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On the exterior problem in 2D for stationary flows of fluids with shear dependent viscosity

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Abstract. On the complement of the unit disk B we consider solutions of the equations describing the stationary flow of an incompressible fluid with shear dependent viscosity. We show that the velocity field u is equal to zero provided $u|_{\partial B} = 0$ and $\lim_{|x|\to\infty} |x|^{1/3}|u(x)| = 0$ uniformly. For slow flows the latter condition can be replaced by $\lim_{|x|\to\infty} |u(x)| = 0$ uniformly. In particular, these results hold for the classical Navier-Stokes case.

Keywords: equations of Navier-Stokes type, stationary case, exterior problem in 2D

Classification: 76D05, 35Q30

1. Introduction

In our note we investigate the following exterior problem for the stationary flow of a generalized Newtonian fluid: let B denote the open unit disk in $\mathbb{R}^2$ and suppose that the velocity field u: $\mathbb{R}^2 \setminus B \to \mathbb{R}^2$ and the pressure π: $\mathbb{R}^2 \setminus B \to \mathbb{R}$ satisfy the equations

$$(1.1) \qquad -\operatorname{div}\left[DH\left(\varepsilon(u)\right)\right] + u^k\partial_k u + \nabla\pi = 0$$

and

$$(1.2) \qquad \operatorname{div} u = 0$$

on $\mathbb{R}^2 \setminus \overline{B}$ together with the boundary condition

$$(1.3) \qquad u = 0 \quad \text{on} \quad \partial B.$$

Here $\varepsilon(u)$ denotes the symmetric gradient of the field u, $u^k\partial_k u$ represents the convective term (the convention of summation is used throughout this paper) and we assume that the stress tensor T is generated by a given potential H in the sense that $T^D = DH$, where T^D is the deviatoric part of T.

We further assume the structural condition

$$(1.4) \qquad H(\varepsilon) = h\left(|\varepsilon|\right)$$

with prescribed function $h : [0, \infty) \to [0, \infty)$ of class C^2. From (1.4) it follows

$$DH(\varepsilon) = \mu\left(|\varepsilon|\right)\varepsilon$$

with viscosity function $\mu(t) := \frac{h'(t)}{t}$ and, together with (1.2), this means that we consider stationary flows of incompressible generalized Newtonian fluids being of shear thickening type if μ is an increasing function, and of shear thinning type if the viscosity decreases.

For further mathematical and also physical explanations the reader is referred to the monographs of Ladyzhenskaya [La], Galdi [Ga1],[Ga2] and Málek, Nečas, Rokyta, Růžička [MNRR] (see also [FuSe]).

In the particular case $h(t) = t^2/2$, the equations (1.1)–(1.3) reduce to the exterior problem for the stationary Navier-Stokes equations, and it is a challenging task to prove (or disprove) that

$$\Theta(R) := \sup_{|x| \geq R} |u(x)| \to 0 \quad \text{as } R \to \infty, \tag{1.5}$$

implies that the velocity field u is identically zero. Further details including the historical background and related problems are presented in Chapter X.3 of Galdi's book [Ga2] and in his paper [Ga3].

Of course we will not give an answer to this open question: our goal is to show that with the help of rather elementary energy estimates one can obtain the following results.

Suppose that the fluid is shear thickening or shear thinning. Let u denote a solution of (1.1)–(1.3). *Then we have $u = 0$ if*

(i) (1.5) *holds and the convective term is neglected* (*"slow flows"*)

or if

(ii) (1.5) *is replaced by the stronger condition*

$$\lim_{R \to \infty} R^{1/3}\Theta(R) = 0. \tag{1.6}$$

In order to make these statements precise, we first have to introduce a reasonable class of solutions.

Definition 1.1. A function $u \in C^1(\mathbb{R}^2 \setminus B)$, i.e. u and ∇u are continuous up to ∂B, is a solution of (1.1)–(1.3), if (1.2) and (1.3) hold in the classical sense and if

$$\int_{\mathbb{R}^2 \setminus \overline{B}} DH\left(\varepsilon(u)\right) : \varepsilon(\varphi)\, \mathrm{d}x + \int_{\mathbb{R}^2 \setminus \overline{B}} u^k \partial_k u^i \varphi^i \, \mathrm{d}x = 0 \tag{1.7}$$

holds for all $\varphi \in C^1_0(\mathbb{R}^2 \setminus \overline{B})$ satisfying $\operatorname{div} \varphi = 0$.

Remark 1.1. Obviously (1.7) is the weak form of (1.1) and in the shear thickening case we can replace Definition 1.1 just by the requirement that u is an element of a suitable local energy space having finite energy on the annulus $1 < |x| < r$.

In the shear thinning case the situation becomes more delicate and we decided to work with Definition 1.1.

From the various hypotheses concerning h and the calculations presented below the reader actually can deduce the minimal requirements concerning the field u in the cases under investigation. However we emphasize that we do not assume the validity of global energy bounds like $\int_{\mathbb{R}^2\setminus\overline{B}} h(|\varepsilon(u)|)\,\mathrm{d}x < \infty$ for our class of solutions.

Next we formulate our hypotheses imposed on the density h occurring in the structural condition (1.4). We suppose that h satisfies:

(A1) h is strictly increasing and convex; we have $h''(0) > 0$ and $\lim_{t\to 0} \frac{h(t)}{t} = 0$.

(A2) There is a constant $a > 0$ such that $h(2t) \le a h(t)$ for all $t \ge 0$ (doubling property).

($\mathrm{A3_I}$) In the shear thickening case we have $\frac{h'(t)}{t} \le h''(t)$ for all $t > 0$.

($\mathrm{A3_{II}}$) In the shear thinning case we have $h''(t) \le \frac{h'(t)}{t}$ for all $t > 0$.

Remark 1.2. (i) From (A1) it immediately follows that $h(0) = h'(0)$ and $h'(t) > 0$ for any $t > 0$.

(ii) By considering $\frac{d}{dt}\frac{h'(t)}{t}$ it is immediate that ($\mathrm{A3_I}$) and ($\mathrm{A3_{II}}$) express the fact that the fluid is shear thickening and shear thinning, respectively.

(iii) (A1) together with (A2) implies the balancing condition

$$c\,t h'(t) \le h(t) \le t h'(t) \quad \text{for all } t \ge 0 \tag{1.8}$$

and for a suitable positive constant c. In fact, $0 = h(0) \ge h(t) - t h'(t)$ holds by convexity, whereas by (A2) and the monotonicity of h'

$$h(t) \ge \frac{1}{a} h(2t) = \frac{1}{a}\int_0^{2t} h'(s)\,\mathrm{d}s \ge \frac{1}{a}\int_t^{2t} h'(s)\,\mathrm{d}s \ge \frac{1}{a} t h'(t).$$

(iv) It is easy to see that from (A2) it follows

$$h(t) \le h(1) t^a \quad \text{for all } t \ge 1,$$

thus

$$h(t) \le c\,[t^a + 1] \quad \text{for all } t \ge 0. \tag{1.9}$$

(v) If we are in the shear thickening case ($A3_I$), then $\frac{h'(t)}{t} \ge \lim_{s\to 0} \frac{h'(s)}{s} = h''(0)$ gives

$$h(t) \ge \frac{1}{2} h''(0) t^2 \quad \text{for all } t \ge 0, \tag{1.10}$$

and (A1) implies on account of $h''(0) > 0$ that our energy is of at least quadratic growth.

(vi) In the shear thinning case we have

$$h(t) \le \frac{1}{2} h''(0) t^2 \tag{1.11}$$

and

$$h'(t)^2 \le c h(t) \tag{1.12}$$

for any $t \ge 0$. For (1.12) we observe $h'(t) \le t h''(0)$, which is an immediate consequence of $h'(t)/t \le \lim_{s\to 0} h'(s)/s$, thus

$$h'(t)^2 \le t h''(0) h'(t) \overset{(1.8)}{\le} c h''(0) h(t).$$

Note that according to (1.11) the condition ($A3_{II}$) implies that the energy has subquadratic growth.

Actually, even the case of linear growth is covered, which means that we can easily give examples of densities h satisfying (A1)–($A3_{II}$) for which $\lim_{t\to\infty} h(t)/t \in (0,\infty)$.

(vii) It is not hard to show that (A1) and ($A3_{II}$) already imply (A2), we refer to the Appendix of [BF].

After these preparations we can state our main theorem:

Theorem 1.1. *Suppose that u is a solution of* (1.1)–(1.3) *in the sense of Definition* 1.1 *with H from* (1.4), *where h satisfies* (A1,2), ($A3_I$) *or* (A1,2), ($A3_{II}$). *Then u is identically zero, if*

(i) $|u(x)| \to 0$ *uniformly as* $|x| \to \infty$, *i.e.* (1.5) *holds, and if $u^k \partial_k u$ is neglected*

(ii) *or if* $|x|^{1/3}|u(x)| \to 0$ *uniformly as* $|x| \to \infty$, *i.e. we have* (1.6).

In the subsequent sections we will present the proof of Theorem 1.1 distinguishing the cases of increasing and decreasing viscosity.

However, in both cases we apply energy estimates originating in the papers [Fu] and [FuZha] dealing with entire solutions of equations (1.1) and (1.2).

We finally remark that our arguments immediately extend to the exterior problem in $\mathbb{R}^n$ leading to appropriate bounds in part a) and b) of Theorem 1.1. The details are left to the reader. Moreover, it should be noted that Theorem 1.1 includes the exterior problem for the stationary Navier-Stokes equations as a special case.

2. Some technical preliminaries

Our first tool is a slight extension (presented in [FuZha]) of the "ε-Lemma" due to Giaquinta and Modica (see Lemma 0.5 in [GM]):

Lemma 2.1. *Let $Q := Q_R(z) := \{x \in \mathbb{R}^2 : |x_i - z_i| < R,\ i = 1,2\}$ denote an arbitrary square. Suppose that we are given non-negative functions f, $f_1, \ldots, f_l$ from the space $L^1(Q)$ and exponents $\alpha_1, \ldots, \alpha_l > 0$. Then we can find a number $\varepsilon_0 > 0$ depending on $\alpha_1, \ldots, \alpha_l$ as follows: if for $\varepsilon \in (0, \varepsilon_0)$ it is possible to calculate a constant $c(\varepsilon) > 0$ such that the inequality*

$$\int_{Q_r(y)} f \,\mathrm{d}x \le \varepsilon \int_{Q_{2r}(y)} f \,\mathrm{d}x + c(\varepsilon) \sum_{j=1}^{l} r^{-\alpha_j} \int_{Q_{2r}(y)} f_j \,\mathrm{d}x$$

holds for all squares $Q_{2r}(y) \Subset Q$, then there is a constant $c > 0$ (independent of Q) with the property

$$\int_{Q_r(y)} f \,\mathrm{d}x \le c \sum_{j=1}^{l} r^{-\alpha_j} \int_{Q_{2r}(y)} f_j \,\mathrm{d}x$$

again for all squares $Q_{2r}(y) \Subset Q$.

In order to construct solenoidal testfunctions, we will make use of the following basic lemma (see, e.g. [Ga1, Chapter III, Section 3]).

Lemma 2.2. *Suppose that we are given numbers $1 < p_1 \le p \le p_2 < \infty$. Then there is a constant $c = c(p_1, p_2)$ with the following property: if $f \in L^p(B_R(x_0))$, $B_R(x_0) := \{x \in \mathbb{R}^2 : |x - x_0| < R\}$, satisfies $\int_{B_R(x_0)} f \,\mathrm{d}x = 0$, then there exists a field v in the Sobolev class $\overset{\circ}{W}{}^1_p(B_R(x_0))$ such that $\operatorname{div} v = f$ on the disk $B_R(x_0)$ together with the estimate*

$$\int_{B_R(x_0)} |\nabla v|^s \,\mathrm{d}x \le c \int_{B_R(x_0)} |f|^s \,\mathrm{d}x \tag{2.1}$$

for any exponent $s \in [p_1, p]$. The same is true if the disk is replaced by a square $Q_R(x_0)$ or an annulus $B_{2R}(x_0) \setminus \overline{B_R(x_0)}$.

For handling the shear thickening case we need the following result stated in Lemma 2.5 of [Fu] and being a consequence of (1.8) and (1.9).

Lemma 2.3. *Let h satisfy* (A1), (A2) *and* ($\mathrm{A3_I}$). *Then there exists a number $\tau \in (1, 2]$ such that*

$$h'(t) \le c\left(h(t)^{1/\tau} + t\right) \quad \textit{for all} \ \ t \ge 0, \tag{2.2}$$

where c denotes a suitable positive constant.

3. Shear thinning case

Let h satisfy (A1), (A2), ($\mathrm{A3_{II}}$) and suppose that we have a solution u in the sense of Definition 1.1 satisfying at least (1.5). Note that in this case u is an element of the space $L^\infty(\mathbb{R}^2)$. We fix a square Q having positive distance to the unit disk B and consider subsquares $Q_{2r}(z) \Subset Q$.

Our first goal is to obtain an estimate (see (3.8)) for the energy $\int_{Q_r(z)} h(|\varepsilon(u)|)\,\mathrm{d}x$. To this purpose we let in equation (1.7) $\varphi = \eta^2 u - v$, where $\eta \in C_0^1(Q_{2r}(z))$, $0 \le \eta \le 1$, $\eta = 1$ on $Q_r(z)$, $|\nabla\eta| \le c/r$.

The field v is defined according to Lemma 2.2 with the choices $s = p_1 = p_2 = 2$, $f = \operatorname{div}(\eta^2 u) \overset{(1.2)}{=} \nabla\eta^2 \cdot u$ and with $B_R(x_0)$ replaced by $Q_{2r}(z)$. We obtain from (1.7)

$$
\begin{aligned}
&\int_{Q_{2r}(z)} \eta^2 DH\left(\varepsilon(u)\right) : \varepsilon(u)\,\mathrm{d}x \\
&\quad + 2\int_{Q_{2r}(z)} \frac{\partial H}{\partial\varepsilon_{i\alpha}}\left(\varepsilon(u)\right)\partial_\alpha\eta\,\eta u^i\,\mathrm{d}x - \int_{Q_{2r}(z)} DH\left(\varepsilon(u)\right) : \varepsilon(v)\,\mathrm{d}x \\
&\quad + \int_{Q_{2r}(z)} u^k\partial_k u^i u^i\eta^2\,\mathrm{d}x - \int_{Q_{2r}(z)} u^k\partial_k u^i v^i\,\mathrm{d}x \\
&= T_1 + T_2 - T_3 + T_4 - T_5 = 0.
\end{aligned}
\tag{3.1}
$$

From (1.4) and (1.8) it follows

$$
T_1 = \int_{Q_{2r}(z)} \eta^2 h'\left(|\varepsilon(u)|\right)\frac{\varepsilon(u)}{|\varepsilon(u)|} : \varepsilon(u)\,\mathrm{d}x \ge c\int_{Q_{2r}(z)} \eta^2 h\left(|\varepsilon(u)|\right)\,\mathrm{d}x. \tag{3.2}
$$

By Young's inequality and again (1.8) we have

$$
\begin{aligned}
|T_2| &\le c\int_{Q_{2r}(z)} h'\left(|\varepsilon(u)|\right)\eta|\nabla\eta||u|\,\mathrm{d}x \\
&= c\int_{Q_{2r}(z)} \left[\frac{h'\left(|\varepsilon(u)|\right)}{|\varepsilon(u)|}\right]^{\frac{1}{2}} |\nabla\eta||u|\eta\big[h'\left(|\varepsilon(u)|\right)|\varepsilon(u)|\big]^{\frac{1}{2}}\,\mathrm{d}x \\
&\le \delta\int_{Q_{2r}(z)} \eta^2 h\left(|\varepsilon(u)|\right)\,\mathrm{d}x + c(\delta)\int_{Q_{2r}(z)} \frac{h'\left(|\varepsilon(u)|\right)}{|\varepsilon(u)|}|\nabla\eta|^2|u|^2\,\mathrm{d}x.
\end{aligned}
$$

If δ is chosen sufficiently small, we deduce from the above estimate in combination with (3.1) and (3.2) and by recalling the inequality stated after (1.12)

$$
\int_{Q_{2r}(z)} \eta^2 h\left(|\varepsilon(u)|\right)\,\mathrm{d}x \le c\left[r^{-2}\int_{Q_{2r}(z)} |u|^2\,\mathrm{d}x + |T_3| + |T_4| + |T_5|\right]. \tag{3.3}
$$

For any $\delta > 0$ it holds on account of (2.1) and (1.12)

$$\begin{aligned} |T_3| &\le \delta \int_{Q_{2r}(z)} h'\left(|\varepsilon(u)|\right)^2 \,\mathrm{d}x + \delta^{-1} \int_{Q_{2r}(z)} |\nabla v|^2 \,\mathrm{d}x \\ &\le c\left[\delta \int_{Q_{2r}(z)} h\left(|\varepsilon(u)|\right) \,\mathrm{d}x + \delta^{-1} r^{-2} \int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x\right], \end{aligned}$$

and if we replace $c\,\delta$ by δ we get from this estimate in combination with (3.3)

$$\begin{aligned} &\int_{Q_r(z)} h\left(|\varepsilon(u)|\right) \,\mathrm{d}x \\ &\qquad \le \delta \int_{Q_{2r}(z)} h\left(|\varepsilon(u)|\right) \,\mathrm{d}x + c\left[\delta^{-1} r^{-2} \int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x + |T_4| + |T_5|\right]. \end{aligned} \tag{3.4}$$

We further have

$$T_4 = \frac{1}{2} \int_{Q_{2r}(z)} u^k \partial_k |u|^2 \eta^2 \,\mathrm{d}x \overset{(1.2)}{=} -\frac{1}{2} \int_{Q_{2r}(z)} u \cdot \nabla \eta^2 |u|^2 \,\mathrm{d}x,$$

hence

$$|T_4| \le \frac{1}{r} \int_{Q_{2r}(z)} |u|^3 \,\mathrm{d}x, \tag{3.5}$$

moreover it holds

$$\begin{aligned} |T_5| &\overset{(1.2)}{=} \left|\int_{Q_{2r}(z)} u^k u^i \partial_k v^i \,\mathrm{d}x\right| \\ &\le \left[\int_{Q_{2r}(z)} |u|^4 \,\mathrm{d}x\right]^{\frac{1}{2}} \left[\int_{Q_{2r}(z)} |\nabla v|^2 \,\mathrm{d}x\right]^{\frac{1}{2}} \\ &\overset{(2.1)}{\le} c\,r^{-1} \left[\int_{Q_{2r}(z)} |u|^4 \,\mathrm{d}x \int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x\right]^{\frac{1}{2}} \\ &\le c\,r^{-1} \left[\int_{Q_{2r}(z)} |u|^4 \,\mathrm{d}x + \int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x\right]. \end{aligned} \tag{3.6}$$

From (3.4)–(3.6) we finally obtain

$$\begin{aligned} \int_{Q_r(z)} h\left(|\varepsilon(u)|\right) \,\mathrm{d}x &\le \delta \int_{Q_{2r}(z)} h\left(|\varepsilon(u)|\right) \,\mathrm{d}x + c\left[\delta^{-1} r^{-2} \int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x\right. \\ &\qquad \left. + r^{-1} \int_{Q_{2r}(z)} \left(|u|^2 + |u|^3 + |u|^4\right) \,\mathrm{d}x\right] \end{aligned} \tag{3.7}$$

being valid for any $\delta > 0$ and all squares $Q_{2r}(z) \subset Q$. Inequality (3.7) shows that we can apply Lemma 2.1 with the result

$$\text{(3.8)} \quad \int_{Q_r(z)} h\left(|\varepsilon(u)|\right) \mathrm{d}x \leq c\left[r^{-2}\int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x + r^{-1}\int_{Q_{2r}(z)} \left(|u|^2 + |u|^3 + |u|^4\right) \mathrm{d}x\right],$$

which holds for all squares $Q_{2r}(z) \subset Q$. Let us consider a square $Q = Q_R(x_0)$ with side length $R > 1$. Choosing $r = R/4$, $z = x_0$ in (3.8) and recalling the boundedness of u we get

$$\text{(3.9)} \qquad \int_{Q_{\frac{R}{4}}(x_0)} h\left(|\varepsilon(u)|\right) \mathrm{d}x \leq cR^{-1}\int_{Q_{\frac{R}{2}}(x_0)} |u|^2 \,\mathrm{d}x.$$

With (3.9) we return to the derivation of (3.7) with the choices $r = R/8$, $z = x_0$, but this time we estimate $|T_5|$ through the quantity $c\,r^{-1}[\int_{Q_{2r}(z)} |u|^4 \,\mathrm{d}x \int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x]^{1/2}$ (compare (3.6)) and again we make use of the boundedness of u, which means that in (3.5) we replace $|u|^3$ by $\mathrm{const}|u|^2$. This yields for any $\delta > 0$:

$$\int_{Q_{\frac{R}{8}}(x_0)} h\left(|\varepsilon(u)|\right) \mathrm{d}x \leq c\left[\delta R^{-1}\int_{Q_{\frac{R}{2}}(x_0)} |u|^2 \,\mathrm{d}x + \delta^{-1}R^{-2}\int_{Q_{\frac{R}{4}}(x_0)} |u|^2 \,\mathrm{d}x + R^{-1}\left[\int_{Q_{\frac{R}{4}}(x_0)} |u|^4 \,\mathrm{d}x \int_{Q_{\frac{R}{4}}(x_0)} |u|^2 \,\mathrm{d}x\right]^{\frac{1}{2}}\right].$$

If we choose $\delta = R^{-1/2}$, this inequality implies

$$\text{(3.10)} \quad \int_{Q_{\frac{R}{8}}(x_0)} h\left(|\varepsilon(u)|\right) \mathrm{d}x \leq c\left[R^{-\frac{3}{2}}\int_{Q_{\frac{R}{2}}(x_0)} |u|^2 \,\mathrm{d}x + R^{-1}\left[\int_{Q_{\frac{R}{2}}(x_0)} |u|^4 \,\mathrm{d}x \int_{Q_{\frac{R}{2}}(x_0)} |u|^2 \,\mathrm{d}x\right]^{\frac{1}{2}}\right].$$

Next we fix an annulus $T_R := B_{2R}(0) \setminus \overline{B_R}(0)$ of very large radius R and cover its closure with a finite number N of squares $Q_{\frac{R}{8}}(x_i)$ having centers x_i in $\overline{T_R}$. Note that N can be chosen independent of the radius R. We apply (3.10) to these squares and estimate $|u|$ on $Q_{\frac{R}{2}}(x_i)$ just through $\Theta(R/4)$ being defined in (1.5).

After summation with respect to i we deduce

$$\int_{T_R} h\left(|\varepsilon(u)|\right) \mathrm{d}x \le c\left[R^{\frac{1}{2}}\Theta\left(\frac{R}{4}\right)^2 + R\,\Theta\left(\frac{R}{4}\right)^3\right]. \tag{3.11}$$

Note that assumption (1.6) immediately implies the vanishing of $\int_{T_R} h(|\varepsilon(u)|)\,\mathrm{d}x$ passing to the limit $R \to \infty$.

In the absence of the convective term this is already true under the weaker hypothesis (1.5): under the assumption $u^k\partial_k u \equiv 0$ inequality (3.8) reduces to

$$\int_{Q_r(z)} h\left(|\varepsilon(u)|\right) \mathrm{d}x \le c\,r^{-2}\int_{Q_{2r}(z)} |u|^2\,\mathrm{d}x,$$

and (3.11) has to be replaced by

$$\int_{T_R} h\left(|\varepsilon(u)|\right) \mathrm{d}x \le c\,\Theta\left(\frac{R}{4}\right)^2.$$

In a next step we show that (1.6) implies

$$\int_{|x|>1} h\left(|\varepsilon(u)|\right) \mathrm{d}x = 0, \tag{3.12}$$

which forces u to be a rigid motion, hence $u = 0$ on account of the boundary condition (1.3).

For proving (3.12) it just remains to verify the validity of

$$\lim_{R\to\infty}\int_{1<|x|<R} h\left(|\varepsilon(u)|\right) \mathrm{d}x = 0 \tag{3.13}$$

under the hypothesis (1.6) (or (1.5) in case $u^k\partial_k u = 0$).

To this purpose we fix a radius $R \gg 1$ and choose

$$\varphi := \begin{cases} u & \text{if } \ 1 \le |x| \le R, \\ \eta^2 u - v & \text{if } \ R \le |x| \le 2R \end{cases}$$

as testfunction in equation (1.7) with $\eta = 1$ on $1 \le |x| \le R$, $0 \le \eta \le 1$ in $1 \le |x| \le 2R$, $\eta = 0$ outside of $|x| \le 2R$ and $|\nabla\eta| \le c/R$.

The field v is defined according to Lemma 2.2 with the choices $s = p_1 = p_2 = 2$, $f = \operatorname{div}(\eta^2 u)$ and for the domain T_R, i.e. $v \in \overset{\circ}{W}{}^1_2(T_R)$, $\operatorname{div} v = f$ on T_R and v satisfies (2.1). Note (recall (1.3)) that φ vanishes on $|x| = 1$, moreover we have

$$\int_{T_R} f\,\mathrm{d}x = 0, \tag{3.14}$$

which justifies the application of Lemma 2.2: in fact, by the choice of η it holds

$$\begin{aligned}\int_{T_R} f\,\mathrm{d}x &= \int_{\partial T_R} \eta^2 u \mathcal{N}_{T_R} \mathrm{d}\mathcal{H}^1 = -\int_{\partial B_R} u\cdot \mathcal{N}_{\partial B_R} \mathrm{d}\mathcal{H}^1 \\ &\overset{(1.3)}{=} -\int_{\partial(B_R\setminus\overline{B})} u\cdot \mathcal{N}_{\partial(B_R\setminus\overline{B})} \mathrm{d}\mathcal{H}^1 \\ &= -\int_{B_R\setminus\overline{B}} \operatorname{div} u\,\mathrm{d}x = 0\end{aligned}$$

and (3.14) follows. Here $\mathcal{N}$ denotes the exterior normal of the domains under consideration and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

Equation (1.7) then yields

$$\begin{aligned}0 = &\int_{1<|x|<R} DH(\varepsilon(u)) : \varepsilon(u)\,\mathrm{d}x + \int_{T_R} DH(\varepsilon(u)) : \varepsilon(\eta^2 u)\,\mathrm{d}x \\ &- \int_{T_R} DH(\varepsilon(u)) : \varepsilon(v)\,\mathrm{d}x + \int_{1<|x|<2R} u^k \partial_k u^i \varphi^i\,\mathrm{d}x\end{aligned}$$

or equivalently

$$\begin{aligned}&\int_{1<|x|<2R} \eta^2 DH(\varepsilon(u)) : \varepsilon(u)\,\mathrm{d}x \\ &= -\int_{T_R} DH(\varepsilon(u)) : \left(\nabla\eta^2 \otimes u\right)\,\mathrm{d}x \\ &\quad + \int_{T_R} DH(\varepsilon(u)) : \varepsilon(v)\,\mathrm{d}x - \int_{1<|x|<2R} u^k \partial_k u^i \varphi^i\,\mathrm{d}x.\end{aligned} \tag{3.15}$$

We have

$$\begin{aligned}\left|\int_{T_R} DH(\varepsilon(u)) : \left(\nabla\eta^2\otimes u\right)\,\mathrm{d}x\right| &\leq \int_{T_R} h'(|\varepsilon(u)|)\,|\nabla\eta||u|\,\mathrm{d}x \\ &\leq c\left[\int_{T_R} h'(|\varepsilon(u)|)^2\,\mathrm{d}x + R^{-2}\int_{T_R}|u|^2\,\mathrm{d}x\right] \\ &\overset{(1.12)}{\leq} c\left[\int_{T_R} h(|\varepsilon(u)|)\,\mathrm{d}x + R^{-2}\int_{T_R}|u|^2\,\mathrm{d}x\right]\end{aligned}$$

as well as

$$\begin{aligned}\left|\int_{T_R} DH(\varepsilon(u)) : \varepsilon(v)\,\mathrm{d}x\right| &\leq \int_{T_R} h'(|\varepsilon(u)|)\,|\varepsilon(v)|\,\mathrm{d}x \\ &\leq c\left[\int_{T_R} h(|\varepsilon(u)|)\,\mathrm{d}x + R^{-2}\int_{T_R}|u|^2\,\mathrm{d}x\right],\end{aligned}$$

where we used Young's inequality and the definition of v. Returning to (3.15) we find (recall (1.8))

$$\begin{aligned}(3.16)\qquad &\int_{1<|x|<R} h\left(|\varepsilon(u)|\right)\,\mathrm{d}x \le c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)\,\mathrm{d}x + R^{-2}\int_{T_R}|u|^2\,\mathrm{d}x + |S|\right],\\ &S := \int_{1<|x|<2R} u^k\partial_k u^i\varphi^i\,\mathrm{d}x.\end{aligned}$$

With (3.11) we immediately see that (3.16) implies our claim (3.13), i.e. finishes the proof in the presence of the convective term, as soon as we can show that

$$(3.17)\qquad \lim_{R\to\infty} S = 0.$$

It holds

$$\begin{aligned}(3.18)\qquad S &= -\int_{1<|x|<2R} u^k u^i\partial_k\varphi^i\,\mathrm{d}x\\ &= -\int_{1<|x|<2R} u^k u^i\partial_k(\eta^2 u^i)\,\mathrm{d}x + \int_{T_R} u^k u^i\partial_k v^i\,\mathrm{d}x\\ &=: -T_1 + T_2,\end{aligned}$$

and for T_2 we have

$$\begin{aligned}|T_2| &\le \int_{T_R}|u|^2|\nabla v|\,\mathrm{d}x\\ &\le \left[\int_{T_R}|u|^4\,\mathrm{d}x\right]^{\frac12}\left[\int_{T_R}|\nabla v|^2\,\mathrm{d}x\right]^{\frac12}\\ &\le cR^{-1}\left[\int_{T_R}|u|^4\,\mathrm{d}x\right]^{\frac12}\left[\int_{T_R}|u|^2\,\mathrm{d}x\right]^{\frac12} \le cR\Theta(R)^3,\end{aligned}$$

thus by (1.6)

$$(3.19)\qquad \lim_{R\to\infty} T_2 = 0.$$

For T_1 we observe the identity (recalling (1.3))

$$\begin{aligned}T_1 &= \int_{1<|x|<2R}\partial_k(u^k u^i\eta^2 u^i)\,\mathrm{d}x - \int_{1<|x|<2R}\partial_k(u^k u^i)\eta^2 u^i\,\mathrm{d}x\\ &= -\int_{1<|x|<2R}\partial_k(u^k u^i)\eta^2 u^i\,\mathrm{d}x = -\frac12\int_{1<|x|<2R} u^k\partial_k|u|^2\eta^2\,\mathrm{d}x\\ &= \frac12\int_{1<|x|<2R} u^k|u|^2\partial_k\eta^2\,\mathrm{d}x,\end{aligned}$$

and this immediately shows

$$\lim_{R\to\infty} T_1 = 0. \tag{3.20}$$

With (3.19) and (3.20) we obtain (3.17), and as outlined before this completes the proof of Theorem 1.1 in the shear thinning case. □

4. Shear thickening case

With h satisfying (A1), (A2) and ($A3_{\mathrm{I}}$) we consider a solution u of the exterior problem (1.1)–(1.3) as explained in Definition 1.1. We further assume the validity of (1.6) (or of (1.5) in the case that $u^k\partial_k u = 0$). The calculations follow the same ideas as in the previous section, for the necessary adjustments we benefit from [Fu, Section 4].

Let $p := \frac{\tau}{\tau-1} \geq 2$ with exponent τ being defined in Lemma 2.3. For $l \in \mathbb{N}$ sufficiently large we let $\varphi := \eta^{2l}u - v$ with η as introduced in front of equation (3.1), but now we choose $v \in \overset{\circ}{W}{}^1_p(Q_{2r}(z))$ such that $\operatorname{div} v = \operatorname{div}(\eta^{2l}u)(= \nabla\eta^{2l}\cdot u)$ on $Q_{2r}(z)$ together with

$$\begin{aligned} &\|\nabla v\|_{L^p(Q_{2r}(z))} \leq c\|\nabla\eta^{2l}\cdot u\|_{L^p(Q_{2r}(z))} \text{ and} \\ &\|\nabla v\|_{L^2(Q_{2r}(z))} \leq c\|\nabla\eta^{2l}\cdot u\|_{L^2(Q_{2r}(z))}. \end{aligned} \tag{4.1}$$

Replacing η^2 by η^{2l} in (3.1) we obtain for the terms T_i, $i = 1,\dots,5$

$$\begin{aligned} T_1 &\geq c\int_{Q_{2r}(z)} \eta^{2l} h\left(|\varepsilon(u)|\right)\,\mathrm{d}x, \\ |T_2| &\leq c\int_{Q_{2r}(z)} h'\left(|\varepsilon(u)|\right)\eta^{2l-1}|\nabla\eta||u|\,\mathrm{d}x \\ &\overset{(2.2)}{\leq} c\int_{Q_{2r}(z)} \eta^{2l-1}|\nabla\eta||u|\left[h\left(|\varepsilon(u)|\right)^{\frac{1}{\tau}} + |\varepsilon(u)|\right]\mathrm{d}x \\ &\leq \delta\int_{Q_{2r}(z)} \eta^{(2l-1)\tau} h\left(|\varepsilon(u)|\right)\mathrm{d}x + c(\delta)\int_{Q_{2r}(z)} |\nabla\eta|^p|u|^p\,\mathrm{d}x \\ &\quad + \delta\int_{Q_{2r}(z)} \eta^{(2l-1)2}|\varepsilon(u)|^2\,\mathrm{d}x + c(\delta)\int_{Q_{2r}(z)} |\nabla\eta|^2|u|^2\,\mathrm{d}x, \end{aligned} \tag{4.2}$$

where we have used Young's inequality with arbitrary $\delta > 0$. Observing (1.10) and selecting l so large that $(2l-1)\tau \geq 2l$, we see that after suitable choice of δ

it follows from (4.2)

$$\int_{Q_{2r}(z)} \eta^{2l} h\left(|\varepsilon(u)|\right) \mathrm{d}x \leq c\left[r^{-p}\int_{Q_{2r}(z)} |u|^p \,\mathrm{d}x + r^{-2}\int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x + |T_3| + |T_4| + |T_5|\right]. \tag{4.3}$$

From (2.2) and Young's inequality we get

$$\begin{aligned} |T_3| &\leq c\int_{Q_{2r}(z)} h'\left(|\varepsilon(u)|\right)|\varepsilon(v)| \;\mathrm{d}x \\ &\leq c\int_{Q_{2r}(z)} \left[h(|\varepsilon(u)|)^{\frac{1}{\tau}} + |\varepsilon(u)|\right]|\varepsilon(v)|\,\mathrm{d}x \\ &\leq \delta\int_{Q_{2r}(z)} h\left(|\varepsilon(u)|\right) \mathrm{d}x + c\delta^{1-p}\int_{Q_{2r}(z)} |\varepsilon(v)|^p \;\mathrm{d}x \\ &\quad +\delta\int_{Q_{2r}(z)} |\varepsilon(u)|^2 \,\mathrm{d}x + c\delta^{-1}\int_{Q_{2r}(z)} |\varepsilon(v)|^2 \;\mathrm{d}x, \end{aligned}$$

and if we use (4.1) and (1.10) we have shown

$$|T_3| \leq \delta\int_{Q_{2r}(z)} h\left(|\varepsilon(u)|\right) \mathrm{d}x + c\left[\delta^{1-p} r^{-p}\int_{Q_{2r}(z)} |u|^p \,\mathrm{d}x + \delta^{-1} r^{-2}\int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x\right]. \tag{4.4}$$

Returning to (4.3) and using (4.4) we obtain in place of (3.4)

$$\int_{Q_r(z)} h\left(|\varepsilon(u)|\right) \mathrm{d}x \leq \delta\int_{Q_{2r}(z)} h\left(|\varepsilon(u)|\right) \mathrm{d}x + c\left[\delta^{1-p} r^{-p}\int_{Q_{2r}(z)} |u|^p \,\mathrm{d}x + \delta^{-1} r^{-2}\int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x + |T_4| + |T_5|\right],$$

and since the estimates for T_4 and T_5 remain unchanged we deduce in place of (3.7)

$$\begin{aligned} \int_{Q_r(z)} h\left(|\varepsilon(u)|\right) \mathrm{d}x &\leq \delta\int_{Q_{2r}(z)} h\left(|\varepsilon(u)|\right) \mathrm{d}x \\ &\quad + c\left[\delta^{1-p} r^{-p}\int_{Q_{2r}(z)} |u|^p \,\mathrm{d}x + \delta^{-1} r^{-2}\int_{Q_{2r}(z)} |u|^2 \,\mathrm{d}x \right. \\ &\quad \left. + r^{-1}\int_{Q_{2r}(z)} \left(|u|^2 + |u|^3 + |u|^4\right) \mathrm{d}x\right]. \end{aligned} \tag{4.5}$$

Inequality (4.5) enables us to use Lemma 2.1, hence

$$\begin{aligned} \int_{Q_r(z)} h\left(|\varepsilon(u)|\right)\,\mathrm{d}x \le c\Bigg[& r^{-p}\int_{Q_{2r}(z)} |u|^p\,\mathrm{d}x + r^{-2}\int_{Q_{2r}(z)} |u|^2\,\mathrm{d}x \\ & + r^{-1}\int_{Q_{2r}(z)} \left(|u|^2+|u|^3+|u|^4\right)\,\mathrm{d}x\Bigg]. \end{aligned} \tag{4.6}$$

With the notation introduced after (3.8) we see that (4.6) implies in a first step the inequality (3.9), that is we obtain

$$\int_{Q_{\frac{R}{4}}(x_0)} h\left(|\varepsilon(u)|\right)\,\mathrm{d}x \le cR^{-1}\int_{Q_{\frac{R}{2}}(x_0)} |u|^2\,\mathrm{d}x. \tag{4.7}$$

With the help of (4.7) we then proceed exactly as done after (3.9) and get (for any $\delta > 0$)

$$\begin{aligned} \int_{Q_{\frac{R}{8}}(x_0)} h\left(|\varepsilon(u)|\right)\,\mathrm{d}x \quad \le \quad c\Bigg[& \delta R^{-1}\int_{Q_{\frac{R}{2}}(x_0)} |u|^2\,\mathrm{d}x + \delta^{1-p}R^{-p}\int_{Q_{\frac{R}{4}}(x_0)} |u|^p\,\mathrm{d}x \\ & + \delta^{-1}R^{-2}\int_{Q_{\frac{R}{4}}(x_0)} |u|^2\,\mathrm{d}x \\ & + R^{-1}\left[\int_{Q_{\frac{R}{4}}(x_0)} |u|^4\,\mathrm{d}x \int_{Q_{\frac{R}{4}}(x_0)} |u|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\Bigg]. \end{aligned}$$

Let $\delta = R^{-1/2}$. The above inequality implies (3.10) with the additional term

$$R^{-\frac{1}{2}-\frac{p}{2}}\int_{Q_{\frac{R}{2}}(x_0)} |u|^p\,\mathrm{d}x$$

on the right-hand side. Therefore we get in place of (3.11)

$$\int_{T_R} h\left(|\varepsilon(u)|\right)\,\mathrm{d}x \le c\left[R^{\frac{1}{2}}\Theta\left(\frac{R}{4}\right)^2 + R\Theta\left(\frac{R}{4}\right)^3 + R^{\frac{3}{2}-\frac{p}{2}}\Theta\left(\frac{R}{4}\right)^p\right],$$

but on account of $p \ge 2$ and the vanishing of Θ it clearly holds

$$R^{\frac{1}{2}}\Theta\left(\frac{R}{4}\right)^2 \ge cR^{\frac{3}{2}-\frac{p}{2}}\Theta\left(\frac{R}{4}\right)^p,$$

and as in Section 3 we obtain

$$\lim_{R\to\infty}\int_{T_R} h\left(|\varepsilon(u)|\right)\,\mathrm{d}x = 0$$

under the assumption (1.6) (or (1.5) for slow flows).

It remains to verify (3.13). We use the same testfunction φ as introduced after (3.13) observing that v satisfies

$$\|\nabla v\|_{L^s(T_R)} \leq c\|\nabla \eta^2 \cdot u\|_{L^s(T_R)} \tag{4.8}$$

for $s = 2$ and $s = p$.

Passing to (3.15) the first two terms on the right-hand side are now estimated as follows:

$$\begin{aligned}
\left|\int_{T_R} DH\left(\varepsilon(u)\right) : \left(\nabla\eta^2 \otimes u\right)\,\mathrm{d}x\right| &\overset{(2.2)}{\leq} c\int_{T_R}\left(h^{\frac{1}{\tau}}(|\varepsilon(u)|) + |\varepsilon(u)|\right)|\nabla\eta||u|\,\mathrm{d}x \\
&\leq c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)\mathrm{d}x + R^{-p}\int_{T_R}|u|^p\,\mathrm{d}x\right. \\
&\qquad \left. + \int_{T_R}|\varepsilon(u)|^2\,\mathrm{d}x + R^{-2}\int_{T_R}|u|^2\,\mathrm{d}x\right] \\
&\overset{(1.10)}{\leq} c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)\mathrm{d}x + R^{-2}\int_{T_R}|u|^2\,\mathrm{d}x\right]
\end{aligned}$$

on account of $p \geq 2$ and the boundedness of u. With (4.8) the same bound is seen to be true for $\int_{T_R} DH(\varepsilon(u)) : \varepsilon(v)\,\mathrm{d}x$, hence we get (3.16) with S being defined there. Clearly (3.17) remains valid, thus we get (3.13), and the proof of Theorem 1.1 is complete. □

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