AC holds iff every compact completely regular topology can be extended to a compact Tychonoff topology

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Abstract. We show that AC is equivalent to the assertion that every compact completely regular topology can be extended to a compact Tychonoff topology.

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Let $\mathbf{X} = (X, T)$ be a topological space. If $A \subset X$, then the subspace topology A inherits from $\mathbf{X}$ will be denoted by T_A.

$\mathbf{X}$ is *compact* iff every open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$.

$\mathbf{X}$ is *regular* iff for every closed set F and any point $x \in X \backslash F$ there exist disjoint open sets U, V such that $F \subset U$ and $x \in V$.

$\mathbf{X}$ is *completely regular* iff for every closed set F and any point $x \in X \backslash F$ there is a continuous function $f : \mathbf{X} \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$ for every $y \in F$. A completely regular T_1 space is called a *Tychonoff space.*

$\mathbf{X}$ is an R_0 space provided that its T_0 reflection is a T_1 space. Equivalently, any two topologically distinguishable points in $\mathbf{X}$ (at least one of them has a neighborhood which is not a neigborhood of the other) can be separated.

$\mathbf{X}$ is a *preregular space*, or R_1 space, provided that its T_0 reflection is a T_2 space. Equivalently, any two topologically distinguishable points x, y can be separated by disjoint neighborhoods.

A subspace $\mathbf{Y}$ of a space $\mathbf{X}$ is C- (resp. C^*-) embedded in $\mathbf{X}$ provided each real valued continuous function on $\mathbf{Y}$ (resp. bounded continuous function on $\mathbf{Y}$) extends continuously over $\mathbf{X}$. The set of all continuous (resp. continuous bounded) functions will be denoted by $C(\mathbf{X})$ (resp. $\mathbf{C}^*(\mathbf{X})$).

For a locally compact, non-compact, R_1 space $\mathbf{X} = (X, R)$, $\mathbf{X}(a)$ will denote the *one-point compactification* of $\mathbf{X}$. ($\mathbf{X}(a) = (X \cup \{a\}, T_a)$, $a \notin X$ and T_a is the topology on $X \cup \{a\}$ in which open neighborhoods of points $x \in X$ are the old R ones whereas open neighborhoods of a leave out a R-compact subset of X.)

(**R**) For every set X, every compact R_1 topology on X can be enlarged to a compact T_2 topology.

(**CRT**) For every set X, every compact regular topology can be enlarged to a compact Tychonoff topology.

(**RT**) For every set X, every compact R_1 topology T on X can be enlarged to a compact Tychonoff topology R.

(**CCRT**) For every set X, every compact and completely regular topology on X can be enlarged to a compact Tychonoff topology.

AC : Every family of non-empty sets has a choice function.

1. Introduction

In [2] it was shown, in **ZFC**, that compact T_1 topologies do not extend to compact T_2 topologies in general. However, if $\mathbf{X} = (X, T)$ is a compact R_1 space then T can always be enlarged to a compact T_2 topology R. Thus, (**R**) is a theorem of **ZFC** but, as expected, (**R**) is not a theorem of **ZF**. In fact, (**R**) depends heavily on **AC** as the following theorem from [2] shows.

Theorem 1 ([2]). **AC** *is equivalent to each one of the following:*

(1) (**R**) *and "$\wp(\mathbb{R})$ is well orderable"* (*Form* 130 *in* [3]);
(2) (**R**) *and "$\mathbb{R}$ is well orderable"* (*Form* 79 *in* [3]);
(3) (**R**) *and "there exists a free ultrafilter on ω"* (*Form* 70 *in* [3]);
(4) (**R**) *and "there exists a free ultrafilter"* (*Form* 206 *in* [3]);
(5) (**R**) *and "$\aleph_1$ is regular (i.e., has cofinality greater than ω)"* (*Form* 34 *in* [3]);
(6) (**R**) *and "there exists some regular ordinal $\aleph$ (i.e., $\aleph$ is infinite and has cofinality greater than ω)";*
(7) (**R**) *and "there exists a non-compact, locally compact T_2 space with exactly one T_2 compactification (namely its Alexandroff one-point compactification)".*

In the same work the following question was asked.

Question 1. Does (**R**) imply **AC**? Equivalently, does there exist in **ZF** a non-compact, locally compact T_2 space with exactly one T_2 compactification?

In addition to Question 1, one may ask the following questions:

Question 2. What other topological properties P can we replace R_1 with in (**R**) in order to have the conclusion valid?

Question 3. What other topological properties P can we replace T_2 with in (**R**) in order to have (**R**) $\leftrightarrow$ **AC**?

Proposition 2. (i) *A regular space* $\mathbf{X}$ *is preregular.*
(ii) *A compact preregular space* $\mathbf{X}$ *is regular.*

Regarding Question 2, in view of Proposition 2, any $P \in$ {regular, completely regular, T_3, Tychonoff} can replace R_1.

Regarding Question 3 we show in Theorem 3 that if we strengthen the conclusion of (**R**) to "a compact Tychonoff" instead of (its equivalent in **ZFC**) "a compact T_2", then the resulting statement (**RT**) is equivalent to **AC**.

Theorem 3. *The following are equivalent:*

(i) **AC**;
(ii) (**RT**);
(iii) (**CRT**);
(iv) (**CCRT**).

PROOF: **AC** $\to$ (**RT**). By Theorem 8 in [2] **AC** $\to$ (**R**) and in **ZFC** a compact T_2 space is Tychonoff.

(**RT**) $\to$ (**CRT**). This, in view of Proposition 2, is clear.

(**CRT**) $\to$ (**CCRT**). This is obvious.

(**CCRT**) $\to$ **AC**. Fix $\mathcal{A} = (A_i)_{i \in I}$ a disjoint family of non-empty sets. Let $\aleph$ be any uncountable cardinal number and $\mathbf{Y} = \mathbf{2}^{\aleph}$, where 2 is the discrete space with underlying set $\{0,1\}$. Let $\mathbf{1}$ be the point of Y satisfying: $\forall i \in \aleph$, $\mathbf{1}(i) = 1$. Let $\mathbf{X}$ be the subspace obtained from Y by removal of the point $\mathbf{1}$. Clearly, $\mathbf{X}$ is completely regular. (In **ZF**, $\mathbf{2}$ hence $\mathbf{2}^{\aleph}$ also, is completely regular. Since subspaces of completely regular spaces are completely regular it follows that $\mathbf{X}$ is completely regular.) In [1, Theorem 2.1], it is shown that the subspace $\mathbf{X}$ of $\mathbf{Y}$ is C-embedded in $\mathbf{Y}$. Furthermore, it has been shown in [4] that $\mathbf{Y}$ is compact. (If $\mathcal{G}$ is a family of closed sets with the fip then via a straightforward transfinite induction on $\aleph$ we can extend $\mathcal{G}$ to a family $\mathcal{F}$ with the fip such that for every $i \in \aleph$ either $\pi_i^{-1}(1) \in \mathcal{F}$ or $\pi_i^{-1}(0) \in \mathcal{F}$ but not both. Then the element $f \in 2^{\aleph}$ satisfying: $f(i) = 1$ if $\pi_i^{-1}(1) \in \mathcal{F}$ and $f(i) = 0$ otherwise is a member of $\cap\mathcal{G}$).

Claim 1. Every Tychonoff compactification $\mathbf{Z}$ of $\mathbf{X}$ is homeomorphic with $\mathbf{Y}$.

PROOF OF CLAIM 1: . Let the embedding $j : \mathbf{X} \to \mathbf{Z}$ be a Tychonoff compactification of $\mathbf{X}$. It suffices to show that $Z - X$ is a singleton. The embedding $e : \mathbf{X} \to \mathbf{Y}$ is a Tychonoff-compact reflection, since $\mathbf{Y}$ is a compact Tychonoff space, e is a C^*-embedding, and each compact Tychonoff space is a closed subspace of some power $[\mathbf{0},\mathbf{1}]^k$ of $[\mathbf{0},\mathbf{1}]$. Thus there exists some continuous extension $h : \mathbf{Y} \to \mathbf{Z}$ of j. Since $h[Y]$ is compact and contains X, it follows that $h[Y] = Z$. Consequently $Z - Y = h(1)$ is a singleton. □

For every $i \in I$, let X_i be the disjoint union of X and A_i. Let also T_i be the topology on X_i generated by the family

$$Q \cup \{O \subset X_i : X_i \backslash O \text{ is compact subset of } \mathbf{X}\}$$

where Q is the original topology of $\mathbf{X}$.

Claim 2. Each $\mathbf{X}_i = (X_i, T_i)$ is a completely regular space.

PROOF OF CLAIM 2: Fix $F \subset X_i$ a closed subset of $\mathbf{X}_i$ and let $x \in F^c$. We consider the following cases:

(1) $F \subset X$ and $x \in X$. As $\mathbf{Y}$ is completely regular there exists a continuous map $f : \mathbf{Y} \to \mathbb{R}$ with $f(x) = \{0\}$ and $f[F] \subseteq \{1\}$. Define $g : \mathbf{X}_i \to \mathbb{R}$ by $g|X = f|X$ and $g|A_i = f|\{\mathbf{1}\}$.

(2) $F \subseteq X$ and $x \in A_i$. In this case $\mathbf{1}$ is not in the closure of F (in the space $\mathbf{Y}$). Thus there exists a continuous map $f : \mathbf{Y} \to \mathbb{R}$ with $f(\mathbf{1}) = \{0\}$ and $f[F] \subseteq \{1\}$. Define $g : \mathbf{X}_i \to \mathbb{R}$ by $g|X = f|X$ and $g|A_i = f|\{\mathbf{1}\}$.

(3) $F \cap A_i \neq \emptyset$. In this case A_i is a subset of F, and thus x belongs to X. Let U be a clopen neighborhood of x in $\mathbf{Y}$ that does not meet $(F \cap X) \cup \{\mathbf{1}\}$. Define $g : \mathbf{X}_i \to \mathbb{R}$ by $g(y) = 0$ if $y \in U$, and $g(y) = 1$ otherwise. As U is clopen, it follows that g is continuous finishing the proof of Claim 2. □

By Claim 2, each X_i is a completely regular space. Hence, the one point compactification $\mathbf{Z}(a)$ of the topological sum $\mathbf{Z}$ of the family $(\mathbf{X}_i)_{i \in I}$ is completely regular. Indeed, if $F \subset Z \cup \{a\}$ is closed and $x \in Z \backslash F$ then $x \in X_i$ for some $i \in I$ or $x = a$. We consider the following cases:

(1) $x \in X_i$ and $F \cap X_i = \emptyset$. Then the function $f : \mathbf{Z} \to \mathbb{R}$, $f(X_i) = \{0\}$ and $f(X_i^c) = \{1\}$ is continuous and separates x and F.

(2) $x \in X_i$ and $F_i = F \cap X_i \neq \emptyset$. As $\mathbf{X}_i$ is completely regular, there exists a continuous real valued function $h : \mathbf{X}_i \to \mathbb{R}$ such that $h(x) = 0$ and $f(F_i) = \{1\}$. Clearly, the function $f : \mathbf{Z}(a) \to \mathbb{R}$ given by $f|\mathbf{X}_i = h$ and $f\{X_i^c\} = \{1\}$ is continuous and separates x and F.

(3) $x = a$. Clearly, F meets only finitely many X_i's, say $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$. It is easy to see that the function $f : \mathbf{Z}(a) \to \mathbb{R}$ given by: $f(G) = \{1\}$, $G = \cup\{X_{i_j} : j \leq k\}$ and $f(G^c) = \{0\}$ is a continuous mapping separating x and F.

Let, by (**CRT**), R be a compact Tychonoff refinement of the topology of $\mathbf{Z}(a)$. Clearly, each X_i with the subspace topology R_{X_i} it inherits from R is a compact Tychonoff space $\mathbf{Y}_i$, and thus the closure $\mathbf{Z}_i$ of $\mathbf{X}$ in $\mathbf{Y}_i$ is a Tychonoff compactification of $\mathbf{X}$. Hence, by Claim 1, for every $i \in I$, $Z_i \backslash X$ is a singleton, say $\{a_i\}$, of A_i. It follows that $(a_i)_{i \in I}$ is a choice function of $\mathcal{A}$ finishing the proof of the theorem. □

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