (4) follows from (2) of Proposition 4.1 and from the similar result for the usual Weyl calculus, see [17]. Finally, Property (5) is an immediate consequence of Proposition 4.1. □

Finally, let us consider the case when G_0 is a complex Lie group. In this case, we have $V = i\mathfrak{k}$ and M is the maximal torus $\exp(i\mathfrak{a})$ of K [19, p. 143 and p. 468].

Moreover, $o(\xi_2)$ reduces to the point ξ_2 , σ is a character of M and $E = \mathbb{C}$. So, the map \mathcal{W} is just the usual Weyl calculus.

Note that the construction of [10] can also be applied in this case. In [10], we have defined a symplectomorphism Ψ_0 from \mathfrak{n}^2 onto $\tilde{O}(\xi_1, \xi_2)$ and an adapted Weyl correspondence W_0 on $O(\xi_1, \xi_2)$. We can easily verify that $\Psi(Y, Z) = \Psi_0(\theta(Y), \theta(Z))$ for each $(Y, Z) \in \bar{\mathfrak{n}} \times \bar{\mathfrak{n}}$ and that the spaces of symbols for W and for W_0 are the same. Moreover, choosing the orthonormal basis for $\bar{\mathfrak{n}}$ in Section 2 and for \mathfrak{n} in [10] in compatible ways, we have that $W_0(f)(\phi \circ \theta) = (W(f)\phi) \circ \theta$ for each S -symbol f on $O(\xi_1, \xi_2)$ and for each $\phi \in C_0(\bar{\mathfrak{n}})$.

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