

On weakly s -permutably embedded subgroups

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Abstract. Suppose G is a finite group and H is a subgroup of G . H is said to be s -permutably embedded in G if for each prime p dividing $|H|$, a Sylow p -subgroup of H is also a Sylow p -subgroup of some s -permutable subgroup of G ; H is called weakly s -permutably embedded in G if there are a subnormal subgroup T of G and an s -permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$. We investigate the influence of weakly s -permutably embedded subgroups on the p -nilpotency and p -supersolvability of finite groups.

Keywords: weakly s -permutably embedded subgroups, p -nilpotent, n -maximal subgroup

Classification: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to be s -permutable (or s -quasinormal) [1] in G if H permutes with every Sylow subgroup of G . From Ballester-Bolinches and Pedraza-Aguilera [2], we know H is said to be s -permutably embedded in G if for each prime p dividing $|H|$, a Sylow p -subgroup of H is also a Sylow p -subgroup of some s -permutable subgroup of G . In recent years, it has been of interest to use supplementation properties of subgroups to characterize properties of a group. For example, Wang [3] introduced the concept of c -normal subgroup. A subgroup H of a group G is said to be c -normal in G if there exists a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H . In 2007, Skiba [5] introduced the concept of weakly s -permutable subgroup. A subgroup H of a group G is said to be weakly s -permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap K \leq H_{sG}$, where H_{sG} is the maximal s -permutable subgroup of G contained in H . As a generalization of above subgroups, Y. Li, S. Qiao and Y. Wang [7] introduced a new subgroup embedding property in a finite group called weakly s -permutably embedded subgroup. In the present paper we characterize p -nilpotency of finite groups with the assumption that some n -maximal subgroups are weakly s -permutably embedded.

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2. Preliminaries

Definition 2.1. A subgroup H of a group G is said to be weakly s -permutably embedded in G if there are a subnormal subgroup T of G and an s -permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$.

Remark. Obviously, s -permutably embedding property implies weakly s -permutably embedding property. The converse does not hold in general. For example, suppose $G = S_4$, the symmetric group of degree 4. Take $H = \langle (34) \rangle$. Then H is weakly s -permutably embedded in G , but not s -permutably embedded in G .

Lemma 2.2 ([7, Lemma 2.5]). *Let H be a weakly s -permutably embedded subgroup of a group G .*

- (1) *If $H \leq L \leq G$, then H is weakly s -permutably embedded in L .*
- (2) *If $N \triangleleft G$ and $N \leq H \leq G$, then H/N is weakly s -permutably embedded in G/N .*
- (3) *If H is a π -subgroup and N is a normal π' -subgroup of G , then HN/N is weakly s -permutably embedded in G/N .*

Lemma 2.3. *Let G be a group and p a prime such that $p^{n+1} \nmid |G|$ for some integer $n \geq 1$. If $(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1$, then G is p -nilpotent.*

PROOF: Suppose that the statement is not true and let G be a counterexample of minimal order. Obviously, every subgroup of G satisfies the hypothesis of the Lemma. The minimal choice of G implies that G is a minimal non- p -nilpotent group. By [11, III, 5.2 and IV, 5.4], $G = P \rtimes Q$ is a subdirect product of two Sylow subgroups. It is easy to see that every proper quotient group of G satisfies the hypothesis. Thus $\Phi(P) = \Phi(G) = 1$ and so P is an elementary abelian p -group. Since $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\text{Aut}(P)$ and $|\text{Aut}(P)|$ divides $(p-1)(p^2-1)\dots(p^n-1)$ for $|P| \leq p^n$, we have $N_G(P)/C_G(P) = 1$. This induces that G is p -nilpotent by [6, Theorem 10.1.8]. The contradiction completes the proof. \square

Lemma 2.4 ([8, A, 1.2]). *Let U, V , and W be subgroups of a group G . Then the following statements are equivalent:*

- (1) $U \cap VW = (U \cap V)(U \cap W)$;
- (2) $UV \cap UW = U(V \cap W)$.

Lemma 2.5 ([9, Lemma 2.3]). *Suppose that H is s -permutable in G , P a Sylow p -subgroup of H , where p is a prime. If $H_G = 1$, then P is s -permutable in G .*

Lemma 2.6 ([9, Lemma 2.4]). *Suppose P is a p -subgroup of G contained in $O_p(G)$. If P is s -permutably embedded in G , then P is s -permutable in G .*

Lemma 2.7 ([18, Lemma A]). *If P is an s -permutable p -subgroup of G for some prime p , then $N_G(P) \geq O^p(G)$.*

Lemma 2.8 ([4, Lemma 2.8]). *Let M be a maximal subgroup of G and P a normal p -subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G .*

3. Main results

Theorem 3.1. *Let G be a group and p a prime such that $(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1$ for some integer $n \geq 1$. If there exists a Sylow p -subgroup P of G such that every n -maximal subgroup (if exists) of P is weakly s -permutably embedded in G , then G is p -nilpotent.*

PROOF: Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G is not a non-abelian simple group.

By Lemma 2.3, $p^n \parallel |P|$ and so there exists a non-identity n -maximal subgroup P_n of P . By the hypothesis, P_n is weakly s -permutably embedded in G . Then there are a subnormal subgroup T of G and an s -permutably embedded subgroup $(P_n)_{se}$ of G contained in P_n such that $G = P_n T$ and $P_n \cap T \leq (P_n)_{se}$. If G is simple, then $T = G$ and so $P_n = (P_n)_{se}$ is s -permutably embedded in G . Thus there is an s -permutable subgroup K of G such that P_n is a Sylow p -subgroup of K . Since G is simple, we have $K_G = 1$. By Lemma 2.5, P_n is s -permutable in G . Therefore $N_G(P_n) \geq O^p(G) = G$ by Lemma 2.7. It follows that $P_n \triangleleft G$, a contradiction.

(2) G has a unique minimal normal subgroup N such that G/N is p -nilpotent. Moreover $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G . Consider G/N . We will show G/N satisfies the hypothesis of the theorem. Since P is a Sylow p -subgroup of G , PN/N is a Sylow p -subgroup of G/N . If $|PN/N| \leq p^n$, then G/N is p -nilpotent by Lemma 2.3. So we may suppose $|PN/N| \geq p^{n+1}$. Let M_n/N be an n -maximal subgroup of PN/N . Then $M_n = N(M_n \cap P)$. Let $P_n = M_n \cap P$. It follows that $P_n \cap N = M_n \cap P \cap N = P \cap N$ is a Sylow p -subgroup of N . Since

$$p^n = |PN/N : M_n/N| = |PN : (M_n \cap P)N| = |P : M_n \cap P| = |P : P_n|,$$

P_n is an n -maximal subgroup of P . By the hypothesis, P_n is weakly s -permutably embedded in G , thus there are a subnormal subgroup T of G and an s -permutably embedded subgroup $(P_n)_{se}$ of G contained in P_n such that $G = P_n T$ and $P_n \cap T \leq (P_n)_{se}$. So $G/N = M/N \cdot TN/N = P_n N/N \cdot TN/N$. Since $(|N : P_n \cap N|, |N : T \cap N|) = 1$, $(P_n \cap N)(T \cap N) = N = N \cap G = N \cap (P_n T)$. By Lemma 2.6, $(P_n N) \cap (TN) = (P_n \cap T)N$. It follows that $(P_n N/N) \cap (TN/N) = (P_n N \cap TN)/N = (P_n \cap T)N/N \leq (P_n)_{se} N/N$. Since $(P_n)_{se} N/N$ is s -permutably embedded in G/N by [2, Lemma 2.1], we have that M_n/N is weakly s -permutably embedded in G . Therefore G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p -nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(3) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by step (2). Since

$$G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$$

is p -nilpotent, we have G is p -nilpotent, a contradiction.

(4) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (2) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p -nilpotent. Since $O_p(G) \cap M$ is normalized by N and M , $O_p(G) \cap M$ is normal in G . The uniqueness of N yields $N = O_p(G)$. Since $P \cap M < P$, there is a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Take an n -maximal subgroup P_n of P such that $P_n \leq P_1$. By the hypothesis, there are a subnormal subgroup T of G and an s -permutably embedded subgroup $(P_n)_{se}$ of G contained in P_n such that $G = P_n T$ and $P_n \cap T \leq (P_n)_{se}$. So there is an s -permutable subgroup K of G such that $(P_n)_{se}$ is a Sylow p -subgroup of K . If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq (P_n)_{se} \leq P_1$, and so $P = N(P \cap M) = NP_1 = P_1$, a contradiction. If $K_G = 1$, by Lemma 2.7, $(P_n)_{se}$ is s -permutable in G . From Lemma 2.7 we have $O^p(G) \leq N_G((P_n)_{se})$. Since $(P_n)_{se}$ is subnormal in G , $P_n \cap T \leq (P_n)_{se} \leq O_p(G) = N$ by [12, Corollary 1.10.17]. Thus, $(P_n)_{se} \leq P_1 \cap N$ and

$$(P_n)_{se} \leq ((P_n)_{se})^G = ((P_n)_{se})^{O^p(G)P} = ((P_n)_{se})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N.$$

It follows that $((P_n)_{se})^G = 1$ or $((P_n)_{se})^G = P_1 \cap N = N$. If $((P_n)_{se})^G = 1$, then $P_n \cap T = 1$ and so $|T|_p = p^n$. Hence T is p -nilpotent by Lemma 2.3. Since $T \triangleleft \triangleleft G$, we have G is p -nilpotent, a contradiction. If $((P_n)_{se})^G = P_1 \cap N = N$, then $N \leq P_1$ and so $P = P_1$, a contradiction.

(5) The final contradiction.

If $N \cap P \leq \Phi(P)$, then N is p -nilpotent by J. Tate's theorem ([11, IV, 4.7]). Hence, by $N_{p'} \text{ char } N \triangleleft G$, $N_{p'} \leq O_{p'}(G) = 1$. It follows that N is a p -group. Then $N \leq O_p(G) = 1$, a contradiction. Consequently, there is a maximal subgroup P_1 of P such that $P = (N \cap P)P_1$. We take an n -maximal subgroup P_n of P such that $P_n \leq P_1$. By the hypothesis, P_n is weakly s -permutably embedded in G . Then there are a subnormal subgroup T of G and an s -permutably embedded subgroup $(P_n)_{se}$ of G contained in P_n such that $G = P_n T$ and $P_n \cap T \leq (P_n)_{se}$. So there is an s -permutable subgroup K of G such that $(P_n)_{se}$ is a Sylow p -subgroup of K . If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $(P_n)_{se} \cap N$ is a Sylow p -subgroup of N . We know that $(P_n)_{se} \cap N \leq P_n \cap N \leq P \cap N$ and $P \cap N$ is a Sylow p -subgroup of N , so $(P_n)_{se} \cap N = P_n \cap N = P \cap N$. Consequently, $P = (N \cap P)P_1 = (P_n \cap N)P_1 = P_1$, a contradiction. Therefore $K_G = 1$. By Lemma 2.5, $(P_n)_{se}$ is s -permutable in G and so $(P_n)_{se} \triangleleft \triangleleft G$. Hence $P_n \cap T \leq (P_n)_{se} \leq O_p(G) = 1$. Since $|T|_p = p^n$, T is p -nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p -complement of T . Then $T_{p'}$ is a normal Hall p' -subgroups of G , a contradiction. \square

Theorem 3.2. *Let p be a prime and \mathcal{F} a saturated formation containing all p -nilpotent groups. Suppose that G is a group with $(|G|, (p-1)(p^2-1) \dots (p^n-1)) = 1$ for some integer $n \geq 1$. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup*

E such that $G/E \in \mathcal{F}$ and E has a Sylow p -subgroup such that every n -maximal subgroup (if exists) of P is weakly s -permutably embedded in G .

PROOF: The necessity is obvious. We need only to prove the sufficiency. Suppose that the assertion is not true and let G be a counterexample of minimal order. By Lemma 2.1, every n -maximal subgroup of P is weakly s -permutably embedded in E . Hence by Theorem 3.1, E is p -nilpotent. Obviously $E \neq G$. Let T be a normal Hall p' -subgroup of E . Now we divide the proof into the following steps:

(1) $T = 1$, and so $P = E \triangleleft G$.

Assume that $T \neq 1$. Because T is a normal Hall p' -subgroup of E and $E \triangleleft G$, $T \triangleleft G$. We claim that G/T (with respect to E/T) satisfies the hypothesis. In fact, $(G/T)/(E/T) \cong G/E \in \mathcal{F}$ and E/T is a p -group. Suppose that M_n/T is an n -maximal subgroup of PT/T and $P_n = M_n \cap P$. Then P_n is an n -maximal subgroup of P and $M_n = P_n T$. By the hypothesis, P_n is weakly s -permutably embedded in G . By Lemma 2.1, $M_n/T = P_n T/T$ is weakly s -permutably embedded in G/T . The minimal choice of G implies that $G/T \in \mathcal{F}$. It is easy to see that $G \in \mathcal{F}$ from [8, Proposition IV. 3.11], a contradiction. Hence $T = 1$ and $P = E \trianglelefteq G$.

(2) Suppose that Q is a Sylow q -subgroup of G , where q is a prime divisor of $|G|$ and $q \neq p$. Then $PQ = P \times Q$.

By (1), $P = E \trianglelefteq G$. So PQ is a subgroup of G . By Lemma 2.1, every n -maximal subgroup of P is weakly s -permutably embedded in PQ . Hence by Theorem 3.1, we have that PQ is p -nilpotent. It follows that $Q \trianglelefteq PQ$ and so $PQ = P \times Q$.

(3) Final contradiction.

Let H be an arbitrary non-identity normal subgroup of G contained in P and G_p a Sylow p -subgroup of G . By (2), we have $HQ = H \times Q$ for any Sylow q -subgroup of G with $q \neq p$. This induces that $O^p(G) \leq C_G(H)$ and $[H, G] = [H, G_p O^p(G)] = [H, G_p] \trianglelefteq G$. We claim that $[H, G_p] < H$. Indeed, if $[H, G_p] = H$, then for any non-negative integer t , $H = [H, G_p, \dots, G_p] \leq G_p^{t+1}$, where the number of G_p in $[H, G_p, \dots, G_p]$ is t , which contradicts [8, Theorem A.10.3]. Thus $[H, G_p] < H$ and consequently there exists a normal subgroup K of G such that H/K is a chief factor of G and $[H, K] \leq K$. This implies that $H/K \leq Z(G/K)$. Then we obtain that $G \in \mathcal{F}$ since $G/P \in \mathcal{F}$. The final contradiction completes the proof. \square

Corollary 3.3. *Let p be the smallest prime dividing the order of a group G . Assume that H is a normal subgroup of G such that G/H is p -nilpotent. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is weakly s -permutably embedded in G , then G is p -nilpotent.*

Corollary 3.4 ([16, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Assume that H is a normal subgroup of G such that G/H is p -nilpotent. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is c^* -normal in G , then G is p -nilpotent.*

Corollary 3.5 ([9, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is s -quasinormally embedded in G , then G is p -nilpotent.*

Corollary 3.6 ([7, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G . If there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is weakly s -permutably embedded in G , then G is p -nilpotent.*

Corollary 3.7 ([19, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G . If there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is weakly s -permutable in G , then G is p -nilpotent.*

Corollary 3.8 ([20, Theorem 3.2]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is weakly s -permutable in G , then G is p -nilpotent.*

Corollary 3.9 ([10, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G . If there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is s -permutably embedded in G , then G is p -nilpotent.*

Corollary 3.10 ([13, Theorem 3.4]). *Let p be the smallest prime dividing the order of a group G . If there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is c -normal in G , then G is p -nilpotent.*

Corollary 3.11 ([17, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p -nilpotent. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is c -normal or s -permutably embedded in G , then G is p -nilpotent.*

Theorem 3.12. *Let p be a prime, G a p -solvable group and H a normal subgroup of G such that G/H is p -supersolvable. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is weakly s -permutably embedded in G , then G is p -supersolvable.*

PROOF: Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) G has a unique minimal normal subgroup N contained in H such that G/N is p -supersolvable.

Let N be a minimal normal subgroup of G contained in H . Since P is the Sylow p -subgroup of H , PN/N is the Sylow p -subgroup of H/N . Let M/N be a maximal subgroup of PN/N ; then $M = (M \cap P)N$. Let $P_1 = M \cap P$. Obviously, P_1 is the maximal subgroup of P . Since G is p -solvable, N is elementary abelian p -group or p' -group. If N is p' -group, then $M/N = P_1N/N$. If N is p -group, then $M/N = P_1/N$. By hypothesis, P_1 is weakly s -permutably embedded in G and so M/N is weakly s -permutably embedded in G/N by Lemma 2.1. Since $(G/N)/(H/N) \cong G/H$ is p -supersolvable, G/N satisfies all the hypotheses of our theorem. It follows that G/N is p -supersolvable by the minimality of G . Clearly,

N is the unique minimal normal subgroup of G contained in H as the class of p -supersolvable group is a saturated formation.

(2) $O_{p'}(G) = 1$.

If $T = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/T$. Clearly, $\overline{G}/\overline{H} \cong G/HT$ is p -supersolvable by the p -supersolvability of G/H , where $\overline{H} = HT/T$. Let $\overline{P}_1 = P_1T/T$ be a maximal subgroup of PT/T . We may assume that P_1 is a maximal subgroup of P . Since P_1 is weakly s -permutably embedded in G , the subgroup P_1T/T is weakly s -permutably embedded in G/T by Lemma 2.1. The minimality of G yields that \overline{G} is p -supersolvable, and so G is also p -supersolvable, a contradiction.

(3) The final contradiction.

Since G is p -solvable, N is an elementary abelian p -group by step (2). If N is contained in all maximal subgroups of G , then $N \leq \Phi(G)$ and so G is p -supersolvable, a contradiction. Hence there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Applying Lemma 2.8, we have $O_p(H) \cap M \triangleleft G$. Therefore $O_p(H) \cap M = 1$ and $N = O_p(H)$. Let G_p be a Sylow p -subgroup of G containing P . Then $G_p = P(G_p \cap M)$ and $G_p \cap M < G_p$. Take a maximal subgroup G_1 of G containing $G_p \cap M$ and set $P_1 = G_1 \cap P$. Then $G_p \cap M = G_1 \cap M$ and $G_1 = P_1(G_p \cap M)$. Moreover, P_1 is maximal in P . By the hypothesis, P_1 is weakly s -permutably embedded in G . Then there are a subnormal subgroup T of G and an s -permutably embedded subgroup $(P_1)_{se}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. So there is an s -permutable subgroup K of G such that $(P_1)_{se}$ is a Sylow p -subgroup of K . If $K_G \neq 1$, then we can take a minimal normal subgroup N_1 of G such that $N_1 \leq K_G$. Since G is p -solvable, from (2), N_1 must be a p -subgroup, so that $N_1 \leq (P_1)_{se} \leq P \leq H$ and indeed $N_1 = N$ by step (1). Furthermore, $G_p = N(G_p \cap M) \leq P_1(G_p \cap M) = G_1$, a contradiction. Therefore $K_G = 1$ and, by Lemma 2.5, $(P_1)_{se}$ is s -permutable in G . By [12, Corollary 1.10.17], $P_1 \cap T \leq (P_1)_{se} \leq N$. Since $|G : T|$ is a number of p -power and $T \triangleleft \triangleleft G$, $O^p(G) \leq T$. We know $G/O^p(G)$ is p -subgroup, so $G/O^p(G)$ is p -supersolvable and $G/(N \cap O^p(G)) \lesssim G/N \times G/O^p(G)$ is p -supersolvable. Then $N \cap O^p(G) \neq 1$. Since N is the minimal subgroup, $N \cap O^p(G) = N$ and $N \leq O^p(G)$. It follows that $N \leq T$. Thus we have $P_1 \cap T = P_1 \cap N = (P_1)_{se}$ is s -permutable in G . Since $G_1 = P_1(G_p \cap M)$ and $P_1 = (P_1 \cap N)(P \cap M)$, we have $G_1 = (P_1 \cap N)(G_p \cap M)$. Now let Q be a Sylow q -subgroup of M with $q \neq p$. Then Q is also a Sylow q -subgroup of G , and hence $(P_1 \cap N)Q = Q(P_1 \cap N)$. Since $G_p \cap M$ is a Sylow p -subgroup of M , the set $(P_1 \cap N)M$ forms a group. The maximality of M implies that either $(P_1 \cap N)M = G$ or $(P_1 \cap N)M = M$. If the former holds, then $G_p = G_1(G_p \cap M) = G_1$, a contradiction. Thus we must have $(P_1 \cap N)M = M$, that is, $P_1 \cap N \leq M$. It follows that $P_1 \cap N = 1$. Since $P_1 \cap N$ is a maximal subgroup of N , we have N is a cyclic of order p . Thus G is p -supersolvable, a final contradiction. \square

Corollary 3.13 ([16, Theorem 3.5]). *Let p be a prime, G a p -solvable group and H a normal subgroup of G such that G/H is p -supersolvable. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is c^* -normal in G , then G is p -supersolvable.*

Corollary 3.14 ([14, Theorem 3.1]). *Let p be a prime, G a p -solvable group and H a normal subgroup of G such that G/H is p -supersolvable. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is c -normal in G , then G is p -supersolvable.*

Corollary 3.15 ([15, Theorem 3.10]). *Let p be a prime, G a p -solvable group and H a normal subgroup of G such that G/H is p -supersolvable. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is s -permutably embedded in G , then G is p -supersolvable.*

Corollary 3.16 ([20, Theorem 3.3]). *Let p be a prime and G a p -solvable group. If there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is s -permutable in G , then G is p -supersolvable.*

Corollary 3.17. *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is weakly s -permutably embedded in G , then G is supersolvable.*

PROOF: Let p is the smallest prime divisor of $|G|$. The supersolvability of G/H implies that G/H is p -nilpotent. By Corollary 3.3, G is p -nilpotent. Furthermore G is solvable. Applying Theorem 3.12, it is easy to see that G is supersolvable. \square

REFERENCES

- [1] Kegel O.H., *Sylow-Gruppen and Subnormalteiler endlicher Gruppen*, Math. Z. **78** (1962), 205–221.
- [2] Ballester-Bolinches A., Pedraza-Aguilera M.C., *Sufficient conditions for supersolvability of finite groups*, J. Pure Appl. Algebra **127** (1998), 113–118.
- [3] Wang Y., *c -Normality of groups and its properties*, J. Algebra **180** (1996), 954–965.
- [4] Wang Y., Wei H., Li Y., *A generalization of Kramer's theorem and its application*, Bull. Austral. Math. Soc. **65** (2002), 467–475.
- [5] Skiba A.N., *On weakly s -permutable subgroups of finite groups*, J. Algebra **315** (2007), 192–209.
- [6] Robinson D.J.S., *A Course in the Theory of Groups*, Springer, New York, 1982.
- [7] Li Y., Qiao S., Wang Y., *On weakly s -permutably embedded subgroups of finite groups*, Comm. Algebra **37** (2009), 1086–1097.
- [8] Doerk K., Hawkes T., *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.
- [9] Li Y., Wang Y., Wei H., *On p -nilpotency of finite groups with some subgroups π -quasinormally embedded*, Acta. Math. Hungar. **108** (2005), 283–298.
- [10] Asaad M., Heliel A.A., *On s -quasinormally embedded subgroups of finite groups*, J. Pure Appl. Algebra **165** (2001), 129–135.
- [11] Huppert B., *Endliche Gruppen I*, Springer, Berlin, 1968.
- [12] Guo W., *The Theory of Classes of Groups*, Science Press-Kluwer Academic Publishers, Beijing-Boston, 2000.

- [13] Guo X., Shum K.P., *On c -normal maximal and minimal subgroups of Sylow p -subgroups of finite groups*, Arch. Math. **80** (2003), 561–569.
- [14] Ramadan M., Mohamed M.E., Heliel A.A., *On c -normality of certain subgroups of prime power order of finite groups*, Arch. Math. **85** (2005), 203–210.
- [15] Heliel A.A., Alharbia S.M., *The influence of certain permutable subgroups on the structure of finite groups*, Int. J. Algebra **4** (2010), 1209–1218.
- [16] Wei H., Wang Y., *On c^* -normality and its properties*, J. Group Theory **10** (2007), 211–223.
- [17] Li S., Li Y., *On s -quasionormal and c -normal subgroups of a finite group*, Czechoslovak. Math. J. **58** (2008), 1083–1095.
- [18] Schmidt P., *Subgroups permutable with all Sylow subgroups*, J. Algebra **207** (1998), 285–293.
- [19] Li Y., Qiao S., Wang Y., *A note on a result of Skiba*, Siberian Math. J. **50** (2009), 467–473.
- [20] Miao L., *On weakly s -permutable subgroups*, Bull. Braz. Math. Soc., New Series **41** (2010), 223–235.

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