# Duality properties and Riesz representation theorem in Besicovitch-Musielak-Orlicz space of almost periodic functions

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*Abstract.* This paper is an extension of the work done in [Morsli M., Bedouhene F., Boulahia F., *Duality properties and Riesz representation theorem in the Besicovitch-Orlicz space of almost periodic functions*, Comment. Math. Univ. Carolin. **43** (2002), no. 1, 103–117] to the Besicovitch-Musielak-Orlicz space of almost periodic functions. Necessary and sufficient conditions for the reflexivity of this space are given. A Riesz type "duality representation theorem" is also stated.

*Keywords:* Orlicz norm, Amemiya norm, conjugate function, Besicovitch-Musielak-Orlicz spaces, almost periodic functions, reflexivity, Riesz theorem

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## 1. Introduction

The Besicovitch-Musielak-Orlicz space of almost periodic functions was recently introduced in [9] and [10], where the authors characterized also some of its metric properties with respect to the Luxemburg norm.

In the present work, this space is endowed with the so-called Orlicz norm. Different properties and formulations of this norm are pointed out.

Next, necessary and sufficient conditions for the reflexivity of the space are given. A Riesz type "duality representation theorem" is also stated in this space.

## 2. Preliminaries

Let $\varphi$ be a Musielak-Orlicz function, i.e. $\varphi : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is such that:

(1) $\forall t \in \mathbb{R}$, $\varphi(t, \cdot)$ is convex on $\mathbb{R}^+$;

(2) $\forall x \in \mathbb{R}^+$, $\varphi(\cdot, x)$ is Lebesgue measurable on $\mathbb{R}$ and $\varphi(t, x) = 0$ iff $x = 0$, $\forall t \in \mathbb{R}$;

(3) $\forall t \in \mathbb{R}$, $\lim\limits_{x\to+\infty} \frac{\varphi(t,x)}{x} = +\infty$ and $\lim\limits_{x\to 0} \frac{\varphi(t,x)}{x} = 0$.

In the sequel we assume that $\varphi$ verifies also the following two conditions:

(4) $\varphi(\cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^+$;

(5) $\forall x \in \mathbb{R}^+$, $\varphi(\cdot, x)$ is periodic with period $T$ independent of $x$ (we may suppose $T = 1$).

We denote by $\psi$ the function complementary to $\varphi$, i.e.

$$\psi(t,x)=\sup_{y\geq 0}\{xy-\varphi(t,y)\},\quad \forall t\in\mathbb{R},\quad \forall x\in\mathbb{R}^+.$$

Recall that $\psi$ is also a Musielak-Orlicz function (see [11]). The pair $(\varphi,\psi)$ satisfies the Young inequality:

$$xy\leq\varphi(t,x)+\psi(t,y)\ \text{ for all }\ t\in\mathbb{R},\ \text{and}\ \ x,y\in\mathbb{R}^+.$$

Let now $\mathcal{M}(\mathbb{R},\mathbb{C})=\mathcal{M}$ be the set of all Lebesgue measurable functions on $\mathbb{R}$ with values in $\mathbb{C}$. The functional

$$\begin{array}{rcl}\rho_\varphi:\mathcal{M} & \longrightarrow & [0,+\infty]\\ f & \longrightarrow & \rho_\varphi(f)=\overline{\lim}_{T\to+\infty}\dfrac{1}{2T}\displaystyle\int_{-T}^{+T}\varphi(t,|f(t)|)\,dt=\overline{M}[\varphi(\cdot,|f(\cdot)|)]\end{array}$$

is a convex pseudo-modular on $\mathcal{M}$ (see [1]). The associated modular space:

$$\begin{aligned}B^\varphi &= \{f\in\mathcal{M}:\lim_{\alpha\to 0}\rho_\varphi(\alpha f)=0\}\\ &= \{f\in\mathcal{M}:\rho_\varphi(\lambda f)<+\infty,\ \text{for some}\ \ \lambda>0\}\end{aligned}$$

is called the Besicovitch-Musielak-Orlicz space. This space is naturally endowed with the Luxemburg norm:

$$\|f\|_\varphi=\inf\left\{k>0,\ \rho_\varphi\left(\frac{f}{k}\right)\leq 1\right\}.$$

We can also consider the so-called Amemiya norm defined as follows:

$$\|f\|_\varphi^A=\inf\left\{\frac{1}{k}(\rho_\varphi(kf)+1),k>0\right\}.$$

These two norms are in fact equivalent:

$$\|f\|_\varphi\leq\|f\|_\varphi^A\leq 2\|f\|_\varphi,\ \text{for all}\ \ f\in B^\varphi\ \ \text{(see [7])}.\tag{2.1}$$

Let $\mathcal{A}$ be the set of all generalized trigonometric polynomials, i.e.,

$$\mathcal{A}=\{P_n(t)=\sum_{j=1}^{j=n}a_je^{i\lambda_jt},\ a_j\in\mathbb{C},\ \lambda_j\in\mathbb{R},\ n\in\mathbb{N}\}.$$

The Besicovitch-Musielak-Orlicz space of almost periodic functions denoted by $B^\varphi a.p.$ is the closure of the set $\mathcal{A}$ with respect to the Luxemburg norm,

$$B^\varphi a.p.=\left\{f\in B^\varphi:\exists\{p_n\}\in\mathcal{A}\ \text{s.t.}\ \lim_{n\to+\infty}\|f-p_n\|_\varphi=0\right\}.$$

When $\varphi(t,x) = |x|$ the space $B^\varphi a.p.$ is denoted by $B^1 a.p.$ The closure of the set $\mathcal{A}$ with respect to the modular $\rho_\varphi$ is the subspace of $B^\varphi$ denoted by $\widetilde{B}^\varphi a.p.$:

$$\widetilde{B}^\varphi a.p. = \{f \in B^\varphi : \exists \{p_n\} \in \mathcal{A} \text{ s.t. } \lim_{n\to+\infty} \rho_\varphi(\alpha(f-p_n)) = 0 \text{ for some } \alpha > 0\}.$$

Let $\{u.a.p.\}$ denote the classical Bohr's algebra of almost periodic functions. It is known that $\{u.a.p.\}$ is the closure of the set $\mathcal{A}$ with respect to the uniform norm. It is easily seen that $\{u.a.p.\} \subseteq B^\varphi a.p. \subseteq B^1 a.p.$ Moreover, in view of Theorem 2.8 in [2] we have the following property:

> If $f \in \{u.a.p.\}$, $\varphi(\cdot,\cdot)$ is continuous and $\varphi(\cdot, x)$ is uniformly almost periodic with respect to $x$ then $\varphi(\cdot, |f(\cdot)|) \in \{u.a.p.\}$.

Therefore a fortiori, this holds true for a Musielak-Orlicz function satisfying the conditions (4) and (5) presented above.

A fundamental result concerning the functions in $B^\varphi a.p.$ is the following:

$$\text{If } f \in B^\varphi a.p., \text{ then } \varphi(\cdot, |f(\cdot)|) \in B^1 a.p. \text{ (see [10])}. \tag{2.2}$$

This property ensures the existence of the limit in the expression of $\rho_\varphi(f)$.

In order to end this introductory section, we define the so-called Orlicz norm in $B^\varphi a.p.$,

$$\|f\|_\varphi^o = \sup\left\{\overline{M}(|fg|),\quad g \in B^\psi a.p.,\quad \rho_\psi(g) \le 1\right\}.$$

Using the Young inequality it is easy to see that

$$\|f\|_\varphi^o \le \|f\|_\varphi^A, \quad \text{for all} \quad f \in B^\varphi a.p. \tag{2.3}$$

## 3. Auxiliary results

The fundamental convergence results of measure theory cannot be used directly in $B^\varphi a.p.$ A key role in our computations is played by the set function $\overline{\mu}$ defined on the $\sigma$-algebra $\Sigma(\mathbb{R}) = \Sigma$ of Lebesgue measurable sets as follows:

$$\overline{\mu}(A) = \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{-T}^{+T} \chi_A(t)\,dt = \overline{\lim_{T\to+\infty}} \frac{1}{2T}\mu(A \cap [-T, +T]),$$

where $\mu$ denotes the Lebesgue measure on $\mathbb{R}$. Note that $\overline{\mu}$ is not a measure.

We list here definitions and some properties concerning convergence type results with respect to the set function $\overline{\mu}$.

Let $\{f_n\}$ be a sequence in $B^\varphi$. We say that:

- $\{f_n\}$ is $\overline{\mu}$ convergent to $f$ (and denote by $f_n \xrightarrow{\overline{\mu}} f$) when, $\forall \eta > 0$, $\lim\limits_{n\to+\infty} \overline{\mu}\{t \in \mathbb{R} : |f_n - f| > \eta\} = 0$;
- $\{f_n\}$ is modular convergent to $f$ when, $\exists \alpha > 0$ such that: $\lim\limits_{n\to+\infty} \rho_\varphi(\alpha(f_n - f)) = 0$.

We have the following relations between the different kinds of convergence:

**Lemma 1.** (1) $\lim_{n\to+\infty}\|f_n - f\|_\varphi = 0$ *iff* $\forall\alpha > 0$, $\lim_{n\to+\infty}\rho_\varphi(\alpha(f_n - f)) = 0$ *(see* [11]*).*

(2) *If* $\lim_{n\to+\infty}\rho_\varphi(f_n - f) = 0$ *then* $f_n \xrightarrow{\overline{\mu}} f$ *(see Lemma* 2 *in* [10]*).*

(3) *Let* $\{f_n\} \subset B^\varphi$ *and* $f \in B^\varphi$*. If* $f_n \xrightarrow{\overline{\mu}} f$ *and* $\max(f_n, f) \leq g$ *with* $g \in B^\varphi a.p.$*, then* $\rho_\varphi(f_n) \longrightarrow \rho_\varphi(f)$ *(see Lemma* 5 *in* [10]*).*

(4) *Let* $\{f_n\} \subset B^\varphi$ *be such that* $f_n \xrightarrow{\overline{\mu}} f$ *and* $g(t,x)$ *a continuous function on* $\mathbb{R}\times\mathbb{R}^+$*, periodic with respect to* $t$*. Then* $g(\cdot, f_n(\cdot)) \xrightarrow{\overline{\mu}} g(\cdot, f(\cdot))$ *(see the proof of Proposition* 1 *in* [10]*).*

The following two results are very useful in our computations and proofs:

**Lemma 2.** *Let* $f \in B^\varphi a.p.$*,* $\rho_\varphi(f) > 0$*, and* $\{f_n\}$ *be a sequence* $\overline{\mu}$ *convergent to* $f$*. Then we have:*

(1) *there exist* $\alpha$*,* $\beta$*,* $\theta$ *with* $0 < \alpha < \beta$*;* $\theta \in ]0,1[$ *and* $G = \{t \in \mathbb{R}, \alpha \leq |f(t)| \leq \beta\}$ *such that* $\overline{\mu}(G) \geq \theta$ *(see* [10]*);*

(2) *there exist* $\alpha'$*,* $\beta'$*,* $\theta'$ *with* $0 < \alpha' < \beta'$*;* $\theta' \in ]0,1[$ *and* $G_n = \{t \in \mathbb{R}, \alpha' \leq |f_n(t)| \leq \beta'\}$ *such that* $\overline{\mu}(G_n) \geq \theta'$ *(see* [8]*).*

In the following we denote by $L^\varphi([0,1])$ the usual Musielak-Orlicz space of functions defined on $[0,1]$. The proposition below shows that $L^\varphi([0,1])$ is isometrically imbedded into the Besicovitch-Musielak-Orlicz space of almost periodic functions $\widetilde{B}^\varphi a.p.$

**Proposition 1** ([10])**.** *Let* $f \in L^\varphi([0,1])$*. Then,*

(1) *if* $\widetilde{f}$ *is the periodic extension of* $f$ *to the whole* $\mathbb{R}$ *(with period* $\tau = 1$*), we have* $\widetilde{f} \in \widetilde{B}^\varphi a.p.$*;*

(2) *the injection map* $i : L^\varphi([0,1]) \hookrightarrow \widetilde{B}^\varphi a.p.$*,* $i(f) = \widetilde{f}$ *is an isometry with respect to the modulars and for the respective Luxemburg norms.*

**Definition 1.** • We say that a function $f \in B^\varphi$ is absolutely integrable when:

$$\forall\varepsilon > 0,\ \ \exists\delta > 0\ \ \forall Q \in \Sigma,\quad \overline{\mu}(Q) \leq \delta \Longrightarrow \rho_\varphi(f\chi_Q) \leq \varepsilon.$$

• A sequence $\{f_n\} \subset B^\varphi$ is said equi-absolutely integrable when:

$$\forall\varepsilon > 0,\ \ \exists\delta > 0\ \ \exists n_0 \in \mathbb{N}\quad \forall Q \in \Sigma,\quad \overline{\mu}(Q) \leq \delta\ \text{ and }\ n \geq n_0 \Longrightarrow \rho_\varphi(f_n\chi_Q) \leq \varepsilon.$$

Remark that all the functions in $B^\varphi a.p.$ are absolutely integrable [10].

**Lemma 3.** *Let* $\{f_n\} \subset B^1$ *be* $\overline{\mu}$ *convergent to* $f \in B^1 a.p.$ *Then, if* $\{f_n\}$ *is equi-absolutely integrable we have* $\rho_1(f_n) \underset{n\to+\infty}{\longrightarrow} \rho_1(f)$*.*

Proof: Fix $\theta > 0$ and consider the set:

$$A_n^\theta = \{t \in \mathbb{R} : |f_n(t) - f(t)| > \theta\}.$$

Since the sequence $\{f_n\}$ is $\overline{\mu}$ convergent to $f$, we have

$$\lim_{n\to+\infty} \overline{\mu}(A_n^\theta) = 0. \tag{3.1}$$

Now

$$\begin{aligned}\rho_1(|f_n - f|) &\leq \rho_1(|f_n - f|\chi_{A_n^\theta}) + \rho_1(|f_n - f|\chi_{CA_n^\theta}) \\ &\leq \rho_1(|f_n|\chi_{A_n^\theta}) + \rho_1(|f|\chi_{A_n^\theta}) + \rho_1(|f_n - f|\chi_{CA_n^\theta}) \\ &\leq \rho_1(f_n\chi_{A_n^\theta}) + \rho_1(f\chi_{A_n^\theta}) + \theta.\end{aligned}$$

Given any $\varepsilon > 0$, the equi-absolute integrability of $\{f_n\}$ ensures the existence of $n_1 \in \mathbb{N}$ and $\delta_1 > 0$ s.t.

$$\forall Q \in \Sigma,\ \ \overline{\mu}(Q) \leq \delta_1 \ \text{ and } \ n \geq n_1 \Longrightarrow \rho_1(f_n\chi_Q) \leq \frac{\varepsilon}{2}\,. \tag{3.2}$$

On the other hand the absolute integrability of $f$ ensures the existence of a $\delta_2$ s.t.

$$\overline{\mu}(Q) \leq \delta_2 \Longrightarrow \rho_1(f\chi_Q) \leq \frac{\varepsilon}{2}\,. \tag{3.3}$$

Put $\delta = \min(\delta_1, \delta_2)$. Then by (3.1) there exists $n_2 \in \mathbb{N}$ s.t. $\forall n \geq n_2$ we have $\overline{\mu}(A_n^\theta) \leq \delta$. Hence for $n_0 = \max(n_1, n_2)$, we get $\forall n \geq n_0$

$$\max(\rho_1(f\chi_{A_n^\theta}), \rho_1(f_n\chi_{A_n^\theta})) \leq \frac{\varepsilon}{2}\,,$$

and then:

$$\forall \varepsilon > 0,\ \ \exists n_0 \in \mathbb{N},\quad \forall n \geq n_0,\ \ \rho_1(|f_n - f|) \leq \varepsilon + \theta.$$

Letting $n$ tending to infinity we get:

$$\overline{\lim_{n\to+\infty}}\ \rho_1(f_n - f) \leq \theta.$$

Finally, since $\theta > 0$ is arbitrary we deduce that $\lim\limits_{n\to+\infty} \rho_1(f_n - f) = 0$, i.e. $\rho_1(f_n) \underset{n\to+\infty}{\longrightarrow} \rho_1(f)$. □

**Remark 1.** Under the same hypothesis, the result of Lemma 3 remains true when $\{f_n\} \subset B^\varphi$ and $f \in B^\varphi a.p.$, i.e. $\rho_\varphi(f_n) \underset{n\to+\infty}{\longrightarrow} \rho_\varphi(f)$.

**Corollary 1.** *Let $\{f_n\} \subset B^\varphi$ be such that $\lim\limits_{n\to+\infty} \|f_n - f\|_\varphi = 0$, with $f \in B^\varphi a.p.$ Then $\rho_\varphi(f_n) \underset{n\to+\infty}{\longrightarrow} \rho_\varphi(f)$.*

Proof: Since the sequence $\{f_n\}$ is $\overline{\mu}$ convergent to $f$, it suffices to show that it is equi-absolutely integrable. In fact: $\rho_\varphi(f_n) \leq \frac{1}{2}\rho_\varphi(2(f_n - f)) + \frac{1}{2}\rho_\varphi(2f)$.
Given any $\varepsilon > 0$, since $2f \in B^\varphi a.p.$ there exists $\delta > 0$ s.t. $\forall Q \in \Sigma$ we have $\overline{\mu}(Q) \leq \delta \Longrightarrow \rho_\varphi(2f\chi_Q) \leq \varepsilon$.

On the other hand there exists $n_0 \in \mathbb{N}$ s.t. $\rho_\varphi(2(f_n - f)) \le \varepsilon, \ \forall n \ge n_0$. Finally, we have the following:

$$\rho_\varphi(f_n\chi_Q) \le \frac{1}{2}\rho_\varphi(2(f_n - f)\chi_Q) + \frac{1}{2}\rho_\varphi(2f\chi_Q) \le \varepsilon, \ \ \forall n \ge n_0, \ \ \forall Q \in \Sigma.$$

□

**Lemma 4.** (1) *If $f \in B^\varphi a.p.$ with $\|f\|_\varphi \neq 0$, then $\rho_\varphi(\frac{f}{\|f\|_\varphi}) = 1$.*

(2) *If $f \in B^\varphi a.p.$, $g \in B^\psi a.p.$, then $f \cdot g \in B^1 a.p.$ Moreover we have the so-called Hölder's inequality*

$$M(|f \cdot g|) \le 2\|f\|_\varphi \cdot \|g\|_\psi.$$

(3) *If $f \in B^\varphi a.p.$, then $\|f\|_\varphi^o \le 2\|f\|_\varphi$.*

Proof: (1) follows immediately from the property: $\|f\|_\varphi = 1$ iff $\rho_\varphi(f) = 1$ for $f \in B^\varphi a.p.$ (see [10]).

(2) Let $f \in B^\varphi a.p.$, $g \in B^\psi a.p.$, $\|f\|_\varphi \neq 0$, $\|g\|_\psi \neq 0$. From the Young inequality we have:

$$\frac{|f(t)|}{\|f\|_\varphi} \cdot \frac{|g(t)|}{\|g\|_\psi} \le \varphi\left(t, \frac{|f(t)|}{\|f\|_\varphi}\right) + \psi\left(t, \frac{|g(t)|}{\|g\|_\psi}\right).$$

Hence

$$\overline{M}\left(\frac{|f \cdot g|}{\|f\|_\varphi \|g\|_\psi}\right) \le \rho_\varphi\left(\frac{f}{\|f\|_\varphi}\right) + \rho_\psi\left(\frac{g}{\|g\|_\psi}\right) \le 2.$$

Then

$$\overline{M}(|f \cdot g|) \le 2\|f\|_\varphi \cdot \|g\|_\psi.$$

On the other hand $f \cdot g \in B^1 a.p.$ Indeed, let $\{p_n\}$, $\{q_n\}$ be two sequences in $\mathcal{A}$ such that $\lim_{n\to+\infty} \|p_n - f\| = 0$ and $\lim_{n\to+\infty} \|q_n - g\| = 0$. Then

$$\begin{aligned}
\overline{M}(|f \cdot g - p_n \cdot q_n|) &= \overline{M}(|f \cdot g - fq_n + fq_n - p_n \cdot q_n|) \\
&\le \overline{M}(|f| \cdot |g - q_n|) + \overline{M}(|q_n| \cdot |f - p_n|) \\
&\le 2\left(\|f\|_\varphi \cdot \|g - q_n\|_\psi + \|q_n\|_\psi \cdot \|f - p_n\|_\varphi\right) \\
&\le 2(\|f\|_\varphi \cdot \|g - q_n\|_\psi + (\sup_n \|q_n\|_\psi) \cdot \|f - p_n\|_\varphi).
\end{aligned}$$

Letting $n$ tending to infinity, we get: $\lim_{n\to+\infty} \overline{M}(|f \cdot g - p_n \cdot q_n|) = 0$. Consequently we have: $f \cdot g \in B^1 a.p.$ and $\overline{M}(f \cdot g) = M(f \cdot g) \le 2\|f\|_\varphi \cdot \|g\|_\psi$.

(3) Let $f \in B^\varphi a.p.$, $\|f\|_\varphi^o = \sup\{M(|fg|), g \in B^\psi a.p., \rho_\psi(g) \le 1\}$.

In view of Hölder's inequality we have:

$$M(|f \cdot g|) \le 2\|f\|_\varphi \cdot \|g\|_\psi, \ \ \forall f \in B^\varphi a.p., \quad \forall g \in B^\psi a.p.$$

Hence

$$\|f\|_\varphi^o \leq 2\|f\|_\varphi. \tag{3.4}$$

□

## 4. Equality between Orlicz norm and Amemiya norm

We are now ready to state the following comparison result:

**Theorem 1.** *If $f \in B^\varphi a.p.$, then*

$$\|f\|_\varphi^o = \inf\left\{\frac{1}{k}(1+\rho_\varphi(kf)), k>0\right\}.$$

*Moreover,*

$$\|f\|_\varphi^o = \frac{1}{k_0}(1+\rho_\varphi(k_0 f)), \quad \textit{for some} \quad k_0 > 0.$$

Proof: I) We suppose first that $\varphi$ is strictly convex with respect to $x$ for all $t \in \mathbb{R}$ and has a continuous derivative $\varphi^{'}(t,x) = \frac{\delta\varphi}{\delta x}(t,x)$ on $\mathbb{R}\times\mathbb{R}^+$. In this case the conjugate function $\psi$ verifies the same properties as $\varphi$.
We prove the theorem in several steps:

**Step 1.** Case where $f = p$ is a generalized trigonometric polynomial. In view of the inequality (2.3) it suffices to show the converse inequality:

$$\|f\|_\varphi^o \geq \frac{1}{k_0}(1+\rho_\varphi(k_0 f)), \quad \text{for some} \quad k_0 > 0.$$

For, consider the function $F(k)$; $k \geq 0$ defined as follows,

$$F(k) = \rho_\psi(\varphi^{'}(\cdot, k|p(\cdot)|)) = \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{-T}^{+T} \psi(t, \varphi^{'}(t, k|p(t)|))\, dt.$$

We claim that $\lim\limits_{k\to+\infty} F(k) = +\infty$. Indeed, from Lemma 2 there exist $\alpha > 0$, $\theta \in ]0,1[$ and a set $G = \{t \in \mathbb{R} : |p(t)| \geq \alpha\}$ such that $\overline{\mu}(G) \geq \theta$. Then

$$\begin{aligned}
F(k) &\geq \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{[-T,+T]\cap G} \psi(t,\varphi^{'}(t,k\alpha))\, dt \\
&\geq \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{[-T,+T]\cap G} \inf_{t\in\mathbb{R}} \psi(t,\varphi^{'}(t,k\alpha))\, dt \\
&\geq \overline{\mu}(G)\psi(t_0,\varphi^{'}(t_0,k\alpha)) \\
&\geq \theta\psi(t_0,\varphi^{'}(t_0,k\alpha)) \\
&\geq \theta\psi\left(t_0, \frac{\varphi(t_0,k\alpha)}{k\alpha}\right).
\end{aligned}$$

Since $\lim\limits_{x\to+\infty}\frac{\varphi(t,x)}{x}=+\infty,\ \forall t\in\mathbb{R}$, we get $\lim\limits_{k\to+\infty}F(k)=+\infty$.

We will show now that $F$ is continuous on $[0,+\infty[$. Let $k_*\in[0,+\infty[$ and $\{k_n\}$ be a sequence in $[0,+\infty[$ convergent to $k_*$. It is clear that

$$k_n|p(\cdot)| \ \text{ is } \ \overline{\mu} \ \text{ convergent to } \ k_*|p(\cdot)|.$$

Moreover, using Lemma 1(4), we get:

$$\varphi^{'}(\cdot,k_n|p(\cdot)|)\xrightarrow{\overline{\mu}}\varphi^{'}(\cdot,k_*|p(\cdot)|).$$

Since $\{k_n\}$ is bounded we have $\max(\varphi^{'}(\cdot,k_n|p(\cdot)|),\varphi^{'}(\cdot,k_*|p(\cdot)|))\leq\varphi^{'}(\cdot,M|p(\cdot)|)$ with $\varphi^{'}(\cdot,M|p(\cdot)|)\in\{u.a.p.\}\subset B^{\psi}a.p.$ Using Lemma 1(3), we deduce that $\lim\limits_{n\to+\infty}\rho_\psi(\varphi^{'}(\cdot,k_n|p(\cdot)|))=\rho_\psi(\varphi^{'}(\cdot,k_*|p(\cdot)|))$. This proves the continuity of $F$. Consequently, since $F(0)=0$ and $\lim\limits_{k\to+\infty}F(k)=+\infty$, there exists a $k_0\in]0,+\infty[$ such that $\rho_\psi(\varphi^{'}(\cdot,k_0|p(\cdot)|))=1$. Now, considering the case of equality in Young's inequality we get

$$\begin{aligned}\|p\|_\varphi^o &\geq \frac{1}{k_0}M\left(k_0|p|\varphi^{'}(\cdot,k_0|p(\cdot)|)\right)\\ &\geq \frac{1}{k_0}\left(\rho_\varphi(k_0|p|)+\rho_\psi(\varphi^{'}(\cdot,k_0|p(\cdot)|))\right)\\ &\geq \frac{1}{k_0}\left(\rho_\varphi(k_0|p|)+1\right).\end{aligned}\tag{4.1}$$

Combining inequalities (4.1) and (2.3) we get

$$\|p\|_\varphi^o=\inf\left\{\frac{1}{k}\left(1+\rho_\varphi(k|p|)\right)\right\}=\frac{1}{k_0}\left(\rho_\varphi(k_0|p|)+1\right).$$

Note also that we have

$$\|p\|_\varphi^o=M(|p(\cdot)|\varphi^{'}(\cdot,k_0|p(\cdot)|)).\tag{4.2}$$

**Step 2.** Now we will prove that the result remains true for $f\in B^\varphi a.p.$

Let $\{p_n\}\subset\mathcal{A}$ be such that $\lim\limits_{n\to+\infty}\|p_n-f\|_\varphi=0$.

From Step 1, for all $n\in\mathbb{N}$, there exists $k_n\in]0,+\infty[$ such that

$$\|p_n\|_\varphi^o=\frac{1}{k_n}\left(\rho_\varphi(k_np)+1\right).\tag{4.3}$$

Using the inequality (3.4), we get

$$\frac{1}{k_n}\leq\|p_n\|_\varphi^o\leq 2\|p_n\|_\varphi\leq 2\sup_n\|p_n\|_\varphi,$$

hence

$$k_n \geq \frac{1}{2\sup_n \|p_n\|_{\varphi}} = C_1 > 0, \quad \forall n \in \mathbb{N}.$$

On the other hand $\{k_n\}$ is bounded from above. Indeed, in the opposite case, there will exist a subsequence still denoted by $\{k_n\}$ such that $\lim_{n\to+\infty} k_n = +\infty$. Then, again from Lemma 2, there exist $\alpha > 0$, $\theta \in ]0,1[$ and $G_n = \{t \in \mathbb{R} : |p_n(t)| \geq \alpha\}$, such that $\overline{\mu}(G_n) \geq \theta$. Thus,

$$\begin{aligned}
1 = \rho_{\psi}\left(\varphi^{'}(\cdot, k_n|p_n(\cdot)|)\right) &\geq \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{[-T,+T]\cap G_n} \psi(t, \varphi^{'}(t, k_n\alpha))\,dt \\
&\geq \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{[-T,+T]\cap G_n} \inf_{t\in\mathbb{R}} \psi(t, \varphi^{'}(t, k_n\alpha))\,dt \\
&\geq \overline{\mu}(G)\psi(t_0, \varphi^{'}(t_0, k\alpha)) \\
&\geq \theta\psi(t_0, \varphi^{'}(t_0, k_n\alpha)) \\
&\geq \theta\psi\left(t_0, \frac{\varphi(t_0, k_n\alpha)}{k_n\alpha}\right),
\end{aligned}$$

and then $\lim_{n\to+\infty} \rho_{\psi}(\varphi^{'}(\cdot, k_n|p_n(\cdot)|)) = +\infty$, a contradiction.

Now $\{k_n\}$ being bounded, there exists a subsequence denoted again by $\{k_n\}$ that converges to some $k_0 > 0$. We have $\lim_{n\to+\infty} \|k_n p_n - k_0 f\|_{\varphi} = 0$ and by Corollary 1 we deduce that

$$\lim_{n\to+\infty} \rho_{\varphi}(k_n p_n) = \rho_{\varphi}(k_0 f).$$

Finally, using inequality (3.4) and letting $n$ tending to infinity in (4.3) we get:

$$\|f\|_{\varphi}^{o} = \lim_{n\to+\infty} \|p_n\|_{\varphi}^{o} = \lim_{n\to+\infty} \left(\frac{1}{k_n}\left(\rho_{\varphi}(k_n p_n + 1)\right)\right) = \frac{1}{k_0}\left(\rho_{\varphi}(k_0 f) + 1\right).$$

II) To complete the proof of the theorem, we will prove that the result remains true for a general Musielak-Orlicz function $\varphi$.

Indeed, for all $\varepsilon > 0$, we can find a Musielak-Orlicz function $\varphi_{\varepsilon}$ with a continuous derivative $\varphi_{\varepsilon}^{'} = \frac{\delta\varphi_{\varepsilon}}{\delta x}(t, x)$ on $\mathbb{R} \times \mathbb{R}^{+}$ verifying the inequality

$$\varphi(t, x) \leq \varphi_{\varepsilon}(t, x) \leq \varphi(t, (1+\varepsilon)x), \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^{+}.$$

An example of such a function $\varphi_{\varepsilon}$ is the following (see [3], [7]),

$$\varphi_{\varepsilon}(t, x)^{*} = \frac{1}{\ln(1+\varepsilon)}\int_{x}^{(1+\varepsilon)x} \frac{\varphi(t, s)}{s}\,ds, \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^{+}.$$

$^{*}\varphi_{\varepsilon}(t, x)$ verifies $\varphi_{\varepsilon}(t, x) = 0$ iff $x = 0$ for all $t \in \mathbb{R}$.

Moreover, defining the new function

$$\varphi^{\varepsilon}(t,x)^{\dagger} = \varphi(t,(1+\varepsilon)x - \varepsilon\ln(x+1)),$$

we can easily check that $\varphi^{\varepsilon}$ is strictly convex with respect to $x \in \mathbb{R}^{+}$ for all $t \in \mathbb{R}$ and satisfies the inequality

$$\varphi(t,x) \le \varphi^{\varepsilon}(t,x) \le \varphi(t,(1+\varepsilon)x), \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^{+}.$$

Summing up, we claim that for all $\varepsilon > 0$, there exists a strictly convex Musielak-Orlicz function $\varphi_{\varepsilon}$ with a continuous derivative $\varphi_{\varepsilon}^{'}(t,x) = \frac{\delta\varphi_{\varepsilon}}{\delta x}(t,x)$ on $\mathbb{R}\times\mathbb{R}^{+}$ satisfying:

$$\varphi(t,x) \le \varphi_{\varepsilon}(t,x) \le \varphi(t,(1+\varepsilon)x). \tag{4.4}$$

From this, it follows immediately (see [3], [7]) that:

$$B^{\varphi}a.p. = B^{\varphi_{\varepsilon}}a.p.$$

and

$$\|f\|_{\varphi}^{A} \le \|f\|_{\varphi_{\varepsilon}}^{A} \le (1+\varepsilon)\|f\|_{\varphi}^{A}\ ; \quad \|f\|_{\varphi}^{o} \le \|f\|_{\varphi_{\varepsilon}}^{o} \le (1+\varepsilon)\|f\|_{\varphi}^{o}. \tag{4.5}$$

Recall that it was proved in Step 1 that,

$$\|f\|_{\varphi_{\varepsilon}}^{o} = \|f\|_{\varphi_{\varepsilon}}^{A}. \tag{4.6}$$

This equality remains true for $\varphi$. Indeed, we already know that $\|f\|_{\varphi}^{o} \le \|f\|_{\varphi}^{A}$. Then using (4.6) and (4.4) we can write

$$\|f\|_{\varphi}^{o} \le \|f\|_{\varphi}^{A} \le \|f\|_{\varphi_{\varepsilon}}^{A} = \|f\|_{\varphi_{\varepsilon}}^{o} \le (1+\varepsilon)\|f\|_{\varphi}^{o}.$$

Finally, since $\varepsilon$ is arbitrarily small, we deduce that

$$\|f\|_{\varphi}^{o} = \|f\|_{\varphi}^{A}.$$

To end the proof, let us show that

$$\|f\|_{\varphi}^{o} = \frac{1}{k_0}\left(\rho_{\varphi}(k_0 f) + 1\right) \quad \text{for some} \quad k_0 > 0.$$

For all $\varepsilon > 0$ we have

$$\|f\|_{\varphi_{\varepsilon}}^{0} = \frac{1}{k_{\varepsilon}}\left(\rho_{\varphi_{\varepsilon}}(k_{\varepsilon} f) + 1\right),$$

for some $k_{\varepsilon} > 0$ such that

$$\rho_{\psi_{\varepsilon}}(\varphi_{\varepsilon}^{'}(\cdot, k_{\varepsilon} f)) = 1.$$

---

[†] Since the function $(1+\varepsilon)x - \varepsilon\ln(1+x)$ is s.c. and $\varphi(t,x) = 0$ iff $x = 0$ then $\varphi^{\varepsilon}(t,x)$ is strictly convex (see [5]).

Using the same reasoning as in the previous part, we can deduce that the sequence $\{k_\varepsilon\}$ is bounded. Then we can extract a subsequence also denoted by $\{k_\varepsilon\}$ convergent to some $k_0 > 0$.

Now, we have from (4.5) that

$$\|f\|_\varphi^o = \lim_{\varepsilon\to 0} \frac{1}{k_\varepsilon}\left(\rho_{\varphi_\varepsilon}(k_\varepsilon f) + 1\right).$$

On the other hand

$$\frac{1}{k_\varepsilon}\left(\rho_\varphi(k_\varepsilon f) + 1\right) \le \frac{1}{k_\varepsilon}\left(\rho_{\varphi_\varepsilon}(k_\varepsilon f) + 1\right) \le \frac{1}{k_\varepsilon}\left(\rho_\varphi(k_\varepsilon(1+\varepsilon) f) + 1\right).$$

Letting $\varepsilon$ tending to *zero* together with the continuity of the function $\alpha \longrightarrow \rho_\varphi(\alpha f)$ we deduce that:

$$\lim_{\varepsilon\to 0} \frac{1}{k_\varepsilon}\left(\rho_{\varphi_\varepsilon}(k_\varepsilon f) + 1\right) = \frac{1}{k_0}\left(\rho_\varphi(k_0 f) + 1\right) = \|f\|_\varphi^o.$$

This completes the proof of Theorem 1. □

**Lemma 5.** *The Orlicz norm is equivalent to the Luxemburg norm in $B^\varphi a.p.$*

$$\|f\|_\varphi \le \|f\|_\varphi^o \le 2\|f\|_\varphi, \quad f \in B^\varphi a.p.$$

Proof: This is an immediate consequence of (2.1) and Theorem 1. We give here another and direct proof of this result. It remains only to prove the left inequality, or equivalently that $\rho_\varphi(\frac{f}{\|f\|_\varphi^o}) \le 1$.

First let $p \in \mathcal{A}$, $p \ne 0$ and let $g \in B^\psi a.p.$; then we have to consider two cases:

- $\rho_\psi(g) \le 1$, in this case we have $M(|p \cdot g|) \le \|p\|_\varphi^o$;
- $\rho_\psi(g) > 1$, in this case we have $\rho_\psi(\frac{g}{\rho_\psi(g)}) \le \frac{1}{\rho_\psi(g)} \cdot \rho_\psi(g) \le 1$, hence $M(|p \cdot \frac{g}{\rho_\psi(g)}|) \le \|p\|_\varphi^o$.

It follows that in each case we have

$$M(|p\cdot g|) \le \max(1, \rho_\psi(g))\|p\|_\varphi^o.$$

Using the case of equality in Young's inequality we get for a suitable $g$,

$$\rho_\varphi\left(\frac{p}{\|p\|_\varphi^o}\right) + \rho_\psi(g) = M\left(\left|\frac{p}{\|p\|_\varphi^o}\cdot g\right|\right) \le \max(1, \rho_\psi(g)),$$

and then $\rho_\varphi(\frac{p}{\|p\|_\varphi^o}) \le 1$.

Let now $f \in B^\varphi a.p.$ satisfy $\|f\| \ne 0$ and take $\{p_n\}$ in $\mathcal{A}$ such that $\lim\limits_{n\to+\infty}\|p_n - f\|_\varphi = 0$.

Let $k_n = \frac{1}{\|p_n\|_\varphi^o}$. From inequality (3.4) we have $\lim\limits_{n\to+\infty} k_n = k_0 = \frac{1}{\|f\|_\varphi^o}$. Then

$$\lim_{n\to+\infty}\|k_n p_n - k_0 f\|_\varphi = 0,$$

and using Corollary 1 we deduce that

$$\rho_\varphi\left(\frac{p_n}{\|p_n\|_\varphi^o}\right) \underset{n\to+\infty}{\longrightarrow} \rho_\varphi\left(\frac{f}{\|f\|_\varphi^o}\right).$$

Now since $\rho_\varphi(\frac{p_n}{\|p_n\|_\varphi^o}) \le 1$, for all $n \in \mathbb{N}$, it follows that $\rho_\varphi(\frac{f}{\|f\|_\varphi^o}) \le 1$. Finally, we have

$$\|f\|_\varphi \le \|f\|_\varphi^o \le 2\|f\|_\varphi.$$

□

## 5. Duality in $B^\varphi a.p.$

### 5.1 Reflexivity of the space $\widetilde{B}^\varphi a.p.$

**Definition 2.** We say that $\varphi$ satisfies the $\Delta_2^{B^1}$ condition if there exist a constant $k > 0$ and a positive function $h$ with $\rho_1(h) < +\infty$ such that,

$$\varphi(t, 2x) \le k\varphi(t, x) + h(t), \quad \text{for almost all } t \in \mathbb{R}, \text{ and all } \ x \in \mathbb{R}^+.$$

We say that $\varphi$ is $\nabla_2^{B^1}$ if $\psi$ is $\Delta_2^{B^1}$.

**Definition 3.** The function $\varphi$ is uniformly convex on $\mathbb{R}^+$ (see [6]) when, $\forall a \in ]0,1[, \exists\delta(a) \in ]0,1[,\ \forall b \in [0,a]$:

$$\varphi(t, \frac{x+by}{2}) \le (1-\delta(a))\frac{\varphi(t,x)+\varphi(t,by)}{2},$$

for almost all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^+$.

**Theorem 2.** *The space $\widetilde{B}^\varphi a.p.$ is reflexive iff $\varphi \in \triangle_2^{B^1} \cap \nabla_2^{B^1}$.*

PROOF: **Necessity**: suppose that $\widetilde{B}^\varphi a.p.$ is reflexive. From [10] we know that $\widetilde{B}^\varphi a.p.$ contains an isometric copy of the Musielak-Orlicz space $L^\varphi[0,1]$. From [6] a necessary and sufficient condition for the reflexivity of $L^\varphi[0,1]$ is that $\varphi$ satisfies the $\triangle_2^{L^1} \cap \nabla_2^{L^1}$ condition‡. In this case $\varphi$ satisfies also the $\triangle_2^{B^1} \cap \nabla_2^{B^1}$ conditions (see the proof of Theorem 1 in [10, p. 457]).

**Sufficiency:** Suppose that $\varphi \in \Delta_2^{B^1} \cap \nabla_2^{B^1}$. One can see directly that $\varphi \in \Delta_2^{L^1} \cap \nabla_2^{L^1}$. Then from [6] there exists a Musielak-Orlicz function $\varphi_1$ defined on $[0,1] \times \mathbb{R}^+$ uniformly convex and equivalent to the restriction of $\varphi$ on $[0,1] \times \mathbb{R}^+$. Now the 1-periodic extension of $\varphi_1$ denoted by $\widetilde{\varphi}_1$§, defined on $\mathbb{R} \times \mathbb{R}^+$ is also uniformly convex and equivalent to $\varphi$. We deduce that $(B^{\widetilde{\varphi}_1} a.p., \ \|\cdot\|_{\widetilde{\varphi}_1})$ is uniformly convex (see [9]) and so reflexive. Hence $B^\varphi a.p.$ is reflexive. □

‡We say that $\varphi$ is $\Delta_2^{L^1}$ if there exist a constant $k > 0$ and a positive function $h$ with $\int_0^1 h(t)\,dt < +\infty$ s.t. $\varphi(t,2x) \le k\varphi(t,x) + h(t)$, for almost all $t \in [0,1]$ and all $x \ge 0$.

§From the construction of $\varphi_1$ made in [6] page 61, we remark that $\widetilde{\varphi}_1$ inherits the continuity of $\varphi$ on $\mathbb{R} \times \mathbb{R}^+$.

**5.2 Riesz representation theorem in $B^{\varphi}a.p.$.** In view of (4.4), we may suppose in the following that $\varphi$ has a continuous derivative $\varphi'$ and is strictly convex (or equivalently that $\varphi'(t,x)$ is strictly increasing with respect to $x \in \mathbb{R}^+$ and all $t \in \mathbb{R}$).

**Lemma 6.** *If $f \in B^{\psi}a.p.$ then*

$$\begin{aligned} \|f\|_{\psi}^{o} &= \sup\{|M(f\cdot g)|, g \in B^{\varphi}a.p., \quad \rho_{\varphi}(g) \le 1\} \\ &= \sup\{|M(f\cdot g)|, g \in \{u.a.p.\}, \quad \rho_{\varphi}(g) \le 1\}. \end{aligned}$$

PROOF: Consider the case $f = p \in \mathcal{A}$. From the Hölder's inequality we have:

$$|M(p\cdot q)| \le M(|p\cdot q|) \le \|p\|_{\psi}^{o}, \quad \forall q \in \{u.a.p.\}, \quad \rho_{\varphi}(q) \le 1.$$

From the proof of Theorem 1, we know that there exists $k_0 > 0$ such that $\rho_{\varphi}(\psi'(\cdot, k_0|p(\cdot)|)) = 1$, and

$$\|p\|_{\psi}^{o} = M\left(|p(\cdot)\psi'(\cdot, k_0|p(\cdot)|)|\right) = M\left(p(\cdot)\operatorname{sign} p(\cdot)\psi'(\cdot, k_0|p(\cdot)|)\right).$$

Note that $\operatorname{sign} p(\cdot)\psi'(\cdot, k_0|p(\cdot)|) \in \{u.a.p.\}$. Indeed, let

$$F(t,u) = \begin{cases} \dfrac{u}{|u|}\cdot \psi'(t, k_0|u|) & \text{if } \ u \neq 0, \\ 0 & \text{else.} \end{cases}$$

Then $F$ is continuous on $\mathbb{R}\times\mathbb{R}^+$ and periodic in $t$ uniformly with respect to $u$. Since

$$\operatorname{sign} p(t)\psi'(t, k_0|p(t)|) = F(t, p(t))$$

the conclusion follows from Theorem 2.8 in ([2]). Summarizing all these, we have $\|p\|_{\psi}^{o} = M(p\cdot q)$, where $q(\cdot) = \operatorname{sign} p(\cdot)\psi'(\cdot, k_0|p(\cdot)|)$. Then we can assert that:

$$\|p\|_{\psi}^{o} = \sup\{|M(p\cdot q)|, \quad q \in \{u.a.p.\}, \quad \rho_{\varphi}(q) \le 1\}. \tag{5.1}$$

Now let us show that (5.1) remains true for $f \in B^{\psi}a.p.$ Let $\{p_n\} \subset \mathcal{A}$ be a sequence such that: $\lim_{n\to+\infty} \|p_n - f\|_{\psi} = 0$ and consider the quantity

$$I(f) = \sup\{|M(f\cdot q)|, \ q \in \{u.a.p.\}, \ \rho_{\varphi}(q) \le 1\}.$$

It is clear that we have

$$I(f) \le \|f\|_{\psi}^{o}.$$

Moreover, for a fixed $\varepsilon > 0$, we have $\|f\|_{\psi}^{o} \le \|p_n\|_{\psi}^{o} + \varepsilon$, $\forall n \ge n_0$, for some $n_0 > 0$. Hence

$$\|f\|_{\psi}^{o} - \varepsilon \le \|p_n\|_{\psi}^{o} = \sup\{|M(p_n\cdot q)|, \ q \in \{u.a.p.\}, \ \rho_{\varphi}(q) \le 1\}.$$

Using Hölder's inequality we assert that:

$$\begin{aligned}\|p_n\|^o_\psi &\le \sup\{\|p_n - f\|_\psi \cdot \|q\|_\varphi,\ q \in \{u.a.p.\},\ \ \rho_\varphi(q) \le 1\} \\ &\quad + \sup\{|M(fq)|,\ q \in \{u.a.p.\},\ \ \rho_\varphi(q) \le 1\} \\ &\le \varepsilon + I(f),\ \forall n \ge n_0.\end{aligned}$$

Then

$$\|f\|^o_\psi \le I(f) + 2\varepsilon.$$

Finally, since $\varepsilon$ is arbitrary, we conclude that

$$\|f\|^o_\psi \le I(f).$$

Consequently,

$$\|f\|^o_\psi = I(f).$$

This completes the proof. □

**Theorem 3.** *If $\varphi \in \Delta_2^{B_1} \cap \nabla_2^{B_1}$, then $(\widetilde{B}^\varphi a.p.,\ \|\cdot\|_\varphi)^*$ is isomorphically isometric to $(\widetilde{B}^\psi a.p.,\ \|\cdot\|^o_\psi)$. More precisely: if $G$ is a linear continuous functional on $\widetilde{B}^\varphi a.p.$ then there exists a unique $g \in \widetilde{B}^\psi a.p.$ such that:*

- *$G(f) = M(fg)$, $\forall f \in \widetilde{B}^\varphi a.p.$ and*
- *$\|G\| = \|g\|^o_\psi$.*

*Conversely, the condition $\varphi \in \Delta_2^{B_1} \cap \nabla_2^{B_1}$ is necessary for this identification.*

Proof: Consider the linear mapping

$$\begin{aligned} A:\ (B^\psi a.p.,\ \|\cdot\|^o_\psi) &\longrightarrow (B^\varphi a.p., \|\cdot\|_\varphi)^* \\ g &\longrightarrow A(g), \quad A(g)(f) = M(f \cdot g).\end{aligned}$$

$A$ is well defined. Moreover, using Lemma 6 we have:

$$\|A(g)\| = \sup_{\|f\|_\varphi \le 1} |A(g)(f)| = \sup_{\rho_\varphi(f) \le 1} |A(g)(f)| = \|g\|^o_\psi.$$

$A$ is then an isometry.

It remains to show that $A$ is surjective. Let $E = A(B^\psi a.p.)$. Then $E$ is a complete subspace of $(B^\varphi a.p.)^*$. From Banach's classical results, it is sufficient to show that for each $F \in (B^\varphi a.p.)^{**}$ such that $F(A(g)) = 0$, $\forall g \in B^\psi a.p.$, we have $F(h) = 0$, $\forall h \in (B^\varphi a.p.)^*$.

Let then $F \in (B^\varphi a.p.)^{**}$ be such that $F(A(g)) = 0$, $\forall g \in B^\psi a.p.$ Since $B^\varphi a.p.$ is reflexive, there exists $f \in B^\varphi a.p.$ such that $\pi(f) = F$, i.e.

$$\pi(f)(A(g)) = A(g)(f) = M(f \cdot g) = 0, \quad \forall g \in B^\psi a.p.$$

Using Lemma 6 we deduce that $\|f\|^o_\varphi = 0$ and so $\|F\| = 0$.

Conversely, if the identification $(\widetilde{B}^{\varphi} a.p.)^* = \widetilde{B}^{\psi} a.p.$ holds, we will also have

$$(\widetilde{B}^{\varphi} a.p.)^{**} = (\widetilde{B}^{\psi} a.p.)^* = \widetilde{B}^{\varphi} a.p.$$

So $\widetilde{B}^{\varphi} a.p.$ is reflexive and, consequently, $\varphi \in \Delta_2^{B_1} \cap \nabla_2^{B_1}$. □

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