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Nonmonotone nonconvolution functions of positive type and applications

Tomáš Bárta

Abstract. We present two sufficient conditions for nonconvolution kernels to be of positive type. We apply the results to obtain stability for one-dimensional models of chemically reacting viscoelastic materials.

Keywords: functions of positive type, nonconvolution integral equation, chemically reacting viscoelastic fluid

Classification: 42A82, 45A05, 45M05, 76A10

1. Introduction

Recently, models of chemically reacting fluids have been studied by several authors. See Bulíček, Málek and Rajagopal [2] for a general existence result and references therein for other works on this topic. The model studied in [2] is of the form

$$(1)\quad \operatorname{div} v = 0, \qquad v_t + \operatorname{div}(v \otimes v) = \operatorname{div} S + f - \nabla p, \qquad c_t + \operatorname{div}(vc) = -\operatorname{div} q_c,$$

where v is the velocity of the fluid, c is concentration of a chemical, p pressure, S the stress tensor, q_c heat flux and f an external force. A model for viscoelastic materials was proposed by Rajagopal and Wineman in [10]. According to [10], the viscoelastic part of the stress tensor depends on the concentration in the following way

$$(2)\quad \int_0^t a(c(t,x), t-s)\nabla v(s,x)\,ds, \text{ in particular } \int_0^t e^{-\lambda(c(t,x))(t-s)}\nabla v(s,x)\,ds,$$

where λ is a positive function. So, we obtain an integrodifferential equation with a nonconvolution kernel. Another situation where such equations appear are models of aging of materials (see Rajagopal and Wineman [9]).

In the theory of integral and integrodifferential equations, kernels of positive type (sometimes called positive definite) play an important role. See Gripenberg, Londen and Staffans [4], Chapter 3, 17 and 20, Prüss [8], Chapter 3 and 7, or Renardy, Hrusa and Nohel [11], Chapter IV.4. For more recent results see Cannarsa

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and Sforza [3] or Tatar [12], for nonconvolutionary case see Halanay [5], Kiffe [6], or Mustapha and McLean [7].

Therefore, we present two sufficient conditions for nonconvolution kernels to be of positive type. Let us mention that sufficient conditions yielding positive definiteness are usually based on monotonicity of the kernel and its derivatives. It is also partially the case of our first condition (Theorem 2.1). However, the second result (Theorem 2.4) needs no monotonicity. It says that if a nonconvolution kernel is a small perturbation of a convolution kernel of strong positive type, then it is itself of (strong) positive type. This is more appropriate for the systems like (1), where no monotonicity of c can be required.

The main abstract results are contained in Section 2 (Theorems 2.1 and 2.4). In Section 3 we show stability resp. exponential stability for two one-dimensional models of chemically reacting viscoelastic materials.

2. Nonconvolution functions of positive type

In this section we give two sufficient conditions for a nonconvolution kernel a to be of positive type.

Let us remind that a convolution kernel $b : \mathbb{R}_+ \to \mathbb{R}$ ($\mathbb{R}_+ = [0, +\infty)$) is called to be *of positive type*, if for every $T > 0$ and every $w \in L^2([0,T])$ the inequality

$$\int_0^T w(t) \int_0^t b(t-s)w(s)\,ds\,dt \geq 0 \tag{3}$$

holds. All positive nonincreasing convex functions are of positive type, but there are other functions of positive type that do not satisfy these monotonicity assumptions (for example $\cos t$ or $e^{-t}\cos t$). Function b is called to be *of strong positive type*, if there exists $\varepsilon > 0$ such that $t \mapsto b(t) - \varepsilon e^{-t}$ is of positive type. For example, $e^{-\delta t}$, $e^{-t}\cos t$ are of strong positive type.

For nonconvolution kernels one usually requires

$$\int_0^T w(t) \int_0^t k(t,s)w(s)\,ds\,dt \geq 0.$$

However, since our applications contain integral terms of the form

$$\int_0^t a(t, t-s)w(s)\,ds,$$

we will use the following definition.

Definition 2.1. Denote by D_T the set $\{(t,s) \in \mathbb{R}^2 : \ 0 \leq s \leq t \leq T\}$ and $D := \{(t,s) \in \mathbb{R}^2 : \ 0 \leq s \leq t\}$. A function $a \in L^2_{\mathrm{loc}}(D)$ is called to be *of c-positive type* if

$$\int_0^t w(s) \int_0^s a(s, s-\sigma)w(\sigma)\,d\sigma\,ds \geq 0$$

for all $t > 0$ and $w \in L^2([0,t])$. A function $a \in L^2_{\text{loc}}(D)$ is called to be *of ε-strong c-positive type* if $(t,s) \mapsto a(t,s) - \varepsilon e^{-s}$ is of c-positive type. The function a is called to be *of strong c-positive type* if it is of ε-strong c-positive type for some $\varepsilon > 0$.

Obviously, there is a correspondence with the usual definition: Function $a \in L^2_{\text{loc}}(D)$ is of c-positive type if and only if $k(t,s) := a(t,t-s)$ is of positive type. Observe further that a function $b \in L^2(\mathbb{R}_+)$ is of positive type, if and only if $a(t,s) := b(s)$ is of c-positive type. If a function b is defined on $[0,S]$ only, we say that it is of positive type, if the extension of b to $\mathbb{R}_+$ by 0 is of positive type (i.e., the inequality (3) holds for all $T \le S$).

Consider the following assumptions on $a \in X(D)$, where

$$X(D) := \{a \in L^2_{\text{loc}}(D) : \ \partial_1 a \in L^2_{\text{loc}}(D)\}$$

(∂_j denotes the derivative with respect to j's variable).

(a1) There is $\varepsilon > 0$ such that for every $T > 0$, $a(T,\cdot)$ is of ε-strong positive type,

(a2) for every $T > 0$, $-\partial_1 a(T,\cdot)$ is of positive type.

Theorem 2.1. *Let $a \in X(D)$ satisfy* (a1), (a2). *Then a is of ε-strong c-positive type.*

PROOF: Let us take $\varepsilon > 0$ from (a1) and write

$$a(t,r) - \varepsilon e^{-r} = a(T,r) - \varepsilon e^{-r} - \int_t^T \partial_1 a(s,r)\, ds.$$

Using this equality we get

$$\begin{aligned} &\int_0^T w(t) \int_0^t [a(t,t-\sigma) - \varepsilon e^{-(t-\sigma)}] w(\sigma)\, d\sigma\, dt \\ &= \int_0^T w(t) \int_0^t \left[a(T,t-\sigma) - \varepsilon e^{-(t-\sigma)} - \int_t^T \partial_1 a(s,t-\sigma)\, ds \right] w(\sigma)\, d\sigma\, dt \\ &= \int_0^T w(t) \int_0^t [a(T,t-\sigma) - \varepsilon e^{-(t-\sigma)}] w(\sigma)\, d\sigma\, dt \\ &\quad + \int_0^T \int_t^T w(t) \int_0^t -\partial_1 a(s,t-\sigma) w(\sigma)\, d\sigma\, ds\, dt =: I_1 + I_2. \end{aligned} \tag{4}$$

Here I_1 is nonnegative by (a1) and I_2 is by Fubini's Theorem equal to

$$\int_0^T \int_0^s w(t) \int_0^t -\partial_1 a(s,t-\sigma) w(\sigma)\, d\sigma\, dt\, ds.$$

This expression is nonnegative by (a2). ∎

Corollary 2.2. *Let $a \in X(D)$ satisfy* (a1) *with $\varepsilon = 0$ and* (a2). *Then a is of c-positive type.*

Example 2.3. Let $a_1 \in W^{1,2}_{\mathrm{loc}}(I)$ be positive and decreasing (nonincreasing), $a_2 \in L^2_{\mathrm{loc}}(I)$ be of positive type and $a(t,s) := a_1(t)a_2(s)$. Then a is of c-positive type. Moreover, if $a_1 \geq \delta$ for some $\delta > 0$ and a_2 of strong positive type, then a is of strong c-positive type. In particular, we can take $a_2(s) := e^{-\lambda s}$, $a(t,t-s) = a_1(t)e^{-\lambda(t-s)}$, $\lambda > 0$.

Consider a second system of assumptions on $a \in Y$, where

$$Y := \{a : \mathbb{R}^2_+ \to \mathbb{R} :\ a, \partial_1 a \in L^2_{\mathrm{loc}}(D),\ \partial_1 a,\ \partial_2^2\partial_1 a \in L^1_{\mathrm{loc}}(\mathbb{R}_+, L^1(\mathbb{R}_+))\}.$$

By $\|\cdot\|_1$ we denote the norm in $L^1(\mathbb{R}_+)$. Assume that

(A) There exists $\delta > 0$ and for every $T > 0$ there exists $\varepsilon(T) > 0$ and v_1, v_2, $v_3 \in L^1([0,T])$ such that
- (A1) $a(T,\cdot)$ is of $\varepsilon(T)$-strong positive type,
- (A2) for a.a. $t \in [0,T]$, $\partial_1 a(t,0) = \lim_{s\to+\infty} \partial_1 a(t,s) = 0$,
- (A3) for a.a. $t \in [0,T]$, $|\partial_2\partial_1 a(t,0)| \leq v_1(t)$, $\lim_{s\to+\infty} \partial_2\partial_1 a(t,s) = 0$,
- (A4) for a.a. $t \in [0,T]$, $\|\partial_1 a(t,\cdot)\|_1 \leq v_2(t)$, $\|\partial_2^2\partial_1 a(t,\cdot)\|_1 \leq v_3(t)$,
- (A5) $\int_0^T v_1(t) + v_2(t) + v_3(t)\,dt + \delta \leq \varepsilon(T)$.

If $a(t,t-s) = \tilde{a}(t-s)$ is independent on the first variable, it is a convolution kernel and assumptions (A2)–(A5) are satisfied trivially. Assumptions (A2)–(A5) mean that $\partial_1 a$ is small, so a is a small perturbation of a convolution kernel.

Theorem 2.4. *If $a \in Y(D)$ satisfies* (A), *then a is of δ-strong c-positive type.*

In the proof we use the same computations as in the proof of Theorem 2.1. But this time, the integral I_2 in (4) can be negative. However, we show that I_2 in (4) is dominated by I_1, so their sum is nonnegative. We start with the following two lemmas.

Lemma 2.5. *Let $b \in W^{2,1}(\mathbb{R}_+)$ satisfy $b(0) = \lim_{t\to+\infty} b(t) = \lim_{t\to+\infty} b'(t) = 0$. Then*

$$(1+\omega^2)|\hat{b}(i\omega)| \leq |b'(0)| + \|b\|_1 + \|b''\|_1 \quad \text{for all } \omega \in \mathbb{R}.$$

Proof: We have

$$\omega^2\hat{b}(i\omega) = i\omega(-i\omega)\int_0^{+\infty} e^{-i\omega s} b(s)\,ds = i\omega\left[[e^{-i\omega s}b(s)]_0^{+\infty} - \int_0^{+\infty} e^{-i\omega s} b'(s)\,ds\right]$$
$$= -i\omega\int_0^{+\infty} e^{-i\omega s} b'(s)\,ds = [e^{-i\omega s}b'(s)]_0^{+\infty} - \int_0^{+\infty} e^{-i\omega s} b''(s)\,ds.$$

Hence,

$$\omega^2|\hat{b}(i\omega)| \leq |b'(0)| + \int_0^{+\infty} |b''(s)|\,ds.$$

Since $|\hat{b}(i\omega)| \leq \|b\|_1$, the assertion follows. □

Lemma 2.6. *Let $a \in L^2([0,T])$ and for each $s \in [0,T]$, $b_s(\cdot) \in L^1([0,s])$ such that the mapping $(s,t) \mapsto b_s(t)$ belongs to $L^2(D_T)$. Let $c, k \in L^1([0,T])$, k nonnegative, $\int_0^T k(s)\,ds \le 1$. Let us define $A(\omega) = \hat{c}(i\omega) \cdot \overline{\hat{c}(i\omega)}$. Assume*

$$\Re \hat{a}(i\omega) \ge A(\omega) \quad \text{and} \quad |\widehat{b_s}(i\omega)| \le k(s)A(\omega) \quad \text{for all} \quad \omega \in \mathbb{R}.$$

Then

$$(5)\quad \int_0^T w(t)\int_0^t a(t-\sigma)w(\sigma)\,d\sigma\,dt + \int_0^T\int_0^s w(t)\int_0^t b_s(t-\sigma)w(\sigma)\,d\sigma\,dt\,ds \ge 0$$

for all $w \in L^2([0,T])$.

Proof: The first integral in (5) is equal to

$$\int_{\mathbb{R}} \langle \widehat{w_T}(i\omega), \widehat{w_T}(i\omega)\hat{a}(i\omega)\rangle\,d\omega = \int_{\mathbb{R}} |\widehat{w_T}(i\omega)|^2 \Re\hat{a}(i\omega)\,d\omega \ge \int_{\mathbb{R}} |\widehat{w_T}(i\omega)|^2 A(\omega)\,d\omega,$$
$$= \int_{\mathbb{R}} \widehat{w_T}(i\omega)\hat{c}(i\omega)\cdot\overline{\widehat{w_T}(i\omega)\hat{c}(i\omega)}\,d\omega = \int_0^T \left|\int_0^t c(t-\sigma)w_T(\sigma)\,d\sigma\right|^2 dt$$

where $w_T := w \cdot \chi_{[0,T]}$. Absolute value of the integral from 0 to s in (5) is equal to

$$\left|\int_{\mathbb{R}} \langle \widehat{w_s}(i\omega), \widehat{w_s}(i\omega)\widehat{b_s}(i\omega)\rangle\,d\omega\right| \le \int_{\mathbb{R}} |\widehat{w_s}(i\omega)|^2 k(s)A(\omega)\,d\omega$$
$$= k(s)\int_0^s \left|\int_0^t c(t-\sigma)w_s(\sigma)\,d\sigma\right|^2 dt = k(s)\int_0^s \left|\int_0^t c(t-\sigma)w_T(\sigma)\,d\sigma\right|^2 dt.$$

Hence, the expression on the left-hand side of (5) is larger or equal to

$$\int_0^T \left|\int_0^t c(t-\sigma)w_T(\sigma)\,d\sigma\right|^2 dt - \int_0^T k(s)\int_0^s \left|\int_0^t c(t-\sigma)w_T(\sigma)\,d\sigma\right|^2 dt\,ds$$
$$= \int_0^T \left|\int_0^t c(t-\sigma)w_T(\sigma)\,d\sigma\right|^2 dt - \int_0^T\int_t^T k(s)\left|\int_0^t c(t-\sigma)w_T(\sigma)\,d\sigma\right|^2 ds\,dt$$
$$\ge \int_0^T \left(1 - \int_t^T k(s)\,ds\right)\cdot\left|\int_0^t c(t-\sigma)w_T(\sigma)\,d\sigma\right|^2 dt \ge 0.$$

□

Now, let us prove Theorem 2.4.

Proof: Take $T > 0$ and $w \in L^2([0,T])$ fixed. Writing

$$a(t,t-s) - \delta e^{-(t-s)} = a(T,t-s) - \delta e^{-(t-s)} - \int_t^T \partial_1 a(r,t-s)dr$$

we can use the same computations as in Theorem 2.1 and rewrite the integral

$$\int_0^T w(t) \int_0^t [a(t,t-\sigma) - \delta e^{-(t-\sigma)}] w(\sigma)\, d\sigma\, dt$$

in the form

$$\int_0^T w(t) \int_0^t [a(T,t-\sigma) - \delta e^{-(t-\sigma)}] w(\sigma)\, d\sigma\, dt \\ + \int_0^T \int_0^s w(t) \int_0^t -\partial_1 a(s,t-\sigma) w(\sigma)\, d\sigma\, dt\, ds.$$

We would like to apply Lemma 2.6 with

$$a(r) := a(T,r) - \delta e^{-r}, \quad b_s(r) := -\partial_1 a(s,r).$$

We will show that

$$k(t) := \frac{1}{\varepsilon(T)-\delta}(v_1(t)+v_2(t)+v_3(t)),\ A(\omega) := \frac{\varepsilon(T)-\delta}{1+\omega^2},\ c(t) := e^{-t}\sqrt{\varepsilon(T)-\delta}$$

satisfy the assumptions of Lemma 2.6.

It is known (see for example the text below Definition 16.4.1 in [4]) that every convolution kernel f of ε-strongly positive type satisfy $\operatorname{Re} \hat{f}(i\omega) \geq \frac{\varepsilon}{1+\omega^2}$. Hence,

$$\operatorname{Re} \widehat{a(T,\cdot)}(i\omega) \geq \frac{\varepsilon(T)}{1+\omega^2}\,.$$

Since

$$\operatorname{Re} \int_0^{+\infty} e^{-i\omega t} \delta e^{-t}\, dt = \operatorname{Re} \frac{\delta}{1+i\omega} = \frac{\delta}{1+\omega^2}\,,$$

we have

$$\operatorname{Re}[\widehat{a(T,\cdot) - \delta} e^{-(\cdot)}(i\omega)] \geq \frac{\varepsilon(T)-\delta}{1+\omega^2} = A(\omega)\,.$$

On the other hand we have from Lemma 2.5 for a.e. $t \in [0,T]$

$$(1+\omega^2)\widehat{\partial_1 a(t,\cdot)}(i\omega) \leq v_1(t)+v_2(t)+v_3(t)$$

(the assumptions of Lemma 2.5 are satisfied because of (A2), (A3), (A4)). Hence,

$$\widehat{b_s}(i\omega) \leq \frac{v_1(s)+v_2(s)+v_3(s)}{1+\omega^2} = k(s)A(\omega).$$

□

Corollary 2.7. *If $a \in Y(D)$ satisfies* (A) *with $\delta = 0$, then a is of c-positive type.*

Example 2.8. Let $a(t,s) = e^{-\lambda(t)s}$ with $\lambda : \mathbb{R}_+ \to [\alpha, \beta]$, $0 < \alpha < \beta$ and $|\lambda'|$ small enough in comparison to α. Then the assumptions of Theorem 2.4 are satisfied. In fact,

$$-\partial_1 a(t,s) = e^{-\lambda(t)s}\lambda'(t)s, \quad -\partial_2\partial_1 a(t,s) = e^{-\lambda(t)s}\lambda'(t)(1-\lambda(t)s)$$

and

$$-\partial_2^2\partial_1 a(t,s) = e^{-\lambda(t)s}\lambda(t)\lambda'(t)(\lambda(t)s-2).$$

Hence, all the smallness assumptions on $\partial_1 a$ are satisfied if $\lambda'(t)$ is small enough for all $t \in \mathbb{R}_+$.

3. Applications

In one-dimensional case, the system (1) (rewritten in Lagrangian coordinates) can be reduced to

$$u_t = \operatorname{div} S + f, \qquad c_t = -\operatorname{div} q_c. \tag{6}$$

Consider $q_c = -\nabla c$ and constitutive relation

$$S(t,x) = \mu\nabla u(t,x) + \int_0^t A(c(t,x), t-s)\nabla u(s,x)\,ds,$$

where $\mu = 0$ or $\mu = 1$. If $A(z, t-s) = e^{-\lambda(z)(t-s)}$ then we obtain the model presented by Rajagopal and Wineman in [10]. We arrive at

$$u_t = \mu\Delta u + \operatorname{div}\int_0^t A(c(t,x), t-s)\nabla u(s,x)\,ds + f, \qquad c_t = \Delta c. \tag{7}$$

Now, we present two stability results that are standard in the convolutionary case and obviously generalizable to the nonconvolutionary case. Then we apply these results to (7). By results of the previous section we can guarantee that $A \circ c$ is of (strong) c-positive type, even if we do not know c.

Problem 1. Consider the initial value problem

$$u_t(t,x) = \operatorname{div}\int_0^t a(t, t-s, x)\nabla u(s,x)\,ds, \quad u(0,x) = u_0(x) \tag{8}$$

with Dirichlet or Neumann boundary conditions. Let $a \in L^\infty(D \times \Omega)$ such that for almost all $x \in \Omega$ the function $a(\cdot,\cdot,x)$ is of c-positive type.

Theorem 3.1. *Let $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, W^{1,2}(\Omega))$, $u_t \in L^2_{\mathrm{loc}}(\mathbb{R}_+, W^{-1,2}(\Omega))$ be a weak solution to* (8). *Then $\|u(t)\|_2$ is bounded on $\mathbb{R}_+$.*

Proof: Taking $u|_{[0,T]}$ as a test function in the weak formulation we obtain

$$\frac{1}{2}(\|u(T)\|^2 - \|u_0\|^2) = -\int_0^T \langle \int_0^t a(t, t-s)\nabla u(s)\,ds, \nabla u(t)\rangle\,dt.$$

The right-hand side is by Fubini's Theorem nonpositive, hence

$$\|u(T)\|^2 \leq \|u_0\|^2.$$

□

Problem 2. Consider the initial value problem

$$(9)\quad u_t(t,x) = \Delta u(t,x) + \operatorname{div} \int_0^t a(t,t-s,x)\nabla u(s,x)\,ds + f(t,x), \quad u(0,x) = u_0(x)$$

with Dirichlet boundary conditions. Let us denote (for $d \in \mathbb{R}$)

$$f^d(t,\cdot) := e^{dt} f(t,\cdot) \quad \text{and} \quad a^d(t,r,\cdot) := a(t,r,\cdot)e^{dr}.$$

We assume that there exists $\delta > 0$ arbitrarily close to 0, such that

$$f^\delta \in L^2(\mathbb{R}_+, W^{-1,2}(\Omega))$$

and $a \in L^\infty(D \times \Omega)$ such that for almost all $x \in \Omega$ the function $a^\delta(\cdot,\cdot,x)$ is of c-positive type.

Theorem 3.2. *Let $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, W^{1,2}(\Omega))$, $u_t \in L^2_{\mathrm{loc}}(\mathbb{R}_+, W^{-1,2}(\Omega))$ be a weak solution to* (9)*. Then $e^{\delta t}\|u(t)\|_2 \to 0$ for some $\delta > 0$.*

PROOF: Take $\delta > 0$ small enough and denote $v(t) := u(t)e^{\delta t}$. Then v is a weak solution to

$$\dot{v}(t) = (\Delta + \delta I)v(t) + \operatorname{div} \int_0^t a(t,t-s)e^{\delta(t-s)}\nabla v(s)\,ds + e^{\delta t} f(t).$$

Taking $v|_{[0,T]}$ as a test function in the weak formulation we obtain

$$\begin{aligned} &\frac{1}{2}(\|v(T)\|^2 - \|u_0\|^2) - \delta \int_0^T \|v(t)\|^2\,dt + \int_0^T \|\nabla v(t)\|^2\,dt \\ &= - \int_0^T \langle \int_0^t a(t,t-s)e^{\delta(t-s)}\nabla v(s)\,ds, \nabla v(t)\rangle\,dt + \int_0^T \langle e^{\delta t} f(t), v(t)\rangle\,dt. \end{aligned}$$

The first term on the right-hand side is by Fubini's Theorem nonpositive, hence by Poincaré inequality applied to the second term on the left-hand side, Cauchy–Schwartz, Hölder and Young inequality applied to the second term on the right-hand side we obtain

$$\|v(T)\|^2 + \int_0^T \|\nabla v(t)\|^2\,dt \leq c(\|u_0\|^2 + \|f^\delta\|^2).$$

Hence,

$$\|u(t)\| \leq Ce^{-\delta t}.$$

□

Application 1. Let c be the solution of the diffusion equation in (7) with Dirichlet or Neumann boundary conditions and an initial value $c(0) = c_0$ smooth enough. If $A(z, t-s) = e^{-\lambda(z)(t-s)}$ with $\lambda : \Omega \to [\alpha, \beta]$, $\alpha > 0$ and $\frac{d}{dt}\lambda(c(t))$ is small, then (by Example 2.8) the function $a(\cdot,\cdot,x)$ defined by $a(t, t-s, x) := A(c(t,x), t-s)$ is of c-positive type for a.a. $x \in \Omega$. Moreover, the function $a^\delta(\cdot,\cdot,x)$ defined by

$$a^\delta(t, r, \cdot) := a(t, r, \cdot)e^{\delta r}$$

is of c-positive type for all $\delta < \alpha$ and a.a. $x \in \Omega$ (also by Example 2.8). We have the following:

If the initial concentration c_0 is small enough or if the dependence on the concentration is small (λ' small) then solutions to (7) are bounded if $\mu = 0$ and exponentially convergent to zero if $\mu = 1$.

Application 2. Theorem 2.1 and Example 2.3 could be applied if we could keep the function $t \mapsto c(t,x)$ decreasing and if the kernel is in the form $A(c(t), t-s) = \nu(c(t))e^{-(t-s)}$. It is not clear whether this case is physically relevant. However, in the model of aging of materials presented by Rajagopal and Wineman in [9] the kernel has this form and monotonicity of the aging function $\nu(c(t))$ seems to be a physically relevant condition.

Application 3. In [1] we have shown existence of global solution for the quasilinear hyperbolic equation

$$u_{tt} = \chi(t, x, u_x)u_{xx} + \int_0^t \partial_3 a(t, x, t-s)\psi(u_x(s))_x\, ds + g, \quad x \in [0,1],\ t \in [0, +\infty)$$

for a of strong c-positive type under appropriate assumptions on ψ, χ and g. This yields global existence for a one-dimensional viscoelastic problem with dependence on the concentration of a chemical

$$\begin{aligned} u_{tt} &= \chi(c, u_x)u_{xx} + \int_0^t k(c(t,x), t-s)\psi(u_x(s))_x\, ds + g, \\ c_t &= c_{xx}, \end{aligned} \tag{10}$$

provided the initial concentration is smooth enough and small enough and $k(z,t) = e^{-\lambda(z)t}$ (then $a(t, x, t-s) := k(c(t,x), t-s)$ is of strong c-positive type by Theorem 2.4 and Example 2.8).

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Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail: barta@karlin.mff.cuni.cz

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