ω–weighted holomorphic Besov spaces on the unit ball in C^n

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Abstract. The ω-weighted Besov spaces of holomorphic functions on the unit ball B^n in C^n are introduced as follows. Given a function ω of regular variation and $0 < p < \infty$, a function f holomorphic in B^n is said to belong to the Besov space $B_p(\omega)$ if

$$\|f\|^p_{B_p(\omega)} = \int_{B^n} (1-|z|^2)^p |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}}\, d\nu(z) < +\infty,$$

where $d\nu(z)$ is the volume measure on B^n and D stands for the fractional derivative of f.

The holomorphic Besov space is described in the terms of the corresponding $L_p(\omega)$ space. Some projection theorems and theorems on existence of the inversions of these projections are proved. Also, explicit descriptions of the duals of the considered Besov spaces are given.

Keywords: weighted Besov spaces, unit ball, projection

Classification: 32C37, 47B38, 46T25, 46E15

1. Introduction and basic constructions

Let C^n denote the complex Euclidean space of dimension n. For any points $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$ in C^n, we define the inner product as $\langle z, \zeta\rangle = z_1\overline{\zeta}_1 + \dots + z_n\overline{\zeta}_n$ and note that $|z|^2 = |z_1|^2 + \dots + |z_n|^2$. By $B^n = \{z \in C^n,\ |z| < 1\}$ and $C^n : S^n = \{z \in C^n,\ |z| = 1\}$ we denote the open unit ball and its boundary, i.e. the unit sphere, in C^n. Further, by $H(B^n)$ we denote the set of holomorphic functions on B^n and by $H^\infty(B^n)$ the set of bounded holomorphic functions on B^n.

If $f \in H(B^n)$, then $f(z) = \sum_m a_m z^m$ $(z \in B^n)$, where the sum is taken over all multiindices $m = (m_1, \dots, m_n)$ with nonnegative integer components m_k and $z^m = z_1^{m_1} \dots z_n^{m_n}$. Assuming that $|m| = m_1 + \dots + m_n$ and putting $f_k(z) = \sum_{|m|=k} a_m z^m$ for any $k \geq 0$, one can rewrite the Taylor expansion of f as

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \tag{1}$$

which is called homogeneous expansion of f, since each f_k is a homogeneous polynomial of the degree k.

Further, for a holomorphic function f the fractional differential D^α is defined as

$$D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(z),$$
$$D^\alpha f(\overline{z}) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(\overline{z}), \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad z \in B^n.$$

We consider the inverse operator $D^{-\alpha}$ defined in the standard way:

$$D^{-\alpha} D^\alpha f(z) = f(z).$$

Particularly, if $\alpha = 1$ we set $D^1 f(z) := Df(z)$.

By $d\nu$ we denote the volume measure on B^n, normalized so that $\nu(B^n) = 1$, and by $d\sigma$ the surface measure on S^n, normalized so that $\sigma(S^n) = 1$. Then following lemma, the proof of which can be found in [12] or [15], reveals the connection between these measures.

Lemma 1. *If f is a measurable function with summable modulus over B^n, then*

$$\int_{B^n} f(z)\, d\nu(z) = 2n \int_0^1 r^{2n-1}\, dr \int_{S^n} f(r\zeta)\, d\sigma(\zeta).$$

Definition 1. By S we denote the well-known class of all non-negative measurable functions ω on $(0,1)$

$$\omega(x) = \exp\left\{ \int_x^1 \frac{\varepsilon(u)}{u}\, du \right\}, \quad x \in (0,1),$$

where $\varepsilon(u)$ is a bounded measurable function on $(0,1)$ and $-\alpha_\omega \le \varepsilon(u) \le \beta_\omega$.

Note that the functions of S are called *functions of regular variation* (see [13]). Throughout the paper, we shall assume that $\omega \in S$. Besides, for any functions f and g by $f \preceq g$ ($f \succeq g$) we shall mean that $|f(z)| \le C|g(z)|$ ($|g(z)| \le C|f(z)|$) and by $f \asymp g$ that $C_1|f(z)| \le |g(z)| \le C_2|f(z)|$ for some positive constants C, C_1, C_2 independent of z.

Proposition 1. *If $1-|z| \asymp 1-|w|$, then $\omega(1-|z|) \asymp \omega(1-|w|)$.*

Proof: Let $C_1(1-r) \le 1-|z| \le C_2(1-r)$ and $1-r = \rho$, $1-|z| = t$. Then we get

$$\omega(\rho) = \omega(t) \exp\left(\int_\rho^t \frac{\varepsilon(u)}{u}\, du \right) \le \omega(t) \exp\left(\beta_\omega \int_\rho^t \frac{du}{u} \right) = \left(\frac{t}{\rho} \right)^{\beta_\omega} \omega(t)$$

and

$$\omega(\rho) \geq \omega(t) \exp\left(-\alpha_\omega \int_\rho^t \frac{du}{u}\right) = \left(\frac{t}{\rho}\right)^{-\alpha_\omega} \omega(t)$$

which proves our statement. □

We define the holomorphic Besov spaces on the unit ball as follows.

Definition 2. Let $p > n+\beta_\omega$. Then a function $f \in H(B^n)$ is said to be in $B_p(\omega)$ if

$$M_f^p(\omega) = \int_{B^n} (1-|z|^2)^p |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}}\, d\nu(z) < +\infty.$$

We introduce the norm on $H(B^n)$ as $\|f\|_{B_p(\omega)} = M_f(\omega)$ ($|f(0)|$ needs not to be added since $Df = 0$ implies $f = 0$ for a holomorphic function f). Besides, it is easy to check that if $p > 1$, $n = 1$ and $\omega(t) = 1$, then $B_p(\omega)$ becomes the classical Besov space (see [1], [2], [8], [14]).

In particular, for $p = +\infty$ we shall write $B_\infty(\omega) = B_\omega$, where B_ω denotes the $\omega-$weighted Bloch space on the ball (see [4]).

In [9], [10], [11], one can see some other definitions and some characterizations of holomorphic Besov spaces on B^n. For a holomorphic Besov space on the polydisc of C^n, see [5], [6].

Proposition 2. *$H^\infty(B^n) \subset B_p(\omega)$ for all $0 < p < \infty$.*

PROOF: Let $f \in H^\infty(B^n)$. Then, using the Cauchy inequality in the ball $\widetilde{B}(z) = \{\zeta, |\zeta - z| < (1-|z|)/2\}$ we get $|Df(z)| \preceq (1-|z|)^{-1}$, and hence $|Df(z)|(1-|z|) \leq$ const. Thus,

$$\int_{B^n} (1-|z|^2)^p |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}}\, d\nu(z) < \infty$$

for $p > n+\beta_\omega$, and hence $f \in B_p(\omega)$. □

By $L_p(\omega)$ we denote the class of all measurable functions on B^n, for which

$$\|f\|^p_{L_p(\omega)} = \int_{B^n} |f(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}}\, d\nu(z) < +\infty$$

and we shall assume that $n + \beta_\omega < 0$. Further, by L_∞ we denote the set of all measurable functions f for which

$$\|f\|_\infty = \sup_{z \in B^n} \{|f(z)|\} < \infty.$$

Besides, we shall consider also the set of holomorphic functions f for which

$$\|f\|^p_{A^p(\omega)} = \int_{B^n} |f(z)|^p \omega(1-|z|)\, d\nu(z) < +\infty, \quad \alpha > -1,\ 0 < p < \infty.$$

The following lemmas will be used for the proof of the main results of the paper.

Lemma 2. *Let $\omega \in S$ and let $f \in B_p(\omega)$ for some $0 < p < \infty$. Then*

$$|Df(z)| \leq \frac{\|f\|_{B_p(\omega)}}{\omega^{1/p}(1-|z|)(1-|z|^2)}, \quad z \in B^n.$$

PROOF: Let $z \in B^n$, and let $B_z^n(r)$ be the disc centered at z, with the radius $r = (1-|z|)/2$. If $w \in B_z^n(r)$, then

$$|w| \leq |w-z| + |z| \leq \frac{1-|z|}{2} + |z| = \frac{1+|z|}{2} < 1.$$

Hence $B_z^n(r) \subset B^n$. Besides, the function $|Df|^p$ is subharmonic and hence

$$|Df(z)|^p \leq \frac{1}{|B_z^n(r)|} \int_{B_z^n(r)} |Df(w)|^p \, d\nu(w).$$

On the other hand, it is not difficult to show that $1-|z| \asymp 1-|w|$. Therefore, $\omega(1-|z|) \asymp \omega(1-|w|)$ by Proposition 1. Consequently,

$$\begin{aligned}
&(1-|z|^2)^p |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}} \\
&\leq \frac{1}{|B_z^n(r)|} \int_{B_z^n(r)} (1-|w|^2)^p |Df(w)|^p \frac{\omega(1-|w|)}{(1-|w|)^{n+1}} \, d\nu(w) \leq \frac{\|f\|^p_{B_p(\omega)}}{|B_z^n(r)|},
\end{aligned}$$

and

$$|Df(z)| \preceq \frac{\|f\|_{B_p(\omega)}}{\omega^{1/p}(1-|z|)(1-|z|)}$$

since $|B_z^n(r)| \asymp (1-|z|)^{n+1}$. □

Lemma 3. *Let $\omega \in S$ and let $f \in B_p(\omega)$ for some $0 < p < \infty$. Then*

$$|f(z)| \leq C(n, \omega, p) \frac{\|f\|_{B_p(\omega)}}{(1-|z|)^\gamma}, \quad z \in B^n,$$

where $\gamma = \frac{2n-1}{p}$, if $2n \geq p+1$; and $\gamma = 0$ if $2n < p+1$.

PROOF: The function $|Df(z)|^p$ is subharmonic in B^n and, hence,

$$|Df(rz)|^p \leq \int_{S^n} |Df(r\zeta)|^p P(z,\zeta) \, d\sigma(\zeta),$$

where $P(z,\zeta)$ is the Poisson kernel which satisfies the estimate

$$P(z,\zeta) = \frac{1-|z|^2}{|\zeta - z|^{2n}} \leq \frac{2}{(1-|z|)^{2n-1}}, \quad \zeta \in S^n.$$

Consequently,

$$|Df(rz)|^p \le \frac{2}{(1-|z|)^{2n-1}} \int_{S^n} |Df(r\zeta)|^p \, d\sigma(\zeta),$$

and it is clear that

$$\int_{S^n} |Df(r_1\zeta)|^p \, d\sigma(\zeta) \le \int_{S^n} |Df(r_2\zeta)|^p \, d\sigma(\zeta), \quad r_1 \le r_2.$$

Hence

$$\begin{aligned}
&\int_r^1 \frac{(1-t^2)^p \omega(1-t^2) t^{2n-1}}{(1-t^2)^{n+1}} \int_{S^n} |Df(r\zeta)|^p \, d\sigma(\zeta) \, dt \\
&\le \int_r^1 \frac{(1-t^2)^p \omega(1-t^2) t^{2n-1}}{(1-t^2)^{n+1}} \int_{S^n} |Df(t\zeta)|^p \, d\sigma(\zeta) \, dt \\
&= \int_r^1 \int_{S^n} |Df(t\zeta)|^p \, d\sigma(\zeta) \frac{(1-t^2)^p \omega(1-t^2) t^{2n-1}}{(1-t^2)^{n+1}} \, dt \\
&\preceq \int_{B^n} |Df(w)|^p \frac{(1-|w|^2)^p \omega(1-|w|^2)}{(1-|w|^2)^{n+1}} \, d\nu(w) = \|f\|_{B_p(\omega)}.
\end{aligned}$$

Consequently, the following estimate is true:

$$|Df(rz)|^p \preceq \frac{2}{(1-|z|^2)^{2n-1}} \left(\int_r^1 \frac{(1-t^2)^p \omega(1-t^2) t^{2n-1}}{(1-t^2)^{n+1}} \, dt \right)^{-1} \|f\|^p_{B_p(\omega)}.$$

Changing here $rz \mapsto z$ and putting $r = (1+2|z|)/3$, we get

$$|Df(z)| \le C(n,\omega,p) \frac{\|f\|^p_{B_p(\omega)}}{(1-|z|^2)^{\frac{2n-1}{p}}} \, .$$

Therefore

$$|f(z)| \le C(n,\omega,p) \|f\|_{B_p(\omega)} \int_0^1 \frac{dr}{(1-r|z|)^{\frac{2n-1}{p}}} \le C(n,\omega,p) \frac{\|f\|_{B_p(\omega)}}{(1-|z|)^{\gamma}} \, ,$$

where $\gamma = \frac{2n-1}{p}$, if $2n \ge p+1$; and $\gamma = 0$, if $2n < p+1$. □

Lemma 4. *Let $\omega \in S$ and let $f \in B_p(\omega)$ for some $0 < p \le 1$. Then*

$$\left(\int_{B^n} |Df(z)| \frac{\omega^{1/p}(1-|z|)}{(1-|z|)^n} \, d\nu(z) \right)^p \le \int_{B^n} |Df(z)|^p \frac{(1-|z|)^p \omega(1-|z|)}{(1-|z|)^{n+1}} \, d\nu(z).$$

Proof: Noting that $|Df(z)| = |Df(z)|^p |Df(z)|^{1-p}$ and using Lemma 2 we get

$$|Df(z)| \le |Df(z)|^p \frac{\|f\|^{1-p}_{B^p(\omega)}}{\omega^{(1-p)/p}(1-|z|)(1-|z|)^{1-p}} \, .$$

Therefore

$$|Df(z)|\frac{(1-|z|)\omega^{1/p}(1-|z|)}{(1-|z|)^{n+1}} \le |Df(z)|^p \|f\|^{1-p}_{B^p(\omega)} \frac{\omega(1-|z|)(1-|z|)^p}{(1-|z|)^{n+1}},$$

and the proof is completed by integration over B^n. □

Corollary 1. *If $0<p<1$, then $B_p(\omega)\subset B_1(\omega^*)$, where $\omega^*(t)=\omega^{1/p}(t)$.*

Lemma 5. *Let $1\le p<\infty$ and let $f\in B_p(\omega)$. Further, let $\alpha>-n/p+\beta_\omega/p$. Then*

$$\int_{B^n}(1-|z|^2)^\alpha |Df(z)|\,d\nu(z)<\infty.$$

PROOF: For $1<p<\infty$ Hölder's inequality gives

$$\begin{aligned}
&\int_{B^n}(1-|z|^2)^\alpha |Df(z)|\,d\nu(z)\\
&=\int_{B^n}(1-|z|^2)|Df(z)|\frac{\omega(1-|z|)(1-|z|^2)^{\alpha+n}}{(1-|z|^2)^{n+1}\omega(1-|z|)}\,d\nu(z)\\
&\le\left(\int_{B^n}(1-|z|^2)^p|Df(z)|^p\frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}}\,d\nu(z)\right)^{1/p}\\
&\quad\times\left(\int_{B^n}(1-|z|^2)^{\alpha q+nq-n-1}\omega^{1-q}(1-|z|)\,d\nu(z)\right)^{1/q}.
\end{aligned}$$

It is obvious that if $\alpha>-n/p-\beta_\omega/p$, then

$$\begin{aligned}
&\int_{B^n}\omega^{1-q}(1-|z|)(1-|z|)^{\alpha q+n(q-1)-1}\,d\nu(z)\\
&\preceq\int_0^1(1-r)^{\alpha q+n(q-1)-1-\beta_\omega(1-q)}\,dr<\infty.
\end{aligned}$$

Now, let $p=1$. Then, evidently,

$$\begin{aligned}
&\int_{B^n}(1-|z|^2)|Df(z)|\frac{\omega(1-|z|)(1-|z|^2)^{\alpha+n}}{(1-|z|^2)^{n+1}\omega(1-|z|)}\,d\nu(z)\\
&\le\int_{B^n}(1-|z|^2)|Df(z)|\frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}}\,d\nu(z)=\|f\|_{B_1(\omega)}.
\end{aligned}$$

□

Corollary 2. *Let $1\le p<\infty$ and let $f\in B_p(\omega)$. Further, let $\alpha>-n/p-\beta_\omega/p$. Then the function $Df(z)$ lies in the space $A^1(\alpha)$ and can be represented as*

$$Df(z)=C(n,\alpha)\int_{B^n}\frac{(1-|\zeta|^2)^\alpha Df(\zeta)}{(1-\langle z,\zeta\rangle)^{n+1+\alpha}}\,d\nu(\zeta),\quad z\in B^n, \tag{2}$$

where $C(n,\alpha)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$.

Proof: (2) is a simple consequence of the well known representation in the one-dimensional case (for details, see [3], [15]). □

Also the following auxiliary lemma will be used.

Lemma 6. *If* $0 < p < \infty$ *and* $f \in B_p(\omega)$*, then*

$$|f(z)| \preceq \int_{B^n} \frac{(1-|\zeta|^2)^\alpha}{|1-\langle z,\zeta\rangle|^{n+\alpha}} |Df(\zeta)|\, d\nu(\zeta)$$

for sufficiently great α.

Proof: Obviously, $f(z) = \int_0^1 Df(rz)\, dr$, and by Corollary 2

$$\begin{aligned} f(z) &= C(n,\alpha) \int_0^1 \int_{B^n} \frac{(1-|\zeta|^2)^\alpha Df(\zeta)}{(1-r\langle z,\zeta\rangle)^{n+1+\alpha}}\, d\nu(\zeta)\, dr \\ &= C(n,\alpha) \int_{B^n} (1-|\zeta|^2)^\alpha Df(\zeta) \int_0^1 \frac{dr}{(1-r\langle z,\zeta\rangle)^{n+1+\alpha}}\, d\nu(\zeta) \\ &= \widetilde{C}(n,\alpha) \int_{B^n} \frac{(1-|\zeta|^2)^\alpha((1-\langle z,\zeta\rangle)^{n+\alpha}-1)}{\langle z,\zeta\rangle(1-\langle z,\zeta\rangle)^{n+\alpha}} Df(\zeta)\, d\nu(\zeta). \end{aligned}$$

Hence the desired statement follows. □

To prove the other main results, we need also the following lemma.

Lemma 7. *Let* $D^m f \in B_p(\omega)$ *for some* $0 < p < +\infty$ *and* $m \in N$*. Then*

$$|f(z)| \preceq \int_{B^n} \frac{(1-|\zeta|^2)^\alpha}{|1-\langle z,\zeta\rangle|^{n+1+\alpha-m}} |D^m f(\zeta)|\, d\nu(\zeta)$$

for sufficiently great α.

Lemma 8. *For any numbers* $\alpha \in \mathbb{N}$ *and* $\beta > 0$

$$D^\alpha\left(\frac{1}{(1-\langle z,\zeta\rangle)^\beta}\right) \asymp \frac{1}{(1-\langle z,\zeta\rangle)^{\beta+\alpha}}, \quad z,\zeta \in B^n.$$

Proof: One can see that $D^\alpha f(z) = D^{\alpha-1} Df(z)$ and $Df(z) = Rf(z) + f(z)$, where

$$Rf(z) = \sum_{k=1}^n z_k \frac{\partial f(z)}{\partial z_k}.$$

On the other hand, $R(1-\langle z,\zeta\rangle)^{-\alpha} = \alpha\langle z,\zeta\rangle(1-\langle z,\zeta\rangle)^{-\alpha-1}$ and we get the proof of the lemma. □

2. Description of the spaces $B_p(\omega)$

The following theorem is one of the main results of the paper.

Theorem 1. *For any $0 < p < \infty$, the space $B_p(\omega)$ is a closed subspace of $L_p(\omega)$.*

PROOF: First, we show that if $f \in B_p(\omega)$, then $f \in L_p(\omega) \cap H(B^n)$. Indeed, if $1 < p < \infty$ and $\gamma > 0$, then by Lemma 4 we obtain

$$\begin{aligned} |f(z)|^p &\le \left(\int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-\gamma-1}\, d\nu(\zeta)}{|1-\langle z,\zeta\rangle|^{\alpha+n}}\right)^{p/q} \\ &\quad\times \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-\gamma-1}(1-|\zeta|^2)^{p+\gamma p}}{|1-\langle z,\zeta\rangle|^{\alpha+n}} |Df(\zeta)|\, d\nu(\zeta) \\ &\preceq (1-|z|^2)^{-\gamma p/q} \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-\gamma-1}(1-|\zeta|^2)^{p+\gamma p}}{|1-\langle z,\zeta\rangle|^{\alpha+n}} |Df(\zeta)|\, d\nu(\zeta). \end{aligned}$$

Hence

$$\begin{aligned} &\|f\|_{L_p(\omega)} \\ &\preceq \int_{B^n} |Df(\zeta)|^p (1-|\zeta|^2)^{(n+1)(p-1)} \int_{B^n} \frac{\omega(1-|z|)\, d\nu(z)\, d\nu(\zeta)}{|1-\langle z,\zeta\rangle|^{(\alpha+n)p}(1-|z|^2)^{n+1}} \\ &\preceq \int_{B^n} |Df(\zeta)|^p (1-|\zeta|^2)^p \frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}}\, d\nu(\zeta) = \|f\|_{B_p(\omega)}. \end{aligned}$$

If $0 < p \le 1$, then by Lemma 4

$$|f(z)|^p \preceq \int_{B^n} \frac{|Df(\zeta)|^p (1-|\zeta|)^{pn+\alpha p+p}}{|1-\langle z,\zeta\rangle|^{(\alpha+n)p}(1-|\zeta|^2)^{n+1}}\, d\nu(\zeta).$$

Consequently,

$$\begin{aligned} &\|f\|_{L_p(\omega)} \\ &\preceq \int_{B^n} |Df(\zeta)|^p (1-|\zeta|^2)^{(n+1)(p-1)} \int_{B^n} \frac{\omega(1-|z|)\, d\nu(z)\, d\nu(\zeta)}{|1-\langle z,\zeta\rangle|^{(\alpha+n)p}(1-|z|^2)^{n+1}} \\ &\preceq \|f\|_{B_p(\omega)} \end{aligned}$$

by [15, Theorem 1.12] and [7, Lemma 1.6].

Next, we show that if $f \in L_p(\omega) \cap H(B^n)$, then $f \in B_p(\omega)$. Indeed, using (2) we obtain

$$|Df(z)|^p \preceq \left(\int_{B^n} \frac{(1-|\zeta|^2)^m |f(\zeta)|}{|1-\langle z,\zeta\rangle|^{m+n+2}}\, d\nu(\zeta)\right)^p$$

for sufficiently great m. If $1 < p < \infty$, then by Hölder's inequality

$$\|f\|_{B_p(\omega)} \preceq \int_{B^n} (1-|\zeta|^2)^m |f(\zeta)|^p \int_{B^n} \frac{\omega(1-|z|)\, d\nu(z)}{|1-\langle z,\zeta\rangle|^{m+n+2}} \preceq \|f\|_{L_p(\omega)}.$$

If $0 < p \le 1$, then by Lemma 4 we obtain

$$|Df(z)|^p \preceq \int_{B^n} \frac{|f(\zeta)|^p (1-|\zeta|)^{p(n+1)+mp}}{|1-\langle z,\zeta\rangle|^{(m+n+2)p}(1-|\zeta|^2)^{n+1}}\,d\nu(\zeta),$$

and hence

$$\begin{aligned}\|f\|_{B_p(\omega)} &\preceq \int_{B^n}(1-|\zeta|^2)^{(n+1)(p-1)+mp}|f(\zeta)| \\ &\quad\times \int_{B^n}\frac{\omega(1-|z|)(1-|z|^2)^p\,d\nu(z)}{|1-\langle z,\zeta\rangle|^{(m+n+2)p}(1-|z|^2)^{n+1}}\,d\nu(\zeta) \\ &\preceq \int_{B^n}|f(\zeta)|^p\frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}}\,d\nu(\zeta) = \|f\|_{L_p(\omega)}.\end{aligned}$$

Now, we shall show that for any sequence $\{f_k\} \subset B_p(\omega)$ such that $\|f_k - f\|_{L_p(\omega)} \to 0$ as $k \to \infty$, the limit function $f \in L_p(\omega)$ is holomorphic in B^n. To this end, suppose that K is a compact set in B^n. Then by Lemma 3 there exists a constant $C(K,n,\omega,p)$ such that $\max_{z\in K}|f(z)| \le C(K,n,\omega,p)\|f\|_{B_p(\omega)}$. Hence

$$\max_{z\in K}|f_k(z)-f_m(z)| \le C(K,n,\omega,p)\|f_k - f_m\|_{B_p(\omega)}$$

for all k, m. Thus, $\{f_k\}$ uniformly converges to a holomorphic function $g(z)$ on all compact subsets of B^n. Since the compact sets are arbitrary, g is holomorphic on B^n. By Riesz' theorem, some subsequence of $\{f_k(z)\}$ pointwise converges to f for almost all $z \in B^n$. Hence $f = g$ for almost all $z \in B^n$, and the desired statement follows. □

Corollary 3. *$B_p(\omega)$ is a Banach space for $1 \le p < \infty$, and a complete metric space for $0 < p < 1$.*

Theorem 2. *The following statements are true for any $0 < p < +\infty$:*

1. *If $f \in B_p(\omega)$ and $f_r(z) = f(rz)$, $0 < r < 1$, then $\|f - f_r\|_{B_p(\omega)} \to 0$ as $r \to 1-0$.*
2. *The set of polynomials is dense in $B_p(\omega)$, i.e. for any $f \in B_p(\omega)$ there is a sequence $\{P_n\}$ of polynomials such that $\|P_n - f\|_{B_p(\omega)} \to 0$ as $n \to \infty$.*

Proof: For completeness, we give a full proof, although it is based on a standard argument.

1. By the inequality $(a+b)^p \le 2^p(a^p + b^p)$ $(a, b > 0)$, for $\delta \in (0,1)$ we get

$$\begin{aligned}\|f - f_r\|_{B_p(\omega)} &\le \int_0^\delta \int_{S^n}|Df(z) - Df_r(z)|^p\frac{(1-|z|^2)^p\omega(1-|z|)}{(1-|z|^2)^{n+1}}\,d\sigma(z)\,dr \\ &\quad+ \int_\delta^1 \int_{S^n}|Df(z) - Df_r(z)|^p\frac{(1-|z|^2)^p\omega(1-|z|)}{(1-|z|^2)^{n+1}}\,d\sigma(z)\,dr.\end{aligned}$$

The function $|Df|^p$ is subharmonic in B^n, and hence

$$\int_{S^n} |Df_r(z)|^p\, d\sigma(z) \le \int_{S^n} |Df(z)|^p\, d\sigma(z).$$

Therefore,

$$\begin{aligned}\|f - f_r\|_{B_p(\omega)} \le &\int_0^\delta \int_{S^n} |Df(z) - Df_r(z)|^p \frac{(1-|z|^2)^p \omega(1-|z|)}{(1-|z|^2)^{n+1}}\, d\sigma(z)\, dr\\ &+ 2^p \int_\delta^1 \int_{S^n} |Df(z)|^p \frac{(1-|z|^2)^p\omega(1-|z|)}{(1-|z|^2)^{n+1}}\, d\sigma(z)\, dr.\end{aligned}$$

The first integral in the right-hand side of this inequality vanishes as $r \to 1-0$, and the second one can be made arbitrarily small by choosing δ close enough to 1.

2. Let $f \in B_p(\omega)$ be an arbitrary function. Then

$$\lim_{r\to 1-0} \|f - f_r\|_{B_p(\omega)} = 0.$$

Further, the function f_r can be uniformly approximated by its Taylor polynomials in a neighborhood of $\overline{B}^n$. Therefore, the function f can be uniformly approximated in norm by a sequence of polynomials. □

The following theorem gives a description of the space $B_p(\omega)$ in the terms of $L_p(\omega)$ $(0 < p < +\infty)$.

Theorem 3. *Let $f \in H(B^n)$. Then for any $0 < p < +\infty$ the inclusion $f \in B_p(\omega)$ is true if and only if the function $g(z) = (1-|z|^2)^\beta D^\beta f(z)$ is in $L_p(\omega)$ for some $\beta \ge 1$. Moreover, there are some constants C_1, C_2 such that*

$$C_1\|g\|_{L_p(\omega)} \le \|f\|_{B_p(\omega)} \le C_1\|g\|_{L_p(\omega)}. \tag{3}$$

Proof: If $f \in B_p(\omega)$, then by (2) and Lemmas 8, 7

$$|D^\beta f(z)| \preceq C(n,m) \int_{B^n} \frac{(1-|\zeta|^2)^m |Df(\zeta)|}{|1-\langle z,\zeta\rangle|^{m+n+\beta}}\, d\nu(\zeta).$$

Let $p > 1$. Then

$$\begin{aligned}&\int_{S^n} |D^\beta f(r\zeta)|^p\, d\sigma(\zeta)\\ &\le C(m,n) \int_{S^n} \left| \int_0^1 (1-\rho^2)^m \rho^{2n-1} \int_{S^n} \frac{|Df(\rho\zeta)|\, d\sigma(\zeta)}{|1-r\rho\langle z,\zeta\rangle|^{m+n+\beta}} \right|^p d\sigma(z)\, d\rho\\ &\le C(m,n) \int_0^1 \int_{S^n} \left| \int_{S^n} \frac{|Df(\rho\zeta)|\, d\sigma(\zeta)}{|1-r\rho\langle z,\zeta\rangle|^{m+n+\beta}} \right|^p d\sigma(z)(1-\rho^2)^{mp}\rho^{(2n-1)p}\, d\rho\end{aligned}$$

$$\leq C(m,n)\int_0^1\int_{S^n}\int_{S^n}\frac{|Df(\rho\zeta)|^p\,d\sigma(\zeta)}{|1-r\rho\langle z,\zeta\rangle|^{m+n+\beta}}$$
$$\times\left(\int_{S^n}\frac{d\sigma(\zeta)}{|1-r\rho\langle z,\zeta\rangle|^{m+n+\beta}}\right)^{p/q}d\sigma(z)(1-\rho^2)^{mp}\rho^{(2n-1)p}\,d\rho.$$

Further, using [15, Theorem 1.12] we get

$$\int_{S^n}|D^\beta f(r\zeta)|^p\,d\sigma(\zeta)$$
$$\leq C(m,n)\int_0^1\int_{S^n}\int_{S^n}\frac{|Df(\rho\zeta)|^p\,d\sigma(\zeta)}{|1-r\rho\langle z,\zeta\rangle|^{m+n+\beta}}\,d\sigma(z)\frac{(1-\rho^2)^{mp}\rho^{(2n-1)p}}{(1-r\rho)^{(m+\beta)p/q}}\,d\rho$$
$$=C(m,n)\int_0^1\int_{S^n}|Df(\rho\zeta)|^p\,d\sigma(\zeta)\int_{S^n}\frac{d\sigma(z)}{|1-r\rho\langle z,\zeta\rangle|^{m+n+\beta}}$$
$$\times\frac{(1-\rho^2)^{mp}\rho^{(2n-1)p}}{(1-r\rho)^{(m+\beta)p/q}}\,d\rho\leq\int_0^1\int_{S^n}|Df(\rho\zeta)|^p\,d\sigma(\zeta)\frac{(1-\rho^2)^{mp}\rho^{(2n-1)p}}{(1-r\rho)^{(m+\beta)p}}\,d\rho.$$

Therefore,

$$\int_0^1\frac{(1-r^2)^{\beta p}\omega(1-r)}{(1-r^2)^{n+1}}\int_{S^n}|D^\beta f(\rho\zeta)|^p\,d\sigma(\zeta)r^{2n-1}\,dr$$
$$\preceq\int_0^1\int_0^1\int_{S^n}|D^\beta f(\rho\zeta)|^p\,d\sigma(\zeta)\frac{(1-\rho^2)^{mp}\rho^{(2n-1)p}r^{2n-1}\omega(1-r)(1-r^2)^{\beta p}}{(1-\rho r)^{(m+\beta)p}(1-r^2)^{n+1}}\,dr\,d\rho$$
$$=\int_0^1\int_{S^n}|D^\beta f(\rho\zeta)|^p(1-\rho^2)^{mp}\rho^{(2n-1)p}$$
$$\times\int_0^1\frac{r^{2n-1}\omega(1-r)(1-r^2)^{\beta p}}{(1-\rho r)^{(m+\beta)p}(1-r^2)^{n+1}}\,dr\,d\rho\,d\sigma(\zeta).$$

Further, by [7, Lemma 1.6] we obtain

$$\int_0^1\frac{r^{2n-1}\omega(1-r)(1-r^2)^{\beta p}}{(1-\rho r)^{(m+\beta)p}(1-r^2)^{n+1}}\,dr\preceq\frac{\omega(1-\rho)}{(1-\rho)^{mp+n}}\,.$$

Consequently,

$$\int_{B^n}|g(w)|^p\frac{\omega(1-|w|)}{(1-|w|^2)^{n+1}}\,d\nu(w)\preceq\|f\|_{B_p(\omega)}.$$

Conversely, if $f\in H(U^n)$ and $g\in L^p(\omega)$, then $f\in B_p(\omega)$. Indeed, using Lemma 7 we obtain

$$|Df(z)|\preceq\int_{B^n}\frac{(1-|\zeta|^2)^m|D^\beta f(\zeta)|}{|1-\langle z,\zeta\rangle|^{m-\beta+n+2}}\,d\nu(\zeta)$$

(recall that m is assumed to be sufficiently great). Using Hölder's inequality, we get

$$|Df(z)|^p \preceq \int_{B^n} \frac{(1-|\zeta|^2)^{m+p\beta}}{|1-\langle z,\zeta\rangle|^{m+n+2-\beta}} |D^\beta f(\zeta)|^p \, d\nu(\zeta) \times \left(\int_{B^n} \frac{(1-|\zeta|^2)^{m-\beta q}\, d\nu(\zeta)}{|1-\langle z,\zeta\rangle|^{m+n+2-\beta}} \right)^{p/q} \preceq (1-|z|^2)^{-(\beta q+1-\beta)p/q} \int_{B^n} \frac{(1-|\zeta|^2)^{m+p\beta}}{|1-\langle z,\zeta\rangle|^{m+n+2-\beta}} |D^\beta f(\zeta)|^p \, d\nu(\zeta),$$

where [15, Theorem 1.12] is used for obtaining the last inequality. Consequently,

$$\|f\|_{B_p(\omega)} \preceq \int_{B^n} |D^\beta f(\zeta)|^p (1-|\zeta|^2)^{\beta p+m} \times \int_0^1 \omega(1-r)(1-r^2)^{p-n-1-(\beta q+1-\beta)p/q} \int_{S^n} \frac{d\nu(z)}{|1-r\langle z,\zeta\rangle|^{m+n+2-\beta}} r^{2n-1} \, d\nu(\zeta)$$
$$\preceq \int_{B^n} |D^\beta f(\zeta)|^p (1-|\zeta|^2)^{\beta p+m} \int_0^1 \frac{\omega(1-r)(1-r^2)^{p-n-1-(\beta q+1-\beta)p/q} r^{2n-1}}{(1-r|\zeta|^2)^{m+2-\beta}} \, dr$$
$$\preceq \int_{B^n} |D^\beta f(\zeta)|^p (1-|\zeta|^2)^{\beta p+m} \frac{\omega(1-|\zeta|)(1-|\zeta|^2)^{p-n-1-(\beta q+1-\beta)p/q} r^{2n-1}}{(1-|\zeta|^2)^{m+1-\beta}} \, d\nu(\zeta)$$
$$= \int_{B^n} |D^\beta f(\zeta)|^p (1-|\zeta|^2)^{\beta p} \frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}} \, d\nu(\zeta) = \|g\|_{L_p(\omega)},$$

where $g(z) = (1-|z|^2)^\beta D^\beta f(z)$. Summing up, we get the proof of (3).

The case $0 < p \le 1$ requires a different proof. Let $f \in B_p(\omega)$. Then by Lemmas 8, 7

$$|D^\beta f(z)|^p \preceq \int_{B^n} \frac{(1-|\zeta|^2)^{mp+(n+1)p-n-1}}{|1-\langle z,\zeta\rangle|^{(m+n+\beta)p}} |Df(\zeta)|^p \, d\nu(\zeta).$$

Therefore,

$$\|g\|_{L_p(\omega)} = \int_{B^n} (1-|z|^2)^{\beta p} |D^\beta f(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}} \, d\nu(z)$$
$$\preceq \int_{B^n} (1-|\zeta|^2)^{mp+(n+1)p-n-1} |Df(\zeta)| \times \int_{B^n} \frac{(1-|z|^2)^{\beta p-n-1}\omega(1-|z|)}{|1-\langle z,\zeta\rangle|^{(m+n+\beta)p}} \, d\nu(\zeta)\, d\nu(z)$$
$$\preceq \int_{B^n} (1-|\zeta|^2)^p |Df(\zeta)|^p \frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}} \, d\nu(\zeta) = \|f\|_{B_p(\omega)} < +\infty.$$

Conversely, if $g \in L_p(\omega)$, then using Lemmas 8 and 7 one more time we get

$$|Df(z)|^p \preceq \int_{B^n} \frac{(1-|\zeta|^2)^{mp+(n+1)p-n-1}}{|1-\langle z,\zeta\rangle|^{(m+n+2-\beta)p}} |D^\beta f(\zeta)|^p \, d\nu(\zeta).$$

Therefore,

$$\begin{aligned}
&\int_{B^n} (1-|z|^2)^p |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|)^{n+1}} \, d\nu(z) \\
&\leq \int_{B^n} (1-|\zeta|^2)^{mp+(n+1)p-n-1} |D^\beta f(\zeta)|^p \int_{B^n} \frac{(1-|z|^2)^{p-n-1}\omega(1-|z|)}{|1-\langle z,\zeta\rangle|^{(m+n+2-\beta)p}} \, d\nu(\zeta) \\
&\preceq \int_{B^n} (1-|\zeta|^2)^{\beta p} |D^\beta f(\zeta)|^p \frac{\omega(1-|\zeta|)}{(1-|\zeta|)^{n+1}} \, d\nu(\zeta) = \|g\|_{L_p(\omega)},
\end{aligned}$$

where [7, Lemma 1.7] and [15, Theorem 1.12] were used for the last inequality. □

3. Bounded and inverse operators on $B_p(\omega)$

Let us consider the linear operator

$$P_\alpha(f)(z) := \int_{B^n} \frac{(1-|\zeta|^2)^\alpha}{(1-\langle z,\zeta\rangle)^{n+1+\alpha}} f(\zeta) \, d\nu(\zeta) \qquad (\alpha > -1). \tag{4}$$

If $f \in L_p(\omega)$, then obviously $P_\alpha(f)$ is holomorphic on B^n. For finding the class of functions to which it belongs, we prove the following theorem stating that P_α is a bounded operator on $L_p(\omega)$.

Theorem 4. *If $\alpha > -\alpha_\omega - n - 1$ and $1 \leq p < \infty$, then $P_\alpha : L_p(\omega) \to B_p(\omega)$ boundedly.*

Proof: We shall prove that if $f \in L_p(\omega)$ and $F = P_\alpha(f)$, then $F \in B_p(\omega)$. It is obvious that

$$|D^\beta F(z)| \preceq \int_{B^n} \frac{(1-|\zeta|^2)^\alpha |f(\zeta)|}{|1-\langle z,\zeta\rangle|^{n+1+\alpha+\beta}} \, d\nu(\zeta),$$

where we assume that $\alpha + \beta + 1 > 0$ and $\beta > n + \beta_\omega$.

If $p > 1$, then by Hölder's inequality

$$\begin{aligned}
&\int_{B^n} (1-|z|^2)^{\beta p} |D^\beta F(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}} \, d\nu(z) \\
&\preceq \int_{B^n} (1-|\zeta|^2)^\alpha |f(\zeta)|^p \int_{B^n} \frac{(1-|z|^2)^{\beta p-\beta p/q-n-1}\omega(1-|z|)}{|1-\langle z,\zeta\rangle|^{n+1+\alpha+\beta}} \, d\nu(\zeta)\, d\nu(z) \\
&= \int_{B^n} (1-|\zeta|^2)^\alpha |f(\zeta)|^p \int_0^1 \int_{S^n} \frac{r^{2n-1}(1-r^2)^{\beta-n-1}\omega(1-r)\, d\sigma(z)}{|1-r\langle z,\zeta\rangle|^{n+1+\alpha+\beta}} \, dr\, d\nu(\zeta) \\
&\preceq \int_{B^n} (1-|\zeta|^2)^\alpha |f(\zeta)|^p \int_0^1 \frac{r^{2n-1}(1-r^2)^{\beta-n-1}\omega(1-r)}{(1-r|\zeta|)^{\alpha+\beta+1}} \, dr\, d\nu(\zeta)
\end{aligned}$$

$$\preceq \int_{B^n} (1-|\zeta|^2)^\alpha |f(\zeta)|^p \frac{\omega(1-|\zeta|)}{(1-|\zeta|)^{n+\alpha+1}}\, d\nu(\zeta) = \|f\|_{L_p(\omega)},$$

where [15, Theorem 1.12] and [7, Lemma 1.7] were used for the last inequality.

If $p=1$, then

$$\int_{B^n} (1-|z|^2)^\beta |D^\beta F(z)| \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}}\, d\nu(z)$$
$$\leq \int_{B^n} (1-|\zeta|^2)^\alpha |f(\zeta)| \int_{B^n} \frac{(1-|z|^2)^\beta \omega(1-|z|)\, d\nu(z)\, d\nu(\zeta)}{|1-\langle z,\zeta\rangle|^{n+1+\beta+\alpha}(1-|z|^2)^{n+1}} \preceq \|f\|_{L_1(\omega)}.$$

□

For the case when $1 \leq p < \infty$ and $\alpha > -1$, $\gamma > 0$, we define the inverse mapping $R_{\alpha,\gamma}$ of P_α by the formula

$$R_{\alpha,\gamma}(f)(z) := (1-|z|^2)^\gamma \int_{B^n} \frac{(1-|\zeta|^2)^\alpha f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+1+\alpha+\gamma}}\, d\nu(\zeta), \quad z \in B^n. \tag{5}$$

One can prove that $P_\alpha R_{\alpha,\gamma}(f)(z) \equiv f(z)$ for all $f \in B_p(\omega)$ if $\alpha > 2n-2$. To this end, observe that a change of integration order gives

$$P_\alpha R_{\alpha,\gamma}(f)(z) = \int_{B^n} \frac{(1-|t|^2)^\alpha f(t)\, d\nu(t)}{(1-\langle \zeta,t\rangle)^{n+1+\alpha+\gamma}}\, d\nu(\zeta)$$
$$= \int_{B^n} (1-|t|^2)^\alpha f(t) \overline{\int_{B^n} \frac{(1-|\zeta|^2)^{\alpha+\gamma}\, d\nu(\zeta)}{(1-\langle t,\zeta\rangle)^{n+1+\alpha+\gamma}(1-\langle \zeta,z\rangle)^{n+1+\alpha}}}\, d\nu(t)$$
$$= \int_{B^n} \frac{(1-|t|^2)^\alpha f(t)}{(1-\langle \zeta,t\rangle)^{n+1+\alpha}}\, d\nu(t) \equiv f(z).$$

Further, we show that $R_{\alpha,\gamma}(f) \in L_p(\omega)$ for all $f \in L_p(\omega)$. Indeed, if $1 < p < \infty$, then by Hölder's inequality

$$|f(z)| \leq \int_{B^n} \frac{(1-|\zeta|^2)^m |Df(\zeta)|}{|1-\langle z,\zeta\rangle|^{m+n}}\, d\nu(\zeta)$$
$$\leq (1-|z|^2)^{\frac{1-m}{q}} \left(\int_{B^n} \frac{(1-|\zeta|^2)^{mp} |Df(\zeta)|^p}{|1-\langle z,\zeta\rangle|^{m+n}}\, d\nu(\zeta)\right)^{1/p}.$$

Consequently,

$$\left(\int_{B^n} \frac{(1-|\zeta|^2)^\alpha f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+1+\alpha+\gamma}}\, d\nu(\zeta)\right)^p$$
$$\leq (1-|z|^2)^{-\gamma p/q} \int_{B^n} \frac{(1-|\zeta|^2)^\alpha |f(\zeta|)}{(1-\langle z,\zeta\rangle)^{n+1+\alpha+\gamma}}\, d\nu(\zeta)$$
$$\preceq \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-(m-1)p/q}}{(1-\langle z,\zeta\rangle)^{n+1+\alpha+\gamma}} \int_{B^n} \frac{(1-|t|^2)^{mp} |Df(t)|^p}{|1-\langle \zeta,t\rangle|^{m+n}}\, d\nu(t)\, d\nu(\zeta),$$

and hence

$$\int_{B^n} \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}} R_\alpha(f)(z)|^p \, d\nu(z) \preceq \int_{B^n} (1-|t|^2)^{mp} |Df(t)|^p$$
$$\times \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-(m-1)p/q}}{|1-\langle\zeta,t\rangle|^{m+n}} \int_{B^n} \frac{(1-|z|)^{\gamma p-\gamma p/q-n-1}}{|1-\langle z,\zeta\rangle|^{n+1+\alpha+\gamma}} \, d\nu(\zeta)\, d\nu(t)$$
$$\preceq \int_{B^n} (1-|t|^2)^{mp} |Df(t)|^p$$
$$\times \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-(m-1)p/q+p\gamma-\gamma p/q}\omega(1-|\zeta|)}{|1-\langle\zeta,t\rangle|^{m+n}(1-|\zeta|^2)^{n+1+\alpha+\gamma}} \, d\nu(\zeta)\, d\nu(t)$$
$$\preceq \int_{B^n} |Df(t)|^p \frac{(1-|t|^2)^{mp+\alpha-(m-1)p/q}\omega(1-|t|)}{(1-|t|^2)^{m+n+\alpha}} \, d\nu(t)$$
$$= \int_{B^n} (1-|t|^2)^p |Df(t)|^p \frac{\omega(1-|t|)}{(1-|t|^2)^{n+1}} \, d\nu(t) = \|f\|_{B_p(\omega)}.$$

If $p=1$, then

$$\int_{B^n} \frac{(1-|\zeta|^2)^\alpha |f(\zeta)|}{|1-\langle z,\zeta\rangle|^{n+1+\alpha+\gamma}}$$
$$\le \int_{B^n} \frac{(1-|\zeta|^2)^\alpha}{|1-\langle z,\zeta\rangle|^{n+1+\alpha+\gamma}} \int_{B^n} \frac{(1-|t|^2)^m |Df(t)|}{|1-\langle t,\zeta\rangle|^{m+n}} \, d\nu(t)\, d\nu(\zeta).$$

Hence

$$\int_{B^n} |R_{\alpha,\gamma}(f)(z)| \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}} \, d\nu(z) \le \int_{B^n} (1-|t|^2)^m |Df(t)|$$
$$\times \int_{B^n} \frac{(1-|\zeta|^2)^\alpha}{|1-\langle t,\zeta\rangle|^{n+m}} \int_{B^n} \frac{\omega(1-|\zeta|)(1-|\zeta|)^\gamma \, d\nu(\zeta)}{|1-\langle t,\zeta\rangle|^{n+1+\alpha+\gamma}(1-|\zeta|^2)^{n+1}} \, d\nu(t)$$
$$\preceq \int_{B^n} (1-|t|^2)^m |Df(t)| \int_{B^n} \frac{\omega(1-|\zeta|)\, d\nu(\zeta)}{|1-\langle t,\zeta\rangle|^{n+m}(1-|\zeta|^2)^{n+1}} \, d\nu(t)$$
$$\preceq \int_{B^n} (1-|t|^2)^m |Df(t)| \frac{\omega(1-|t|)\, d\nu(t)}{(1-|t|)^{m-1+n+1}}$$
$$= \int_{B^n} |Df(t)| \frac{\omega(1-|t|)\, d\nu(t)}{(1-|t|)^n} = \|f\|_{B_p(\omega)}.$$

Thus, we proved the following

Theorem 5. *If $1 \le p < \infty$ and $\alpha+\gamma > -1$, then R_α is a bounded operator from $B_p(\omega)$ to $L_p(\omega)$.*

For the case $0 < p < 1$, we consider the harmonic subspace $b_p(\omega)$ of $L_p(\omega)$. Repeating the argument of [3, Lemma 2.3] one can prove the following lemma.

Lemma 9. *Let $0 < p < \infty$ and let g be harmonic on the ball $B_r^n = B_r^n(z_0)$. Then there is a constant C_p depending only on p and such that*

$$|u(z_0)|^p \leq \frac{C_p}{\pi r^{n+1}} \int_{B_r} |u(z)|^p \, d\nu(z).$$

As corollaries of the above lemma, one can prove also the following two statements.

Lemma 10. *Let $0 < p < \infty$ and let $g \in b_p(\omega)$. Then*

$$|g(z)| \leq \frac{\|g\|_{b_p(\omega)}}{\omega^{1/p}(1-|z|)}, \quad z \in B^n.$$

Lemma 11. *Let $0 < p < \infty$ and let $g \in b_p(\omega)$. Then*

$$\int_{B^n} |g(z)| \frac{\omega^{1/p}(1-|z|)}{(1-|z|^2)^{n+1}} \, d\nu(z) \leq \int_{B^n} |g(z)|^p \frac{\omega(1-|z|)}{(1-|z|)^{n+1}} \, d\nu(z).$$

The last lemma gives a possibility to consider the operator $P_\alpha(f)$ on $b_p(\omega)$ in the case when $0 < p < 1$. Assuming that $f \in b_p(\omega)$, we shall show that $P_\alpha(f) \in B_p(\omega)$. To this end, we use Lemma 11 and obtain

$$\begin{aligned}
&\int_{B^n} |P_\alpha f(z)|^p \frac{\omega(1-|z|)}{(1-|z|)^{n+1}} \, d\nu(z) \\
&\leq \int_{B^n} |f(\zeta)|^p \frac{(1-|\zeta|^2)^{(\alpha+n+1)p}}{(1-|\zeta|^2)^{n+1}} \int_{B^n} \frac{\omega(1-|z|)(1-|z|^2)^{-n-1}}{|1-\langle z,\zeta\rangle|^{(n+1+\alpha)p}} \, d\nu(z)\, d\nu(\zeta) \\
&\preceq \int_{B^n} \frac{\omega(1-|\zeta|)(1-|\zeta|^2)^{\alpha p}}{(1-|\zeta|^2)^{n+1+\alpha p}} |f(\zeta)|^p \, d\nu(\zeta) = \|f\|_{b_p(\omega)}.
\end{aligned}$$

Thus, we proved

Theorem 6. *If $0 < p < 1$, then P_α is a bounded operator $b_p(\omega) \to B_p(\omega)$.*

For $0 < p < 1$, the inverse operator R_α defined by (5) maps $B_p(\omega)$ to $L_p(\omega)$. Indeed, by Lemma 11

$$\begin{aligned}
&\int_{B^n} (1-|z|^2)^{\gamma p} \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1}} \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha p+(n+1)p} |f(\zeta)|^p \, d\nu(\zeta)\, d\nu(z)}{|1-\langle z,\zeta\rangle|^{(n+1+\alpha+\gamma)p}(1-|\zeta|^2)^{n+1}} \\
&= \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha p+(n+1)p}}{(1-|\zeta|^2)^{n+1}} |f(\zeta)|^p \int_{B^n} \frac{(1-|z|^2)^{\gamma p}\omega(1-|z|)\, d\nu(\zeta)\, d\nu(z)}{(1-|z|^2)^{n+1}|1-\langle z,\zeta\rangle|^{(n+1+\alpha+\gamma)p}} \\
&\preceq \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha p+(n+1)p}(1-|\zeta|^2)^{\gamma p}\omega(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}(1-|\zeta|^2)^{(\alpha+\gamma+n+1)p}} |f(\zeta)|^p \, d\nu(\zeta) = \|f\|_{L_p(\omega)}.
\end{aligned}$$

Summing up, we come to

Theorem 7. *If $0 < p < 1$, then the operator R_α boundedly maps $b_p(\omega)$ to $B_p(\omega)$.*

4. Duals of $B_p(\omega)$ spaces

Theorem 8. *If $1 < p < \infty$ and $\alpha > -(n+\beta_\omega)/p$, then the dual of the space $B_p(\omega)$ under the pairing*

$$\langle f, g\rangle = \int_{B^n} Df(\zeta)\overline{Dg(\zeta)}(1-|\zeta|^2)^\alpha \, d\nu(\zeta) \tag{6}$$

is isomorphic to $B_q(\widetilde{\omega})$, where $\widetilde{\omega}(t) = \omega^{-q/p}(t)t^{(\alpha+n-1)q}$, $1/p + 1/q = 1$.

Proof: Let $\Phi \in (B_p(\omega))^*$. In virtue of Theorem 2, we can consider $B_p(\omega)$ as a subspace of $L_p(\omega)$. Then, by the Hahn-Banach theorem, we can assume that $\Phi \in L_p(\omega)^*$, and hence there is a function $G \in L_q(\omega)$ $(1/q + 1/p = 1)$ such that

$$\Phi(F) = \int_{B^n} F(\zeta)\overline{G(\zeta)}\frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}} \, d\nu(\zeta),$$

and $\|\Phi\| = \|G\|_{L_q(\omega)}$. Taking $F(z) = (1-|z|^2)Df(z)$, where $f \in B_p(\omega)$, we get

$$\Phi(F) = \int_{B^n} Df(\zeta)\overline{G(\zeta)}\frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^{n}} \, d\nu(\zeta).$$

Using the fact that $\alpha > -(n+\beta_\omega)/p$ we can write (2) for Df and get

$$\Phi(F) = \int_{B^n} (1-|w|^2)^\alpha Df(w) \int_{B^n} \frac{\omega(1-|\zeta|)\overline{G(\zeta)}\, d\nu(\zeta)}{(1-\langle\zeta, w\rangle)^{\alpha+n+1}(1-|\zeta|^2)^n} \, d\nu(w).$$

Let $G_1(w)$ be the middle integral:

$$\overline{G_1(w)} = \int_{B^n} \frac{\omega(1-|\zeta|)\overline{G(\zeta)}\, d\nu(\zeta)}{(1-\langle\zeta, w\rangle)^{n+1+\alpha}(1-|\zeta|^2)^n}\,.$$

It is clear that $G_1(w)$ is a holomorphic function and there exists a function $g(w)$ such that $\overline{Dg(w)} = \overline{G_1(w)}$ (for example $g(z) = \int_0^1 G_1(rz)\, dr$). Next we show that $g \in B_q(\omega)$. By Hölder's inequality we get

$$|G_1(w)|^q \le \frac{\omega^{q/p}(1-|w|)}{(1-|w|^2)^{(\alpha+n)q/p}} \int_{B^n} \frac{\omega(1-|\zeta|)|G(\zeta)|^q \, d\nu(\zeta)}{|1-\langle\zeta, w\rangle|^{n+1+\alpha}(1-|\zeta|^2)^n}\,.$$

Then

$$\begin{aligned}
&\int_{B^n} (1-|z|^2)^q |Dg(z)|^q \frac{\widetilde{\omega}(1-|z|)}{(1-|z|^2)^{n+1}} \, d\nu(z) \\
&\le \int_{B^n} |G(\zeta)|^q \frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^n} \int_{B^n} \frac{\widetilde{\omega}(1-|z|)|\omega^{q/p}(1-|z|)(1-|z|^2)^{q-n-1}\, d\nu(z)}{|1-\langle z, \zeta\rangle|^{n+1+\alpha}(1-|z|^2)^{(\alpha+n)q/p}} \, d\nu(\zeta) \\
&\preceq \int_{B^n} |G(\zeta)|^q \frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^n} \int_{B^n} \frac{(1-|z|^2)^{\alpha-1}\, d\nu(z)}{|1-\langle z, \zeta\rangle|^{n+1+\alpha}} \, d\nu(\zeta)
\end{aligned}$$

$$\preceq \int_{B^n} |G(\zeta)|^q \frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}}\, d\nu(z) = \|G\|_{L_p(\omega)}.$$

Conversely, if $g \in B_q(\widetilde{\omega})$, then by Hölder's inequality one can prove that the functional Φ of the form (6) is bounded on $B_p(\omega)$:

$$\begin{aligned}|\Phi(f)| &\leq \int_{B^n} |Df(\zeta)||Dg(\zeta)| \frac{\omega^{1/p}(1-|\zeta|)(1-|\zeta|^2)^{2+\alpha+n+1}}{(1-|\zeta|^2)^{n+3}\omega^{1/p}(1-|\zeta|)}\, d\nu(\zeta)\\ &\leq \left(\int_{B^n} (1-|\zeta|^2)^p |Df(\zeta)|^p \frac{\omega(1-|\zeta)}{(1-|\zeta|^2)^{n+1}}\, d\nu(\zeta)\right)^{1/p}\\ &\times \left(\int_{B^n} (1-|\zeta|^2)^q |Dg(\zeta)|^q \frac{(1-|\zeta|^2)^{(n+1)q+\alpha q}\, d\nu(\zeta)}{\omega^{q/p}(1-|\zeta)(1-|\zeta|^2)^{2q}}\right)^{1/q} = \|f\|_{B_p(\omega)}\|g\|_{B_q(\widetilde{\omega})}.\end{aligned}$$

□

The case $0 < p \leq 1$ calls for a different statement connected with the defined below holomorphic Bloch space on the unit ball of C^n (for details see [4]).

Definition 3. Let $\omega \in S$. We say that a function $f \in H(B^n)$ belongs to the Bloch space B_ω if

$$M_f = \sup_{z \in B^n} \frac{(1-|z|^2)|Df(z)|}{\omega(1-|z|)} < \infty.$$

Theorem 9. *If $0 < p \leq 1$, then the dual of the space $B_p(\omega)$ under the pairing*

$$\text{(7)} \qquad \langle f, g\rangle = \int_{B^n} Df(\zeta)\overline{Dg(\zeta)}(1-|\zeta|^2)^\alpha\, d\nu(\zeta) \qquad (\alpha > n/p - \beta_\omega/p)$$

is isomorphic to the holomorphic weighted Bloch space $B_{\widetilde{\omega}}$ with $\widetilde{\omega}(t) = \omega^{1/p}(t)t^{-n-\alpha+1}$.

Proof: Let $f \in B_1(\omega)$ and let $g \in B_{\widetilde{\omega}}$ and Φ be the functional generated by g, i.e. $\Phi(f) = \langle f, g\rangle$. Then using Lemma 4 we obtain

$$\begin{aligned}|\Phi(f)| &\leq \sup_{z\in B^n} \frac{(1-|\zeta|)|Dg(\zeta)|}{\widetilde{\omega}(1-|\zeta|)} \int_{B^n} |Df(\zeta)|\widetilde{\omega}(1-|\zeta|)(1-\zeta|^2)^{\alpha-1}\, d\nu(\zeta)\\ &\leq \|g\|_{B_{\widetilde{\omega}}} \int_{B^n} |Df(\zeta)| \frac{\widetilde{\omega}^{1/p}(1-|\zeta|)}{(1-\zeta|^2)^n}\, d\nu(\zeta)\\ &\preceq \|g\|_{B_{\widetilde{\omega}}} \left(\int_{B^n} (1-|\zeta|^2)^p |Df(\zeta)|^p \frac{\widetilde{\omega}(1-|\zeta|)}{(1-\zeta|^2)^{n+1}}\, d\nu(\zeta)\right)^{1/p}\\ &= \|g\|_{B_{\widetilde{\omega}}}\|f\|_{B_p(\omega)}\end{aligned}$$

Hence $\|\Phi\| \preceq \|g\|_{B_{\widetilde{\omega}}}$.

Conversely, if $\Phi \in (B_p(\omega))^*$, $f \in B_p(\omega)$, then $f \in B_1(\omega^*)$ by Corollary 1. Considering $B_1(\omega^*)$ as a subspace of $L_1(\omega^*)$ and using the Hahn-Banach theorem we get $\Phi \in (L_1(\omega^*))^*$. Therefore, there is a function $G \in L_\infty(B^n)$ such that

$$\Phi(f) = \int_{B^n} F(\zeta)\overline{G(\zeta)}\frac{\omega^{1/p}(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}}\, d\nu(\zeta)$$

and $\|\Phi\| = \|G\|_{L_\infty}$. Particularly, taking $F(\zeta) = (1-|\zeta|^2)Df(\zeta)$, $f \in B_p(\omega)$ we obtain

$$\Phi(f) = \int_{B^n} F(\zeta)\overline{G(\zeta)}\frac{\omega^{1/p}(1-|\zeta|)}{(1-|\zeta|^2)^{n+1}}\, d\nu(\zeta).$$

If $\alpha > n/p - \beta_\omega/p$, then $Df \in A^1(\alpha)$. Therefore, by (2)

$$\Phi(f) = \int_{B^n}(1-|w|^2)^\alpha Df(w)\int_{B^n}\frac{\overline{G(\zeta)}\omega^{1/p}(1-|\zeta|)\, d\nu(\zeta)\, d\nu(w)}{(1-\langle w,\zeta\rangle)^{n+\alpha+1}(1-|\zeta|^2)^n}.$$

As in the case $p > 1$, we consider the inner integral separately, as a function $\overline{G_1(w)} = \overline{Dg(w)}$. Then we show that $g \in B_{\widetilde{\omega}}$. To this end, observe that the following estimate is true:

$$|G_1(w)| \le \|G\|_{L_\infty}\int_{B^n}\frac{\omega^{1/p}(1-|\zeta|)\, d\nu(\zeta)}{|1-\langle w,\zeta\rangle|^{n+1+\alpha}(1-|\zeta|^2)^n} \preceq \|G\|_{L_\infty}\frac{\omega^{1/p}(1-|w|)}{(1-|w|)^{n+\alpha}}.$$

Hence

$$\sup_{w\in B^n}\frac{(1-|w|^2)^{n+\alpha}|Dg(w)|}{\omega^{1/p}(1-|w|)} = \sup_{w\in B^n}\frac{(1-|w|^2)|Dg(w)|}{\widetilde{\omega}(1-|w|)} < \infty,$$

where $\widetilde{\omega}(t) = \omega^{1/p}(t)t^{-n-\alpha+1}$. □

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(*Received* April 23, 2010)