# On Boman's theorem on partial regularity of mappings

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*Abstract.* Let $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ and $k$ be a positive integer. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally bounded map such that for each $(\xi, \eta) \in \Lambda$, the derivatives $D_\xi^j f(x) := \frac{d^j}{dt^j} f(x + t\xi)\Big|_{t=0}$, $j = 1, 2, \dots k$, exist and are continuous. In order to conclude that any such map $f$ is necessarily of class $C^k$ it is necessary and sufficient that $\Lambda$ be *not* contained in the zero-set of a nonzero homogenous polynomial $\Phi(\xi, \eta)$ which is linear in $\eta = (\eta_1, \eta_2, \dots, \eta_m)$ and homogeneous of degree $k$ in $\xi = (\xi_1, \xi_2, \dots, \xi_n)$.

This generalizes a result of J. Boman for the case $k = 1$. The statement and the proof of a theorem of Boman for the case $k = \infty$ is also extended to include the Carleman classes $C\{M_k\}$ and the Beurling classes $C(M_k)$ (Boman J., *Partial regularity of mappings between Euclidean spaces*, Acta Math. **119** (1967), 1–25).

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A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ that is differentiable when restricted to arbitrary differentiable curves is not necessarily differentiable as a function of several variables (see [12]). Indeed, there are discontinuous functions $f : \mathbb{R}^n \to \mathbb{R}$ whose restrictions to arbitrary analytic arcs are analytic [2]. But a $C^\infty$ function $f : \mathbb{R}^n \to \mathbb{R}$ whose restriction to every line segment is real analytic is necessarily real analytic ([13]). In [8], [9], [10] and [11] this result was extended by considering restrictions to algebraic curves and surfaces of functions belonging to more general classes of infinitely differentiable functions. It is also well known that a function $f : \mathbb{R}^n \to \mathbb{R}$ that is infinitely differentiable in each variable separately may be no better than measurable ([7]). In [4], the obverse problem is considered; for vector valued functions hypothesis is made on the source as well as the target space. In this note, Theorem 4 of [4] is generalized to $C^k$, $k \geq 1$, the class of functions that have continuous derivatives up to order $k$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally bounded map. For $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$, set

$$D_\xi \langle f, \eta \rangle (x) := \frac{d}{dt} \langle f(x + t\xi), \eta \rangle \Big|_{t=0} \quad \text{in the sense of distributions,}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^m$. By the Leibniz Integral rule, we have

$$\frac{d}{dt} \int \langle f(x + t\xi), \eta \rangle \, dx = \int \frac{d}{dt} \langle f(x + t\xi), \eta \rangle \, dx.$$

Let $k$, $1 \le k < \infty$, be fixed. For $\xi \in \mathbb{R}^n$, denote by $C_\xi^k(\mathbb{R}^n)$ the space of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the derivatives $D_\xi^j f(x) := \frac{d^j}{dt^j} f(x+t\xi)\big|_{t=0}$, $j = 1,2,\dots k$, exist and are continuous. Similarly, $C_\xi^\infty(\mathbb{R}^n) := \bigcap_{k=0}^\infty C_\xi^k(\mathbb{R}^n)$.

We are interested in finding the necessary and sufficient conditions on a subset $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ to have the following property:

$$\text{if } \; f : \mathbb{R}^n \to \mathbb{R}^m \; \text{ is locally bounded}$$
$$\text{such that } \; \langle f, \eta \rangle \in C_\xi^k(\mathbb{R}^n), \forall (\xi,\eta) \in \Lambda, \; \text{ then } \; f \in C^k(\mathbb{R}^n).$$

The case $k = 1$ and $k = \infty$ was dealt in [4].

Let $\mathbb{Z}_+^n$ denote all $n$-tuples of nonnegative integers. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_+^n$, set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The set $\mathbb{Z}_+^n$ of multi-indices is assumed to be ordered lexicographically i.e. for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}_+^n$, define $\alpha \prec \beta$ if there is $i, 1 \le i \le n$, such that $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i$.

Let $k_n = \binom{k+n-1}{k}$ denote the number of monomials of degree $k$ in $n$ variables. Then for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\int D_\xi \langle f, \eta \rangle (x) \varphi(x)\, dx = \frac{d}{dt} \int \langle f(x+t\xi), \eta \rangle \varphi(x)\, dx \Big|_{t=0} \\
&= \frac{d}{dt} \left\langle \int f(x) \varphi(x - t\xi)\, dx, \eta \right\rangle \Big|_{t=0} = \left\langle \int f(x) \frac{d}{dt} \varphi(x - t\xi)\, dx, \eta \right\rangle \Big|_{t=0} \\
&= -\sum_i \xi_i \left\langle \int f(x) \partial_i \varphi(x - t\xi)\, dx, \eta \right\rangle \Big|_{t=0} = \sum_{i,j} \xi_i \eta_j \int \partial_i f_j(x) \varphi(x)\, dx.
\end{aligned}$$

By iteration, we obtain the formula for higher-order distributional derivatives:

$$D_\xi^p \langle f, \eta \rangle (x) = \sum_{|\alpha| = p} \sum_{j=1}^m \xi^\alpha \eta_j \partial^\alpha f_j(x). \tag{1}$$

Let

$$\mathcal{B}_k := \left\{ \Phi(\xi,\eta) = \sum_{j=1}^m \sum_{|\alpha|=k} \varphi_{\alpha j} \xi^\alpha \eta_j : \varphi_{\alpha j} \in \mathbb{R}, \alpha \in \mathbb{Z}_+^n, j \in \mathbb{Z}_+ \right\}.$$

For any function $\Phi(\xi,\eta)$, set $\|\Phi\| := \max_{\|\xi\| \le 1, \|\eta\| \le 1} |\Phi(\xi,\eta)|$. For a subset $K \subset\subset \Lambda$, ($\subset\subset$ denotes the compact inclusion) put $\|\Phi\|_K := \max_{(\xi,\eta) \in K} |\Phi(\xi,\eta)|$.

**Theorem 1.** *Let $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ be a subset and $k$ be a positive integer. The following conditions are equivalent:*

(i) *$\Lambda$ is* not *contained in an algebraic hypersurface defined by an element of $\mathcal{B}_k$ i.e.*

$$\Phi \in \mathcal{B}_k,\ \Phi|_\Lambda \equiv 0 \Rightarrow \Phi \equiv 0;$$

(ii) *there exists a set consisting of* $m \cdot k_n$ *points*

$$(\xi^*, \eta^*) = \left\{ \left(\xi^{(p)}, \eta^{(p)}\right) \in \Lambda,\ p = 1, 2, \ldots, mk_n \right\} \quad \textit{such that} \quad \det \Delta\left(\xi^*, \eta^*\right) \neq 0,$$

*where*

$$\Delta\left(\xi^*, \eta^*\right) := \left[\left(\xi^{(p)}\right)^{\alpha} \eta_j^{(p)}\right]_{|\alpha|=k, 1\leq j\leq m, 1\leq p\leq mk_n};$$

(iii) *if* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is locally bounded and* $\langle f, \eta\rangle \in C^k_\xi(\mathbb{R}^n)$, $\forall\,(\xi, \eta) \in \Lambda$, *then* $f \in C^k(\mathbb{R}^n, \mathbb{R}^m)$.

*If any one of the above equivalent conditions is satisfied, then there exists a constant* $B$ *depending only on* $\Lambda$ *such that the following inequality holds for all locally bounded maps* $f : \mathbb{R}^n \to \mathbb{R}^m$*:*

$$\max_{1\leq j\leq m}\ \max_{|\alpha|=k} |\partial^\alpha f_j(x)| \leq B \cdot \sup_{(\xi,\eta)\in\Lambda} \left|D^k_\xi \left\langle f, \eta\right\rangle (x)\right|, \forall\, x \in \mathbb{R}^n. \tag{2}$$

Proof: We will prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii). Suppose $\det \Delta(\xi^*, \eta^*) = 0$ for every set of $mk_n$ elements $(\xi^*, \eta^*) = \{(\xi^{(p)}, \eta^{(p)})\}_{1\leq p\leq mk_n}$ in $\Lambda$. Fix one such set $(\xi^*, \eta^*)$ so that the rank $l := \operatorname{rank} \Delta(\xi^*, \eta^*)$ is positive. Let $\Delta^{(l)}$ denote some $l \times l$ submatrix of $\Delta(\xi^*, \eta^*)$ such that the minor $\det \Delta^{(l)}$ is nonzero. Let $\Delta^{(l+1)}$ be a $(l+1)\times(l+1)$ submatrix of $\Delta(\xi^*, \eta^*)$ that contains $\Delta^{(l)}$ as a submatrix. Replace the point $(\xi^{(p_0)}, \eta^{(p_0)})$ in $\Delta^{(l+1)}$ which does not appear in $\Delta^{(l)}$ by variables $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$. By expanding $\Delta^{(l+1)}$ along the row where the replacement took place we obtain an element

$$\Phi(\xi, \eta) = \sum_{\alpha,j} \varphi_{\alpha j} \xi^\alpha \eta_j,$$

of $\mathcal{B}_k$ which is nonzero since one of its coefficients coincides with $\det \Delta^{(l)}$ up to a sign.

Since $\Delta(\xi^*, \eta^*)$ has rank $l$, we find that $\Phi(\xi, \eta) = 0$ for all $(\xi, \eta) \in (\xi^*, \eta^*)$. If $\Phi(\xi, \eta) = 0$ for all $(\xi, \eta) \in \Lambda$, we are done. Otherwise, choose a point $(\widetilde{\xi}, \widetilde{\eta}) \in \Lambda \setminus (\xi^*, \eta^*)$ with $\Phi(\widetilde{\xi}, \widetilde{\eta}) \neq 0$.

Let $(\widetilde{\xi^*}, \widetilde{\eta^*})$ be the set which is obtained from $(\xi^*, \eta^*)$ by replacing the point $(\xi^{(p_0)}, \eta^{(p_0)})$ by $(\widetilde{\xi}, \widetilde{\eta})$. Then, the rank $\Delta(\widetilde{\xi^*}, \widetilde{\eta^*}) \geq l + 1$. By repeating above procedure, we find a sequence of subsets $(\xi^*, \eta^*)^{(i)} \subset \Lambda$, $i = 1, 2, 3, \ldots$, each with $mk_n$ elements such that the rank $\Delta(\xi^*, \eta^*)^{(j)}$ is a strictly increasing sequence of nonnegative integers. After finitely many steps we obtain a nonzero element of $\mathcal{B}_k$ which vanishes on the entire $\Lambda$.

(ii)⇒(iii). Let $(\xi^*,\eta^*)=\{(\xi^{(p)},\eta^{(p)})\in\Lambda\}_{1\le p\le mk_n}$ be a set of points such that $\det\Delta(\xi^*,\eta^*)\neq 0$. By applying Cramer's rule to (1), we get

$$\partial^\alpha f_j(x)=\sum_{p=1}^{mk_n}\frac{\det\Delta_{\alpha j}^{(p)}}{\det\Delta}D^k_{\xi^{(p)}}\left\langle f,\eta^{(p)}\right\rangle(x)\ \text{ in the distributional sense,}$$

where $\Delta_{\alpha j}^{(p)}$ denotes the cofactor obtained by deleting the $(\alpha,j)$-th row and the $p$-th column. Since $D^k_\xi\langle f,\eta\rangle\in C^0$ for all $(\xi,\eta)\in\Lambda$, we have

$$\partial^\alpha f_j(x)=\sum_{p=1}^{mk_n}\frac{\det\Delta_{\alpha j}^{(p)}}{\det\Delta}D^k_{\xi^{(p)}}\left\langle f,\eta^{(p)}\right\rangle(x)\in C^0.$$

Furthermore, there exists a constant $B=B(k,f,\Lambda)$ such that

$$|\partial^\alpha f_j(x)|\le\sum_{p=1}^{mk_n}\left|\frac{\det\Delta_{\alpha j}^{(p)}}{\det\Delta}\right|\left|D^k_{\xi^{(p)}}\left\langle f,\eta^{(p)}\right\rangle(x)\right|\le B\cdot\sup_{(\xi,\eta)\in\Lambda}\left|D^k_\xi\left\langle f,\eta\right\rangle(x)\right|,$$

for all $\alpha$ with $|\alpha|=k$, and all $j=1,2,\dots,m$.

(iii)⇒(i). Suppose (i) does not hold. Let $\Phi\in\mathcal{B}_k$ be such that $\Phi\big|_\Lambda\equiv 0$. We can write $\Phi(\xi,\eta)=\langle\varphi.(\xi),\eta\rangle$, where $\varphi.(\xi):=(\varphi_1(\xi),\varphi_2(\xi),\dots,\varphi_m(\xi))$ and $\varphi_j(\xi)=\sum_{|\alpha|=k}\varphi_{\alpha j}\xi^\alpha$, $j=1,2,\dots,m$, homogeneous polynomials of degree $k$.

Define the map

$$f(x):=\begin{cases}(\ln|\ln|x||)\,\varphi.(x) & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

Clearly $f\notin C^k$ and $f$ is $C^\infty$ in $\{x\in\mathbb{R}^n:0<|x|<1\}$. We will prove that $D^k_\xi\langle f(x),\eta\rangle$ exists at $x=0$, for all $(\xi,\eta)\in\Lambda$. It is easy to see that here are constants $C_\alpha$ such that

$$|\partial^\alpha\ln|\ln|x|||\le\frac{C_\alpha}{|x|^{|\alpha|}\,|\ln|x||},\forall\,\alpha,|\alpha|\ge 1.$$

Since the $\varphi_j(x)$'s are homogeneous polynomials of degree $k$, when the Leibniz's formula is applied to the products $(\ln|\ln|x||)\varphi_j(x)$, it is clear that all terms in $D^p_\xi\langle f(x),\eta\rangle$, $1\le p\le k$, except possibly

$$(\ln|\ln|x||)\left\langle D^k_\xi\varphi.(x),\eta\right\rangle \tag{3}$$

tend to 0 as $x\to 0$. We only need to prove that the function in (3) also tends to 0 as $x\to 0$. By expanding $(x_1+t\xi_1)^{\alpha_1}(x_2+t\xi_2)^{\alpha_2}\dots(x_n+t\xi_n)^{\alpha_n}$ binomially, we can write

$$\varphi.(x+t\xi):=\varphi.(x)+P(x,\xi,t)+\varphi.(\xi)t^k.$$

But since $(\xi,\eta) \in \Lambda$,

$$\left\langle D_\xi^k \varphi_\cdot(x), \eta \right\rangle = k! \left\langle \varphi_\cdot(\xi), \eta \right\rangle = 0.$$

It follows that $|D_\xi^p \langle f(0), \eta\rangle| = 0$ for $p \leq k$. Thus, $f \in C_\xi^k$ for all $(\xi,\eta) \in \Lambda$, but $f \notin C^k$. $\square$

**Remark 1** (cf. [6]). Suppose (i) is satisfied for all $k \geq 0$. It would be of interest to know whether there exists a constant $\rho = \rho(\Lambda)$, depending only on some appropriate notion of capacity of $\Lambda$, so that (2) is satisfied with $B = (\rho(\Lambda))^{-k}$ for all $f$ and all $k$.

**Remark 2.** Suppose $\Lambda$ satisfies (i) or (ii). The proof of Theorem 1 shows that if $f$ is continuous and $D_\xi^k\langle f, \eta\rangle = 0$, $\forall\,(\xi,\eta) \in \Lambda$, then $f$ is a polynomial. The assumption of continuity of $f$ is not necessary but our proof is valid only if $f$ is continuous (see [4]).

**Remark 3.** If $\Lambda$ satisfies (i), then $\Lambda$ contains at least $mk_n$ elements. Furthermore, if (i) holds for $k$ then (i) also holds for all $j \leq k$. Suppose there exists $\Phi \in B_j, j < k$ such that $\Phi\big|_\Lambda \equiv 0$ but $\Phi \not\equiv 0$. Then, $\xi_1^{k-j}\Phi \in \mathcal{B}_k$, $\xi_1^{k-j}\Phi\big|_\Lambda \equiv 0$ but this is a contradiction.

Let $\{M_k\}_{k=0}^\infty$, be a sequence of nonnegative numbers. For $h > 0$ and $K \subset\subset \mathbb{R}^n$ define the seminorm on $C^\infty(\mathbb{R}^n)$,

$$p_{h,K}(f) = \sup_{\alpha \in \mathbb{Z}_+^n} \sup_{x \in K} \frac{|\partial^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}.$$

The spaces

$$C\{M_k\} = \{f \in C^\infty(\mathbb{R}^n) : \forall\, K \subset\subset \mathbb{R}^n, \exists h > 0, \text{ s.t. } p_{h,K}(f) < \infty\}$$

and

$$C\,(M_k) = \{f \in C^\infty(\mathbb{R}^n) : p_{h,K}(f) < \infty, \forall\, K \subset\subset \mathbb{R}^n, \forall\, h > 0\}$$

are called the Carleman and Beurling classes, respectively. The classes $C\{(k!)^\nu\}$, $\nu > 1$, known as Gevrey classes, are especially important in partial differential equations and harmonic analysis. The class $C\{k!\}$ is precisely the class of real analytic functions.

We assume that

$$M_0 = 1 \text{ and } M_k \geq k!, \forall\, k; \tag{4}$$

$$M_k^{1/k} \text{ is strictly increasing;} \tag{5}$$

(6) $$\exists C > 0 \text{ such that } M_{k+1} \le C^k M_k,\ \forall\, k.$$

These conditions insure that the classes $C\{M_k\}$ and $C(M_k)$ are nontrivial and are closed under product and differentiation of functions. For more properties of these spaces, see [5], [11] and references therein.

It is well known that $f \in C^\infty(\mathbb{R}^n)$ if and only if $\sup_{\xi\in\mathbb{R}^n} |\xi|^j |\widehat{\chi f}(\xi)| < \infty, \forall\, \chi \in C_c^\infty(\mathbb{R}^n)$, $j \ge 1$. A similar characterization is also available for $C\{M_k\}$ (see [5]) a routine modification of which yields an analogous characterization of $C(M_k)$.

Let $r > 0$. Choose a sequence of cut-off functions $\chi_{(j)} \in C_c^\infty$, $j = 1, 2, \ldots$, such that $\chi_{(j)}(x) = 1$ if $|x - x_0| < r$, $\chi_{(j)}(x) = 0$ if $|x - x_0| > 3r$ and

$$\left|\partial^\alpha \chi_{(j)}(x)\right| \le (C_1 j)^{|\alpha|}, \forall\, j, \forall\, |\alpha| \le j, \forall\, x,$$

where the constant $C_1$ is independent of $j$.

Then $f \in C\{M_k\}$ (*resp.* $C(M_k)$) in a neighborhood of $x_0 \in \mathbb{R}^n$ if and only if there exists a constant $\hbar > 0$ (*resp.* for every $\hbar > 0$) such that

$$\sup_{\xi\in\mathbb{R}^n} \sup_{j\ge 1} \hbar^{-j} M_j^{-1} |\xi|^j |\widehat{f\chi_{(j)}}(\xi)| < \infty.$$

Call a subset $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ a determining set for bilinear forms of rank 1 if there is no nonzero bilinear form $\varphi(\xi, \eta), \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m$ of rank 1 such that $\varphi(\xi, \eta) = 0$ for all $(\xi, \eta) \in \Lambda$.

Clearly $\Lambda$ is a determining set for bilinear forms of rank 1 if and only if

$$\langle u, \xi\rangle \langle v, \eta\rangle = 0, \forall\, (\xi, \eta) \in \Lambda \Rightarrow |u||v| = 0$$

(here $\langle u, \xi\rangle$ and $\langle v, \eta\rangle$ are dot products on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively), or equivalently,

$$\bigcap_{(\xi,\eta)\in\Lambda} \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \langle u, \xi\rangle\langle v, \eta\rangle = 0\} = (\mathbb{R}^n \times 0) \cup (0 \times \mathbb{R}^m).$$

Since $\mathbb{R}[u, v]$ is a Noetherian ring, $\Lambda$ contains a finite subset $\Lambda'$ such that the sets $\{\langle u, \xi\rangle\langle v, \eta\rangle : (\xi, \eta) \in \Lambda\}$ and $\{\langle u, \xi\rangle\langle v, \eta\rangle : (\xi, \eta) \in \Lambda'\}$ generate the same ideal in $\mathbb{R}[u, v]$ and thus define the same varieties:

$$\begin{aligned}\bigcap_{(\xi,\eta)\in\Lambda} &\{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \langle u, \xi\rangle\langle v, \eta\rangle = 0\}\\ &= \bigcap_{(\xi,\eta)\in\Lambda'} \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \langle u, \xi\rangle\langle v, \eta\rangle = 0\}.\end{aligned}$$

Thus, any determining set for bilinear forms of rank 1 contains a finite determining set for bilinear forms of rank 1.

Let $C\{M_k\}(\xi)$ (*resp.* $C(M_k)(\xi)$) denote the set of all $f \in C^\infty_\xi(\mathbb{R}^n)$ such that for every subset $K \subset\subset \mathbb{R}^n$, $\sup_{j,x\in K} |D^j_\xi f(x)| \hbar^{-j} M_j^{-1} < \infty, \forall\, j$, for some $\hbar > 0$ (*resp.* for every $\hbar > 0$).

**Theorem 2.** *Let $\{M_k\}_{k=0}^\infty$ be a sequence of nonnegative numbers satisfying the conditions* (4), (5) *and* (6). *The following statements are equivalent:*

(i) *$\Lambda$ is a determining set for bilinear forms of rank* 1;

(ii) *for any locally bounded map $f : \mathbb{R}^n \to \mathbb{R}^m$,*

$$\langle \eta, f \rangle \in C\{M_k\}(\xi), \forall\, (\eta,\xi) \in \Lambda \Rightarrow f \in C\{M_k\};$$

(iii) *for any locally bounded map $f : \mathbb{R}^n \to \mathbb{R}^m$,*

$$\langle \eta, f \rangle \in C(M_k)(\xi), \forall\, (\eta,\xi) \in \Lambda \Rightarrow f \in C(M_k);$$

(iv) *for any locally bounded map $f : \mathbb{R}^n \to \mathbb{R}^m$,*

$$\langle \eta, f \rangle \in C^\infty(\xi), \forall\, (\eta,\xi) \in \Lambda \Rightarrow f \in C^\infty.$$

Proof: (cf. Theorem 4 in [4]) Assume (i) holds. By the remark above, by replacing $\Lambda$ by a subset, if necessary, we may assume $\Lambda$ is finite. Suppose for every $(\eta,\xi) \in \Lambda$, $\langle \eta, f\rangle \in C\{M_k\}(\xi)$ (resp. $\langle \eta, f\rangle \in C(M_k)(\xi)$). Now for a suitable function $f$,

$$\begin{aligned}\langle \xi, z\rangle \widehat{\langle \eta, f\rangle}(z) &= \langle \xi, z\rangle \left\langle \eta, \widehat{f}(z)\right\rangle = \left\langle \eta, i \int \left[\langle \xi, \partial_x\rangle e^{-i\langle x,z\rangle}\right] f(x)\, dx \right\rangle \\ &= \left\langle \eta, -i \int e^{-i\langle x,z\rangle} \langle \xi, \partial_x f\rangle (x) dx \right\rangle = \left\langle \eta, -i \int e^{-i\langle x,z\rangle} D_\xi f(x)\, dx \right\rangle.\end{aligned}$$

Let $g_{(j)} := f\chi_{(j)} \in C\{M_k\}$ near a fixed point $x_0$. Assume, without loss of generality, $x_0 = 0$. By assumption, for all $(\xi,\eta) \in \Lambda$ there exist constants $C = C_{\xi\eta}$ and $\hbar = \hbar_{\xi\eta} > 0$ (*resp.* for all $(\xi,\eta) \in \Lambda$ and for all $\hbar > 0$ there exists a constant $C = C_{\xi\eta,\hbar}$) such that

$$\begin{aligned}\left|\widehat{\langle \eta, g_{(j)}\rangle}(\zeta)\right| |\langle \xi, \zeta\rangle|^j = \left|\langle \eta, \widehat{g_{(j)}}(\zeta)\rangle\right| |\langle \xi,\zeta\rangle|^j \le C\hbar^j M_j,\\ \forall\ (\xi,\eta) \in \Lambda, \zeta \in \mathbb{R}^n, j \in \mathbb{Z}_+.\end{aligned}$$

The function

$$\mathbb{R}^n \times \mathbb{R}^m \ni (u,v) \to \sum_{(\xi,\eta)\in\Lambda} |\langle \eta, v\rangle|\, |\langle \xi, u\rangle|^l, \tag{7}$$

is homogeneous of degree 1 in $v$, of homogeneous degree $l$ in $u$. Since none of the terms $|\langle \eta, v||\langle \xi, u\rangle|$ can vanish on all of $\Lambda$, the function in (7) has a positive minimum on the compact set $\{(u,v) : |u| = 1, |v| = 1\}$. Thus, there is an $\varepsilon > 0$ such that

$$\sum_{(\xi,\eta)\in\Lambda} |\langle \eta, v\rangle| \, |\langle \xi, u\rangle|^l \geq \varepsilon |v| |u|^l,$$

(see [Lemma 1][4]). Applying this to $u = \zeta$, $v = \widehat{g_{(j)}}(\zeta)$, we get

$$\left|\widehat{g_{(j)}}(\zeta)\right| |\zeta|^l \leq \varepsilon^{-1} \sum_{(\xi,\eta)\in\Lambda} \left|\langle \eta, \widehat{g_{(j)}}(\zeta)\rangle\right| |\langle \xi, \zeta\rangle|^l \leq C\hbar^j M_j,$$

where $\hbar = \max_{(\xi,\eta)\in\Lambda} \hbar_{\xi\eta}$ (*resp.* for all $\hbar > 0$) and $C = \varepsilon^{-1} \sum_{(\xi,\eta)\in\Lambda} C_{\xi\eta}$. Thus (ii) and (iii) hold. By setting $\hbar = 1$ and $M_j = 1, \forall j$, in the above argument, it is clear that (iii) holds as well.

Conversely if $\Lambda$ is not a determinant set for bilinear forms of rank 1, there exist $u \neq 0$ and $v \neq 0$ such that

$$\langle u, \xi\rangle \langle v, \eta\rangle = 0, \ \forall \ (\xi, \eta) \in \Lambda.$$

Let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous function. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined as $f(z) = h(\langle u, z\rangle) \cdot v$. Then

$$\left(\frac{d}{dt} \langle \eta, f(z + t\xi)\rangle\right)\Big|_{t=0} = \langle \eta, v\rangle \langle u, \xi\rangle h'(\langle u, z + t\xi\rangle)|_{t=0} \equiv 0.$$

Thus $\langle \eta, f\rangle \in C(M_k)(\xi) \subset C\{M_k\}(\xi) \subset C^\infty(\xi), \forall (\xi, \eta) \in \Lambda$ but $f$ need not be even differentiable. □

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