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A game and its relation to netweight and D-spaces

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Abstract. We introduce a two player topological game and study the relationship of the existence of winning strategies to base properties and covering properties of the underlying space. The existence of a winning strategy for one of the players is conjectured to be equivalent to the space have countable network weight. In addition, connections to the class of D-spaces and the class of hereditarily Lindelöf spaces are shown.

Keywords: topological game, network, netweight, weakly separated, D-space

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1. Introduction

Let us introduce two closely related topological games: Given a space X we let $G(X)$ (resp., $G'(X)$) denote the following two player game of length ω on X played by SET and POINT. In the first inning of the game:

SET plays $D_0 \subseteq X$ and a neighborhood assignment $\{V_x : x \in D_0\}$, and

POINT plays $x_0 \in D_0$.

A play of the game is a sequence $D_0, x_0, \ldots D_n, x_n, \ldots$, where at inning n of the game

SET plays $D_n \subseteq D_{n-1}$, and

POINT plays $x_n \in D_n$.

Let $D = \bigcap\{D_n : n \in \omega\}$. We say *POINT wins in* $G(X)$ if

$$\bigcup\{V_{x_n} : n \in \omega\} \supseteq \bigcup\{V_x : x \in D\}$$

and *POINT wins in* $G'(X)$ if

$$\bigcup\{V_{x_n} : n \in \omega\} \supseteq D.$$

Otherwise *SET wins*.

The games $G(X)$ and $G'(X)$ originated in an attempt to understand the relationship between hereditarily Lindelöfness and the D-space property. A T_1 space X is said to be a *D-space* if for each open neighborhood assignment $\{U_x : x \in X\}$ there is a closed and discrete subset $D \subseteq X$ such that $\{U_x : x \in D\}$ covers the space. The notion is due to van Douwen, first studied with Pfeffer in [4], and the open question whether every regular Lindelöf space is a D-space has been attributed to van Douwen [6]. Indeed very few examples of regular spaces with even very weak covering properties that are not D-spaces are known. Recently a Hausdorff example of a hereditarily Lindelöf space that is not a D-space was constructed assuming $\diamondsuit$ [8]. However, it may be consistent or even a ZFC result that every hereditarily Lindelöf regular space is a D-space.

While the games are closely related to the van Douwen question, a perhaps more interesting question is whether POINT having a winning strategy in $G(X)$ or $G'(X)$ is equivalent to X having countable network weight. Consideration of this question leads us to a generalization of the notion of weakly separated subsets of a space and to an open question of M. Tkachenko. Recall that a subset Y of a space X is *weakly separated* if there is a neighborhood assignment $\{V_y : y \in Y\}$ such that for all $y \neq z$ from Y, if $y \in V_z$ then $z \notin V_y$. Tkachenko asked whether it is consistent that every space with no uncountable weakly separated subspaces has countable network weight [9].

2. Main results

Lemma 1. *Suppose SET has no winning strategy in $G'(X)$. Then SET has no winning strategy in $G'(Y)$ in any subspace Y of X.*

Proof: Suppose SET has no winning strategy in $G(X)$, let Y be a subspace of X, and let σ be a strategy for SET in $G(Y)$. We show σ is not winning by defining a corresponding strategy σ^* in $G(X)$ such that σ^* not winning implies σ not winning. Let D_0 and $\{V_x : x \in D_0\}$ be SET's initial play in $G(Y)$ using the strategy σ. For each $x \in D_0$, let V_x^* be an open neighborhood of x in X such that $V_x^* \cap Y = V_x$, and define D_0 and $\{V_x^* : x \in D_0\}$ to be SET's initial play in $G(X)$ using the strategy σ^*. Then if $x_0 \in D_0$ is POINT's initial play in $G(X)$, let D_1 be SET's response using σ if SET pretends x_0 is POINT's play in $G(Y)$, and let this same set D_1 be SET's response to x_0 in $G(X)$. And so on. Since σ^* is not winning, there is a sequence $x_0, x_1, \dots$ of plays by POINT in $G(X)$ such that

$$\bigcup\{V_{x_n}^* : n \in \omega\} \supseteq D$$

where $D = \bigcap_{n\in\omega} D_n$. But then this same sequence of plays wins for POINT in $G(Y)$. Hence σ is not winning. □

Theorem 2. *If SET has no winning strategy in $G(X)$ or $G'(X)$, then X is hereditarily Lindelöf and hereditarily a D-space.*

Proof: First note that if SET has no winning strategy in $G(X)$, then SET has no winning strategy in $G'(X)$ either, since a win for SET in $G'(X)$ is a win in $G(X)$ too. Thus it suffices to show that SET having no winning strategy in $G'(X)$ implies X is hereditarily a Lindelöf D-space.

Suppose then that X has no winning strategy in $G'(X)$. By the lemma, we only need to prove X is Lindelöf and a D-space, which we do by showing that if $\{U_x : x \in X\}$ is a neighborhood assignment, then there is a countable closed discrete set D such that $\{U_x : x \in D\}$ covers X. Consider the strategy for SET, with initial play $D_0 = X$ and the given neighborhood assignment, and where at the nth inning, SET plays $D_n = X \setminus \bigcup_{i<n} U_{x_i}$ where $(x_i : i < n)$ is the sequence of POINT's plays up to that point. Since this strategy is not winning for SET, there is a sequence of points $\{x_n : n \in X\}$ such that $x_n \notin \bigcup\{U_{x_i} : i < n\}$ and $\{U_{x_n} : n \in \omega\}$ covers all of X. It follows that if $D = \{x_n : n \in \omega\}$, then D is closed discrete and $\{U_x : x \in D\}$ covers X. □

Proposition 3. *If X has a countable network, then POINT has a winning strategy in $G(X)$.*

Proof: Let $\mathcal{F} = \{F_n : n \in \omega\}$ be a network for X. We describe a strategy for POINT. Suppose that SET plays $D_0 \subseteq X$ and $\{V_x : x \in D_0\}$. Then POINT plays some $x_0 \in D_0$ such that $V_{x_0} \supset F_{n_0}$, where n_0 is least possible. At inning $k > 0$ of the game, choose n_k minimal such that $n_k \notin \{n_i : i < k\}$ and there is an $x_k \in D_k$ with $V_{x_k} \supset F_{n_k}$; then POINT plays x_k. To see that POINT wins this play of the game, let $D = \bigcap_{n\in\omega} D_n$ and let $y \in \bigcup\{V_x : x \in D\}$. Then for some $m \in \omega$ and $x \in D$, $y \in F_m \subset V_x$. Then by the way the n_i's were chosen, we must have $m = n_k$ for some k, and hence $y \in \bigcup_{i\in\omega} V_{x_i}$. So POINT wins the game. □

We conjecture that the converse to Proposition 3 is also true:

Question 1. *If POINT has a winning strategy in $G(X)$ or $G'(X)$, does it follow that X has a countable network?*

As indicated by this question, we also do not know of a space X in which POINT has a winning strategy in $G'(X)$ but not in $G(X)$. A counterexample to Question 1 would need to be a hereditarily Lindelöf space without a countable network. Most (all?) known examples of such spaces can be shown to have the property that POINT does not have a winning strategy. Indeed, this is closely related to the Tkachenko's question whether consistently every space with no uncountable weakly separated subset has a countable network ([9]; see also Problem 378, [7]). The following generalization of weak separation will help us show that POINT has no winning strategy in certain examples of hereditarily Lindelöf spaces.

Definition 4. A subset A of a hereditarily Lindelöf topological space (X, τ) is dually weakly separated, if there is another hereditarily Lindelöf topology τ' on X and two neighborhood assignments $\{V_x : x \in A\} \subseteq \tau$ and $\{W_x : x \in A\} \subseteq \tau'$ such that

(1) $x \in V_x \cap W_x$ for all $x \in A$, and
(2) for all $x \neq y$ in A, if $y \in W_x$ then x is not in the τ' closure of V_y.

Note that if $\tau = \tau'$ in the previous definition then we obtain, for regular spaces, a statement equivalent to "A is weakly separated".

Proposition 5. *Suppose POINT has a winning strategy in $G'(X)$ on a space (X, τ). Then no uncountable subset of X is dually weakly separated with respect to any hereditarily Lindelöf topology τ'.*

Proof: Suppose that σ is a strategy for POINT, and by Theorem 2 we may assume that (X,τ) is hereditarily Lindelöf. By way of contradiction suppose that τ' is another hereditarily Lindelöf topology on X and $A \subseteq X$ is uncountable and $\{V_x : x \in A\} \subseteq \tau$ and $\{W_x : x \in A\} \subseteq \tau'$ witness that A is dually weakly separated.

Fix M an elementary submodel of some H_κ for κ sufficiently large such that M contains everything relevant. Fix $z \in X \setminus M$. For each $x \in X$, let y_x be POINT's response to an opening play of $D_0^x = W_x \setminus \{x\}$. Let $U_x = (W_x \setminus \overline{V_{y_x}})$ (here the closure is taken wrt τ'. The sets U_x form a τ'-open cover of X, so has a countable subcover $\{U_x : x \in A_0\}$. By elementarity we may assume that $A_0 \in M$ and since it is an open cover, we may find $x_0 \in A_0$ such that $z \in U_{x_0}$ and so $z \in D_0^{x_0}$. For each $x \in D_0^{x_0}$ let $D_1^x = D_0^{x_0} \cap (W_x \setminus \{x\})$ and let y_x^1 be POINT's response to this play where POINT follows its strategy σ. By assumption, we have that the sets $U_x^1 = W_x \setminus \overline{V_{y_x^1}}$ form a τ'-open cover of $D_0^{x_0}$. Since X must be hereditarily Lindelöf, it follows that we have a countable A_1 such that $\{U_x : x \in A_1\}$ covers $D_0^{x_0}$. By elementarity, we may assume that $A_1 \in M$, and may find $x_1 \in A_1$ with $z \in U_x^1$. It follows that $z \in D_1^{x_1}$. Continuing in this fashion we find a sequence of plays by SET of the form $D_n = D_n^{x_n} = D_{n-1}^{x_{n-1}} \cap W_{x_n} \setminus \{x_n\}$ with the property that for all n, $z \in D_n$ and $z \notin V_{y_{x_n}^n}$ where $y_{x_n}^n$ is POINT's response to this play D_n. This implies that the play is losing for POINT, so POINT does not have a winning strategy. □

Proposition 5 can be used to show that POINT has no winning strategy on many interesting examples of heredetarily Lindelöf spaces: For example, for any space with an uncountable weakly separated subspace (e.g., any uncountable subspace of the Sorgenfrey line or any L-space), POINT has no winning strategy.

There are consistent examples of hereditarily Lindelöf spaces with no uncountable weakly separated subspaces, yet using Proposition 5 we can see that POINT has no winning strategy.

Example 1. We recall an example mentioned in [5, p. 303]. An uncountable set of reals E is called 2-*entangled* if every uncountable monotone function from a subset of E to E has a fixed point. Such sets exist assuming CH and are consistent with MA+$\neg$CH [2]. Now let f be any uncountable one-to-one function from a subset of E to E with no fixed point, and consider the plane with the topology τ refining the usual Euclidean topology by adding "bowtie" neighborhoods of the form $V_{(x_1,x_2)} = \{y : y_1 \leq x_1 \text{ and } x_2 \leq y_2 \text{ or } y_1 \geq x_1 \text{ and } x_2 \geq y_2\}$. Let X be the graph of f as a subspace of the plane with this topology, and let X' be the graph of f with the topology τ' obtained by rotating the bowtie neighborhoods by 90 degrees. Both X and X' are hereditarily Lindelöf, but neither has a countable network because $\{(x,x) : x \in f\}$ is easily seen to be a discrete subspace of $X \times X'$. Now note that if $B(x)$ is a bowtie neighborhood of x in τ, and $B'(x)$ its rotation by 90^o, then $\{(B(x), B'(x)) : x \in f\}$ witnesses that X is dually weakly separated. So POINT has no winning strategy in $G'(X)$.

Example 2. K. Ciesielski constructed an example of space with network weight ω_2 but any subspace of cardinality ω_1 has a countable network [3]. Clearly, no uncountable subset of this space could be weakly separated, however, the entire space is dually weakly separated. The example is obtained by forcing a generic graph on $F : [\omega_2]^{\leq 2} \to 2$ with the stipulation that $F(\{x\}) = 0$ for every $x \in X = \omega_2$. Then $\tau = \tau_F$ is the topology obtained by taking the sets $U^F_{x,i} = \{y : F(\{x,y\}) = i\}$ as a subbasis. Ciesielski constructs a further forcing extension where this topology is the required example. To see that this space is dually weakly separated, define another function $G : [\omega_2]^{\leq 2} \to 2$ by $G(\{x,y\}) = F(\{x,y\})$ for all $x \neq y$ and $G(\{x\}) = 1$ for all $x \in X$. Defining a subbasis with respect to G in the same way, one obtains an alternate topology $\tau' = \tau_G$. The proof that τ' is hereditarily Lindelöf is the same as Ciesielski's proof for τ. Note that $U^G_{x,1} = (X \setminus U^F_{x,0}) \cup \{x\}$. So, $U^G_{x,1}$ is τ-closed. By symmetry, it also follows that each $U^F_{x,0}$ is τ'-closed. Also, if $y \in U^G_{x,1}$ and $x \neq y$ then $F(\{x,y\}) = G(\{x,y\}) = 1$, so $y \notin U^F_{x,0}$. So y is not in the τ' closure of $U^F_{x,0}$. Therefore the sets $W_x = U^G_{x,1}$, $V_x = U^F_{x,0}$ form a dual weak separation of X.

Question 2. *If a hereditarily Lindelöf space includes no uncountable dually weakly separated subset, must it have a countable network?*

If so, then POINT having a winning strategy implies countable network weight.

Finally, we point out that being hereditarily Lindelöf is not characterized by SET not having a winning strategy:

Proposition 6. *SET has a winning strategy on the Sorgenfrey line.*

PROOF: For each $x \in \mathbb{R}$, let $U_x = [x, \infty)$. Let SET play as follows: $D_0 = (0, \infty)$ and $\{U_x : x \in (0, \infty)\}$ is the opening play. Assume that in the nth inning, SET and POINT have played a sequence $\{D_i, x_i : i \leq n\}$ such that $D_i = (y_i, x_{i-1})$ where $0 = y_0 < y_1 < \cdots < y_n < x_{n-1} < \cdots < x_0$. Then if point responds by choosing $x_n \in D_n = (y_n, x_{n-1})$, SET responds with $D_{n+1} = (y_{n+1}, x_n)$. Using compactness, it is easy to see that this is a winning strategy for SET. □

Of course, the square of the Sorgenfrey line is not Lindelöf. And, moreover, the example of [8] is a T_2 example of a space with the property that every subspace has each finite power Lindelöf, but it is not a D-space. This raises the natural question whether X^ω being hereditarily Lindelöf implies that X is a D-space, or even more:

Question 3. *If X is regular and X^ω is hereditarily Lindelöf, is it the case that SET has no winning strategy in $G(X)$?*

Of course, if X^n is hereditarily Lindelöf for each n, then so is X^ω, however, the assumptions of the following question might be weaker than the previous.

Question 4. *Suppose X that is regular and for every subspace $Y \subseteq X$, we have every finite power of Y is Lindelöf. Does it follow that SET has no winning strategy in $G(X)$?*

If we only assume Hausdorff in the previous question then we have a consistent negative answer [8].

The Star Game. Analyzing Arhangel'skii and Buzyakova's proof that spaces with a point countable base are D-spaces, L. Aurichi defined a topological game, called *the star game*, as follows (see [1]). Given a space X with basis $\mathcal{B}$, PLAYER I chooses $x_0 \in X$ and PLAYER II chooses $A_0 \subseteq X$ and basic open sets $\{V_x : x \in A_0 \cup \{x_0\}\}$ such that $x \in V_x$ for each x. At stage α, having chosen $\{x_\xi : \xi < \alpha\}$ and $\{A_\xi : \xi < \alpha\}$:

If $\{x_\xi : \xi < \alpha\}$ is not closed discrete, then I wins if $\bigcup_{\xi<\alpha} A_\xi$ does not include all limit point of $\{x_\xi : \xi < \alpha\}$, otherwise II wins.

If $\{x_\xi : \xi < \alpha\}$ is closed discrete and $\{V_{x_\xi} : \xi < \alpha\}$ covers X, then I wins.

Otherwise, the game continues and I chooses $x_\alpha \in X \setminus \{x_\xi : \xi < \alpha\}$ and II chooses A_α along with neighborhoods $V_x \in \mathcal{B}$ for each $x \in \{x_\alpha\} \cup A_\alpha$ subject to the rule that if $x \in (\{x_\alpha\} \cup A_\alpha) \cap (\bigcup_{\xi<\alpha} A_\xi)$ then V_x fixed at stage α is the same as the V_x chosen in the previous stage of the game.

Theorem 7 ([1])**.** *If X has a point countable base, then PLAYER I has a winning strategy in the star game. If PLAYER II has no winning strategy in the star game on a space X then X is a D-space.*

Theorem 8. *If POINT has a winning strategy in the game $G'(X)$, then PLAYER II has no winning strategy in the star game.*

PROOF: Suppose that POINT has a winning strategy in the game $G'(X)$, and PLAYER II in the star game employs some fixed strategy. We define a response by PLAYER I that will defeat this strategy. Let $f : \omega \to \omega$ be any function such that $f(n) < n$ for all $n > 0$, and $f^{-1}(k)$ is infinite for all $k \in \omega$.

In inning $n = 0$, PLAYER I plays any $x_0 \in X$. Let A_0 and the neighborhood assignment $\{V_x : x \in A_0 \cup \{x_0\}\}$ be PLAYER II's response following her strategy.

Now consider $A_0 \setminus V_{x_0}$ with the neighborhood assignment given from II's move in the star game as SET's first move in a game $G'(X)$, which we will call the 0*th auxiliary game*, and let x_1 be POINT's reply in $G'(X)$ to this move using her winning strategy, and let it also be I's reply to II's first move in the star game.

At stage $n > 0$ of the star game, we have a partial play $x_0, A_0, x_1, \dots, x_{n-1}, A_{n-1}$. We have also defined partial plays (some of which may be empty) ending in a move of POINT in n auxiliary games of type $G'(X)$. We will also have the neighborhoods V_{x_i}, $i < n$, chosen by II's strategy, and I's plays $\{x_i\}_{i<n}$ will always be such that $x_i \notin \bigcup_{j<i} V_{x_j}$.

Define I's response x_n to this partial play as follows. Suppose $f(n) = k$. We then extend the kth auxiliary game by one round. If it has not started yet, let $A_k \setminus \bigcup_{i<n} V_{x_i}$ with the neighborhood assignment given from the star game be SET's first move in $G'(X)$. If it has started, and B is SET's last move in that game, then let $B \setminus \bigcup_{i<n} V_{x_i}$ be SET's next move. Now let x_n be POINT's reply in $G'(X)$ as well as I's reply to the given partial play of the star game. (If SET's

move defined as above happens to be empty, then let x_n be an arbitrary element of $X \setminus \bigcup_{i<n} V_{x_i}$.)

Note that at stage ω all the auxiliary games will have been completed, and every play by POINT in these games will be among the x_n's. Since POINT used her winning strategy, and SET's plays in the kth auxiliary game have the form A_k minus some finite union of the V_{x_i}'s, it follows that $A_k \subset \bigcup_{n\in\omega} V_{x_n}$ for all $k \in \omega$.

Since $x_n \notin \bigcup_{i<n} V_{x_i}$, any limit point of the x_n's lies outside of $\bigcup_{n\in\omega} V_{x_n}$. Since $\bigcup_{n\in\omega} V_{x_n}$ contains all of the A_n's, if $\{x_n\}_{n\in\omega}$ has a limit point, it is not in any A_n and hence Player I has won the game. If on the other hand $\{x_n\}_{n\in\omega}$ is closed discrete, then either the V_{x_n}'s cover X, in which case I again wins, or the game continues.

If the game continues, for the next ω rounds Player I continues similarly to the first ω rounds. That is, I first chooses any $x_\omega \in X \setminus \bigcup_{n\in\omega} V_{x_n}$. II plays A_ω and a neighborhood assignment $\{V_x : x \in A_\omega \cup \{x_\omega\}\}$. Then consider $A_\omega \setminus \bigcup_{n\leq\omega} V_{x_n}$ with the neighborhood assignment given from II's move in the star game as SET's first move in the ωth auxiliary game $G'(X)$, and let $x_{\omega+1}$ be POINT's reply in $G'(X)$ to this move using her winning strategy, and let it also be I's reply to II's ωth move in the star game. And so on out to stage $\omega+\omega$. If the game is still not over, continue in like manner.

Since we are assuming POINT has a winning strategy in $G'(X)$, X is hereditarily Lindelöf and the game must end at some countable stage α. If $\{x_\beta\}_{\beta<\alpha}$ is closed discrete and the game is over, then I has won. If $\{x_\beta\}_{\beta<\alpha}$ has a limit point, then since $x_\beta \notin \bigcup_{\gamma<\beta} V_{x_\gamma}$, and for $\beta < \alpha$, $\{x_\gamma\}_{\gamma<\beta}$ is closed discrete, any limit point of the x_β's lies outside of $\bigcup_{\beta<\alpha} V_{x_\beta}$. But then said limit point cannot be in any A_β since (arguing as in the first ω rounds) A_β is covered by the V_{x_γ}'s, $\gamma < \alpha$. So I wins again and Player II's strategy is defeated. □

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