On extension of functors

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Abstract. A. Chigogidze defined for each normal functor on the category Comp an extension which is a normal functor on the category Tych. We consider this extension for any functor on the category Comp and investigate which properties it preserves from the definition of normal functor. We investigate as well some topological properties of such extension.

Keywords: Chigogidze extension of functors, 1-preimage preserving property

Classification: 18B30, 54B30, 57N20

Introduction

The general theory of functors acting in the category Comp of compact Hausdorff spaces (compacta) and continuous mappings was founded by E.V. Shchepin [15]. He distinguished some elementary properties of such functors and defined the notion of normal functor that has become very fruitful. The class of normal functors includes many classical constructions: the hyperspace exp, the functor of probability measures P, the power functor and many other functors (see [13], [9] for more details). But some important functors do not satisfy some of the properties from the Shchepin list. Omitting some properties we obtain wider classes of functors such as weakly normal functors and almost normal functors.

The properties from the definition of normal functor could be easily generalized for the functors on the category Tych of Tychonov spaces and continuous maps. Let us remark that Tych contains Comp as a subcategory. A. Chigogidze defined for each normal functor on the category Comp an extension which is a normal functor on the category Tych [6]. This extension could be considered for any functor on the category Comp. But the situation is more complicated for wider classes of functors. For example, the extension of the projective power functor (which is weakly normal) does not preserve embeddings, which makes such extension useless (see for example [13, p. 67]). However, if we apply the Chigogidze extension to such weakly normal functors as the functor O of order-preserving functionals, the functor G of inclusion hyperspaces, the superextension, we obtain functors on the category Tych which preserve embeddings.

The main aim of this paper is to investigate which properties from the definition of normal functor are preserved by Chigogidze extension, specially we concentrate our attention on the preserving of embeddings. The results devoted to this problem are contained in Section 2. We define in this section the 1-preimages preserving property which is crucial for preserving of embeddings. In Section 3 we consider which functors have the 1-preimages preserving property.

T. Banakh and R. Cauty obtained topological classification of the Chigogidze extension of the functor of probability measures for separable metric spaces. We generalize this result to convex functors in Section 4.

§1

All spaces are assumed to be Tychonov, all mappings are continuous. All functors are assumed to be covariant. In the present paper we will consider functors acting in two categories: the category Tych and its subcategory Comp.

Let us recall the definition of normal functor. A functor $F : \mathsf{Comp} \to \mathsf{Comp}$ is called *monomorphic* (*epimorphic*) if it preserves embeddings (surjections). For a monomorphic functor F and an embedding $i : A \to X$ we shall identify the space $F(A)$ and the subspace $F(i)(F(A)) \subset F(X)$.

A monomorphic functor F is said to be *preimage-preserving* if for each map $f : X \to Y$ and each closed subset $A \subset Y$ we have $(F(f))^{-1}(F(A)) = F(f^{-1}(A))$.

For a monomorphic functor F the *intersection-preserving* property is defined as follows: $F(\bigcap\{X_\alpha \mid \alpha \in \mathcal{A}\}) = \bigcap\{F(X_\alpha) \mid \alpha \in \mathcal{A}\}$ for every family $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ of closed subsets of X.

A functor F is called *continuous* if it preserves the limits of inverse systems $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, \mathcal{A}\}$ over a directed set $\mathcal{A}$. Let us also note that for any continuous functor $F : \mathsf{Comp} \to \mathsf{Comp}$ the map $F : C(X,Y) \to C(FX,FY)$ (the space $C(X,Y)$ is considered with the compact-open topology) is continuous.

Finally, a functor F is called *weight-preserving* if $w(X) = w(F(X))$ for every infinite $X \in \mathsf{Comp}$.

A functor F is called *normal* [15] if it is continuous, monomorphic, epimorphic, preserves weight, intersections, preimages, singletons and the empty space. A functor F is said to be *weakly normal* (*almost normal*) if it satisfies all the properties from the definition of a normal functor except perhaps the preimage-preserving property (epimorphicity) (see [13] for more details).

Similarly, one can define the same properties for a functor $F : \mathsf{Tych} \to \mathsf{Tych}$ with the only difference that the property of preserving surjections is replaced by the property of sending k-covering maps to surjections (recall that $f : X \to Y$ is a k-covering map if for any compact set $B \subset Y$ there exists a compact set $A \subset X$ with $f(A) = B$) (see [13, Definition 2.7.1]).

A. Chigogidze defined an extension construction of a functor in Comp onto Tych the following way [6]. For any normal functor $F : \mathsf{Comp} \to \mathsf{Comp}$ and any $X \in \mathsf{Tych}$, the space

$$F_\beta(X) = \{a \in F(\beta X) \mid \text{ there exists a compact set } A \subset X \text{ with } a \in F(A)\}$$

is considered with the topology induced from $F(\beta X)$, where βX is the Stone-Čech compactification of the space X. Next, given any continuous mapping $f : X \to Y$ between Tychonov spaces, put $F_\beta(f) = F(\beta f)|_{F_\beta(X)}$. Then F_β forms a covariant functor in the category Tych. Chigogidze showed that in case F is normal, the functor F_β is also normal.

§2

Let us modify the Chigogidze construction for any functor $F : \mathsf{Comp} \to \mathsf{Comp}$. For $X \in \mathsf{Tych}$ we put

$$F_\beta(X) = \{a \in F(\beta X) \mid \text{ there exists a compact set } A \subset X \text{ with } a \in F(i_A)(F(A))\}$$

where by i_A we denote the natural embedding $i_A : A \hookrightarrow X$ (we do not assume that the map $F(i_A)$ is an embedding). Evidently F_β preserves empty set and one-point space iff F does.

Now we consider the problem when F_β preserves embeddings. Extension of any normal functor preserves embeddings, but, if we drop the preimage preserving property, the situation could be different. However, the examples from the introduction show that the preimage-preserving property is not necessary. We define some weaker property which will give us a necessary and sufficient condition.

Definition 1. We say that a monomorphic functor $F : \mathsf{Comp} \to \mathsf{Comp}$ *preserves* 1-*preimages*, if for any $f : X \to Y$, where $X, Y \in \mathsf{Comp}$, any closed $A \subset Y$ such that $f|_{f^{-1}(A)}$ is a homeomorphism, we have that $(Ff)^{-1}(FA) = F(f^{-1}(A))$. (Let us remark it is equivalent to the condition that the map $Ff \mid (Ff)^{-1}(FA)$ is a homeomorphism.)

Let us note that this definition was independently introduced by T. Banakh, M. Klymenko and A. Kucharski [3].

Proposition 1. *If F is a monomorphic functor that preserves 1-preimages in the class of open mappings, then F preserves 1-preimages.*

Proof: Take any mapping $f : X \to Y$ such that $f|_{f^{-1}(A)}$ is a homeomorphism for some closed subset $A \subset Y$. Let $i_1 : X \to X \times Y$ be the embedding defined by the formula $i_1(x) = (x, f(x))$. Denote $Z = X \times Y/\varepsilon$, where the relation ε is given by $\varepsilon = \{\mathrm{pr}_Y^{-1}(a) \mid a \in A\}$ ($\mathrm{pr}_Y : X \times Y \to Y$ is the respective projection). Let $q : X \times Y \to Z$ be the quotient mapping. The map $h : Z \to Y$ given by the conditions $h(z) = y$ for any $z = (x, y) \in Z \setminus q(X \times A)$ and $h(z) = a$ for any $z = q(\mathrm{pr}_Y^{-1}(a))$, $a \in A$, is open and satisfies the following two conditions: $\mathrm{pr}_Y = h \circ q$, $h|_{h^{-1}(A)}$ is a homeomorphism. Apparently, the map $i = q \circ i_1$ is an embedding, moreover, $h \circ i = f$. Since F preserves 1-preimages in the class of open mappings, we have $(Fh)^{-1}(FA) = F(h^{-1}(A))$, which gives us the equality $(Ff)^{-1}(FA) = F(f^{-1}(A))$. □

Proposition 2. *If F is a monomorphic functor that preserves 1-preimages, then F_β preserves embeddings.*

Proof: Take any embedding $f : X \to Y$. Then the map $F_\beta(f)$ is closed as the restriction of a closed map onto a full preimage and, moreover, injective, hence an embedding. $\square$

For any $X \in \mathsf{Tych}$ and any its compactification bX we can define

$$F_b(X) = \{a \in F(bX) \mid \text{ there is a compact subset } A \subset X \text{ with } a \in F(A)\} \subset F(bX)$$

and consider it with the respective subspace topology.

Corollary 1. *If F is a monomorphic, 1-preimage-preserving functor, then $F_\beta(X) \cong F_b(X)$ for any Tychonov space X and its compactification bX.*

Proposition 3. *If F is monomorphic, preserves 1-preimages and weight, then F_β preserves weight.*

Proof: The statement follows from the previous corollary and the fact that for any $X \in \mathsf{Tych}$ there exists its compactification bX which has the same weight as X. $\square$

As the following proposition shows, the reverse implication to that of Proposition 2 also holds.

Proposition 4. *Let F be a continuous functor such that F_β preserves embeddings. Then F preserves 1-preimages.*

Proof: Assume the contrary. Then there exist a map $f : X \to Y$ and a closed subset $A \subset Y$ such that $f|_{f^{-1}(A)}$ is a homeomorphism and $Ff^{-1}(FA) \neq F(f^{-1}(A))$. We can suppose that the map f is open by Proposition 1. There exist $\nu \in FA$ and $\mu \in FX \backslash F(f^{-1}(A))$ such that $Ff(\mu) = \nu$. We will construct a space $S \in \mathsf{Tych}$ and its compactification γS such that the map $F_\beta(\mathrm{id}_S) : F_\beta(S) \to F_\beta(\gamma S) = F(\gamma S)$ is not an embedding, where $\mathrm{id}_S : S \to \gamma S$ is an identity embedding.

First put $Z = X \times \alpha\mathbb{N}$, where the space of natural numbers $\mathbb{N}$ is considered with the discrete topology and $\alpha\mathbb{N} = \mathbb{N} \cup \{\xi\}$ is the one-point compactification of $\mathbb{N}$. Define a continuous function $g : Z \to Y$ by $g(x, n) = f(x)$ for any $x \in X$, $n \in \alpha\mathbb{N}$. Let $T = Z/\varepsilon$ be a quotient space, where ε is an equivalence relation defined by its classes of equivalence $\{\{x\} \mid x \in (X \setminus f^{-1}(A)) \times \mathbb{N}\} \cup \{g^{-1}(y) \cap X \times \{\xi\} \mid y \in Y \setminus A\} \cup \{\{a\} \times \alpha\mathbb{N} \mid a \in f^{-1}(A)\}$. By $q : Z \to T$ we denote the respective quotient mapping. Then the map $h : T \to Y$ defined by the equality $g = h \circ q$ is continuous. The set $D = q(X \times \{\xi\})$ is compact as a continuous image of a compact set and moreover $h|_D$ is one-to-one, hence a homeomorphism between D and Y. We denote by $j : Y \to T$ the inverse embedding. Also, for any $n \in \mathbb{N}$ the space $S_n = q(X \times \{n\})$ is homeomorphic to X and we define $j_n : X \to T$ by $j_n(x) = q(x, n)$. Then we have $h \circ j_n = f$. Finally note that T is a compactification of the space $S = T \backslash q((X \setminus f^{-1}(A)) \times \{\xi\})$.

Put $\mu_n = F(j_n)(\mu)$ for $n \in \mathbb{N}$. The sequence j_n converges to $j \circ f$ in the space $C(X,T)$. Since F is continuous, the sequence $F(j_n)$ converges to $F(j \circ f)$ in the space $C(FX, FT)$. Hence the sequence μ_n converges to $F(j \circ f)(\mu) = F(j)(\nu) \in F(q(f^{-1}(A) \times \alpha\mathbb{N}))$.

Now consider $F_\beta(S)$ as a subspace of $F(\beta S)$. Define a map $s_1 : S \to X$ by the condition $s_1 \circ j_n = \mathrm{id}_X$ for all n. Let us show the continuity of s_1. Consider any point $t \in S$ and any open neighborhood U of $s_1(t)$ in X. Since the map f is open, the set $q(U \times \alpha\mathbb{N}) = q((U \times \mathbb{N}) \cup (f^{-1}(f(U)) \times \{\xi\}))$ is an open set in T which contains the point t. The set $V = q(U \times \alpha\mathbb{N}) \cap S$ is an open neighborhood of t such that $s_1(V) \subset U$.

Let $s : \beta S \to X$ be the extension of s_1. Then $Fs(\mu_n) = \mu \notin F(f^{-1}(A))$. Then the sequence μ_n does not converge to any element of $F(q(f^{-1}(A) \times \alpha\mathbb{N}))$. The proposition is proved. □

Propositions 2 and 4 yield the following

Theorem 1. *For any continuous monomorphic functor F the functor F_β preserves embeddings if and only if F preserves 1-preimages.*

The proof of the following proposition is a routine checking and we omit it.

Proposition 5. *Let $F : \mathsf{Comp} \to \mathsf{Comp}$ be a functor.*

(1) *If F preserves embeddings, 1-preimages and intersections then F_β preserves intersections.*
(2) *If F preserves embeddings and preimages then F_β preserves preimages.*
(3) *If F preserves surjections then F_β sends k-covering maps to surjections.*

Now let us consider continuity of the Chigogidze extension. The following example shows that in the absence of the preimage-preserving property of the functor F, it is difficult to speak of continuity of F_β, since even the extension of such known weakly normal functor as G does not possess it.

Example. Let us define the inclusion hyperspace functor G. Recall that a closed subset $\mathcal{A} \in \exp^2 X$ (where $X \in \mathsf{Comp}$) is called an inclusion hyperspace, if for every $A \in \mathcal{A}$ and every $B \in \exp X$ the inclusion $A \subset B$ implies $B \in \mathcal{A}$. Then GX is the space of all inclusion hyperspaces with the induced topology from $\exp^2 X$. For any map $f : X \to Y$ define $Gf : GX \to GY$ by $Gf(\mathcal{A}) = \{B \in \exp Y \mid f(A) \subset B \text{ for some } A \in \mathcal{A}\}$. The functor G is weakly normal (see [13] for more details). In the next section we will see that the functor G preserves 1-preimages.

Let us show that the functor G_β is not continuous. Consider the following inverse system. For any $n \in \mathbb{N}$ put $X_n = \mathbb{N} \times \{1, \dots, n\}$ (here the spaces $\mathbb{N}$ and $\{1, \dots, n\}$ are considered with the discrete topology). Define $p_n^m : X_m \to X_n$, where $m \geq n$, in the following way: $p_n^m(x,k) = (x, \min\{k, n\})$. We obtained the inverse system $S = \{X_m, p_n^m, \mathbb{N}\}$. Then the limit space $X = \lim S$ is homeomorphic to the space $\mathbb{N} \times A$ (here $A = \alpha\mathbb{N} = \mathbb{N} \cup \{\xi\}$ is the one-point compactification of $\mathbb{N}$, i.e. a convergent sequence; also we put ξ to be greater than any natural number), and the limit projections $p_n : X \to X_n$ can be given by

$p_n(x,k) = (x, \min\{k,n\})$, $k \in \mathbb{N}$. The continuity of G_β means that $\lim G_\beta(p_n) : G_\beta(\lim S) \to \lim G_\beta(S)$ is a homeomorphism. Here both $G_\beta(\lim S)$ and $\lim G_\beta(S)$ can be thought as subspaces of $G(bX)$, where b is a compactification of X with the property $bX = \lim \beta S$. The first inclusion follows from Corollary 1, and the second inclusion is due to continuity of G (hence $G(\lim \beta S) = G(bX) = \lim G(\beta S)$) and existence of the embedding $\lim G_\beta(S) \hookrightarrow \lim G(\beta S)$ which is the limit of a morphism that naturally embeds each $G_\beta(X_n)$ into $G(\beta X_n)$.

Now we will construct $K \in \lim G_\beta(S)$ which does not belong to $\lim G_\beta(p_n)$ $(G_\beta(\lim S))$. Consider the space X embedded into its compactification bX. For any $n \in A\backslash\{\xi\}$ put $K_n = \{1, \dots, n\} \times \{n\}$. If we want to obtain a closed family of sets, the set $K_\xi = \overline{\mathbb{N} \times \{\xi\}}$ must be added to the family $\widetilde{K} = \{K_n\}_{n\in\mathbb{N}}$. Now put $K = \{B \subset bX \mid K_n \subset B$ for some $n \in A\}$. Then $K \in \lim G_\beta(S)$. However, there is no element $C \in G_\beta(\lim S)$ with $\lim G_\beta(p_n)(C) = K$. Indeed, the projection of any compact set $B \subset X$ onto the factor $\mathbb{N}$ of $\mathbb{N} \times A$ is finite, hence $\lim G_\beta(p_n)(C)$ does not contain K_ξ or contains some finite subsets in $\mathbb{N} \subset \mathbb{N} \times \{\xi\}$. Hence, $\lim G_\beta(p_n)$, being not surjective, is not a homeomorphism.

§3

We start this section with definitions of some functors we deal with in this paper. Let X be a compactum. By $C(X)$ we denote the Banach space of all continuous functions $\phi : X \to \mathbb{R}$ with the usual sup-norm. We consider $C(X)$ with the natural order. Let $\nu : C(X) \to \mathbb{R}$ be a functional (we do not suppose a priori that ν is linear or continuous). We say that ν is 1) *non-expanding* if $|\nu(\varphi) - \nu(\psi)| \le d(\varphi, \psi)$ for all $\varphi, \psi \in C(X)$; 2) *weakly additive* if for any function $\phi \in C(X)$ and any $c \in \mathbb{R}$ we have $\nu(\phi + c_X) = \nu(\phi) + c$ (by c_X we denote the constant function); 3) *preserves order* if for any $\varphi, \psi \in C(X)$ such that $\varphi \le \psi$ the inequality $\nu(\varphi) \le \nu(\psi)$ holds; 4) *linear* if for any α, $\beta \in \mathbb{R}$ and for any two functions ψ, $\phi \in C(X)$ we have $\nu(\alpha\phi + \beta\psi) = \alpha\nu(\phi) + \beta\nu(\psi)$.

Now for any space X denote $VX = \prod_{\varphi \in C(X)} [\min \varphi, \max \varphi]$. For any mapping $f : X \to Y$ define the map Vf as follows: $Vf(\nu)(\varphi) = \nu(\varphi \circ f)$ for every $\nu \in VX$, $\varphi \in C(Y)$. Then V is a covariant functor in the category Comp [11].

Let us remark that the space VX could be considered as the space of all functionals $\nu : C(X) \to \mathbb{R}$ with the only condition $\min \varphi(X) \le \nu(\varphi) \le \max \varphi(X)$ for every $\varphi \in C(X)$. By EX we denote the subset of VX defined by the condition 1) (non-expanding functionals; see [5] for more details), by EAX the subset defined by the conditions 1) and 2). The conditions 2) and 3) define the subset OX (order-preserving functionals, see [10]); finally, the conditions 3) and 4) define the well-known subset PX (probability measures, see for example [13]). For a map $f : X \to Y$ the mapping Ff, where F is one of P, O, EA, E, is defined as the restriction of Vf on FX. It is easy to check that the constructions P, O, EA and E define subfunctors of V. It is known that the functors O and E are weakly normal (see [10] and [5]). Using the same arguments one can check that EA is weakly normal too.

The question arises naturally which of the functors defined above have the property of preserving 1-preimages. It is easy to check that we have the inclusions $PX \subset OX \subset EAX \subset EX \subset VX$. We will show that the functor EA satisfies this property and E does not. Since subfunctors inherit the 1-preimages preserving property, this is the complete answer. Let us also remark that the results of [11] and [12] show that many other known functors could be considered as subfunctors of EA, for example the superextension, the hyperspace functor, the inclusion hyperspace functor etc. This shows that the class of functors with the 1-preimages preserving property is wide enough.

We start with a definition of an AR-compactum. Recall that a compactum X is called an absolute retract (briefly $X \in AR$) if for any embedding $i : X \to Z$ of X into compactum Z the image $i(X)$ is a retract of Z.

The next lemma will be needed in the following discussion.

Lemma 1. *Let F be a monomorphic subfunctor of V which preserves intersections and B be a closed subset of a compactum X. Then $\nu \in FB$ iff $\nu(\varphi_1) = \nu(\varphi_2)$ for each $\varphi_1, \varphi_2 \in C(X)$ such that $\varphi_1|_B = \varphi_2|_B$.*

PROOF: *Necessity.* The inclusion $\nu \in FB \subset FX$ means that there exists $\nu_0 \in FB$ with $F(i_B)(\nu_0) = \nu$, where $i_B : B \to X$ is the natural embedding. Hence, for any $\varphi_1, \varphi_2 \in C(X)$ such that $\varphi_1|_B = \varphi_2|_B$ we have $\nu(\varphi_1) = \nu_0(\varphi_1 \circ i_B) = \nu_0(\varphi_2 \circ i_B) = \nu(\varphi_2)$.

Sufficiency. We can find an embedding $j : B \hookrightarrow Y$, where $Y \in AR$. Define Z to be the quotient space of the disjoint union $X \cup Y$ obtained by attaching X and Y by B. Denote by $r : Z \to Y$ a retraction mapping.

Now take any $\nu \in FX \subset FZ$ with the property $\nu(\varphi_1) = \nu(\varphi_2)$ for each φ_1, $\varphi_2 \in C(X)$ such that $\varphi_1|_B = \varphi_2|_B$. We claim that $F(r)(\nu) = \nu$. Indeed, take any $\varphi \in C(Z)$. Then $F(r)(\nu)(\varphi) = \nu(\varphi \circ r) = \nu(\varphi)$ since $\varphi \circ r|_Y = \varphi|_Y$. Hence, $\nu \in FX \cap FY = FB$. $\square$

Proposition 6. *The functor EA preserves 1-preimages.*

PROOF: Let $f : X \to Y$ be a continuous open map between compacta X and Y and B be a closed subset of Y such that $f|_{f^{-1}(B)}$ is a homeomorphism. Choose any $\nu \in EA(B) \subset EA(Y)$. Using Lemma 1 we can define $\mu_0 \in EA(f^{-1}(B))$ by the condition $\mu_0(\varphi) = \nu(\psi)$ for each $\varphi \in C(X)$ and $\psi \in C(Y)$ such that $\psi \circ f \mid f^{-1}(B) = \varphi|_{f^{-1}(B)}$.

It is enough to show that for each $\mu \in (EA(f))^{-1}(\nu)$ we have $\mu = \mu_0$. Suppose the contrary. Then there exist $\varphi \in C(X)$ and $\psi \in C(Y)$ such that $\psi \circ f \mid f^{-1}(B) = \varphi|_{f^{-1}(B)}$ and $\mu(\varphi) \neq \nu(\psi)$. We can suppose that $\mu(\varphi) > \nu(\psi)$. Define a function $\psi' : Y \to \mathbb{R}$ by $\psi'(y) = \max \varphi f^{-1}(y)$ for any $y \in Y$. The function ψ' is continuous since f is open. Also, since $\psi'|_B = \psi|_B$, we have that $\nu(\psi') = \nu(\psi)$. Put $\xi = (\psi' - D) \circ f$, where $D = \sup\{\max \varphi f^{-1}(y) - \min \varphi f^{-1}(y) \mid y \in Y\}$. Then $d(\xi, \varphi) \leq D$ but $\mu(\varphi) - \mu(\xi) = \mu(\varphi) - \mu((\psi' - D) \circ f) = \mu(\varphi) - \nu(\psi') + D = \mu(\varphi) - \nu(\psi) + D > D$ and we obtain a contradiction. The proof is similar for the case $\mu(\varphi) < \nu(\psi)$.

Hence, EA preserves 1-preimages in the class of open mappings, and, by Proposition 1, we are done. $\square$

Proposition 7. *The functor of nonexpanding functionals E does not preserve 1-preimages.*

Proof: Consider the mapping $f : X \to Y$ between discrete spaces $X = \{x, y, s, t\}$ and $Y = \{a, b, c\}$ which is defined as follows: $f(x) = a$, $f(y) = b$, $f(s) = f(t) = c$. Put $A = \{\varphi \in C(X) \mid \varphi(s) = \varphi(t)\}$. Define the functional $\nu : A \to \mathbb{R}$ as follows: $\nu(\varphi) = \min\{\varphi(x), \varphi(y)\}$ if $\varphi|_{\{x,y\}} \geq 0$, $\nu(\varphi) = \max\{\varphi(x), \varphi(y)\}$ if $\varphi|_{\{x,y\}} \leq 0$, and $\nu(\varphi) = 0$ otherwise. One can check that ν is nonexpanding. Now take the function $\psi : X \to \mathbb{R}$ defined as follows $\psi(x) = 1$, $\psi(y) = -1$, $\psi(s) = 0$, $\psi(t) = 4$. One can check that we can extend ν to a nonexpanding functional on $A \cup \{\psi\}$ by defining its value on ψ to be -1. This new functional can be further extended to a nonexpanding functional on the whole $C(X)$ [5]. Denote this extension by $\widetilde{\nu}$. Evidently, $Ef(\widetilde{\nu}) \in E(\{a, b\})$. On the other hand, $\widetilde{\nu} \notin E(\{x, y\})$. $\square$

§4

We consider in this section a monomorphic continuous functor F which preserves intersections, weight, empty set, point and 1-preimages. We investigate the topology of the space $F_\beta Y$ where Y is a metrizable separable non-compact space. We consider Y as a dense subset of a metrizable compactum X. It follows from Corollary 1 that $F_\beta Y$ is homeomorphic to $F_b Y \subset FX$ (where X is considered as a compactification bY of Y) and in what follows we identify $F_\beta Y$ with $F_b Y$. Also, the properties we impose on F imply that $F_\beta Y$ is a dense proper subspace of FX.

T. Banakh proved in [1] that $F_\beta Y$ is a F_σ-subset of FX when Y is locally compact; $F_\beta Y$ is $F_{\sigma\delta}$-subset when Y is a G_δ-subset. If Y is not a G_δ-subset, then $F_\beta Y$ is not analytic.

We consider in the Hilbert cube $Q = [-1, 1]^\omega$ the following subsets: $\Sigma = \{(t_i) \in Q \mid \sup_i |t_i| < 1\}$; $\sigma = \{(t_i) \in Q \mid t_i \neq 0 \text{ for finitely many } i\}$ and $\Sigma^\omega \subset Q^\omega \cong Q$.

It is shown in [2] that any analytic $P_\beta Y$ is homeomorphic to one of the spaces σ, Σ or Σ^ω. We generalize this result for convex functors.

By Conv we denote the category of convex compacta (compact convex subsets of locally convex topological linear spaces) and affine maps. Let $U : \mathsf{Conv} \to \mathsf{Comp}$ be the forgetful functor. A functor F is called *convex* if there exists a functor $F' : \mathsf{Comp} \to \mathsf{Conv}$ such that $F = UF'$. It is easy to see that the functors V, E, EA, O and P are convex. It is shown in [14] that for each convex functor F there exists a unique natural transformation $l : P \to F$ such that the map $lX : PX \to FX$ is an affine embedding for each compactum X.

Lemma 2. $P_\beta Y = (lX)^{-1}(F_\beta Y)$.

Proof: Take any measure $\mu \in P(X)$ such that $lX(\mu) = \mu' \in F_\beta Y$. By the definition of $F_\beta Y$ it means that $\mu' \in FB$ for some compactum $B \subset Y$. We will show that $\mu \in PB \subset P_\beta Y$. Choose a compact absolute retract T which contains B and define Z to be the quotient space of the disjoint union $X \cup T$

obtained by attaching X and T by B. By $r : Z \to T$ denote the retraction. Since l is a natural transformation and r is the identity on $T \subset Z$, we have that $F(r) \circ lZ(\mu) = \mu' = lT \circ P(r)(\mu)$. Hence, $\mu = P(r)(\mu) \in P(T)$ due to injectivity of lZ. Therefore, $\mu \in PX \cap PT = PB$. The lemma is proved. □

We need some notions from infinite-dimensional topology. See [4] for more details. All spaces are assumed to be metrizable and separable. A closed subset A of a compactum T is called Z-set if there exists a homotopy $H : T \times [0;1] \to T$ such that $H|_{T\times\{0\}} = \mathrm{id}_{T\times\{0\}}$ and $H(T \times (0,1]) \cap A = \emptyset$; a countable union of Z-sets of T is called a σZ-set.

We do not know if $F_\beta Y$ is contained in a σZ-set of FX for any convex functor F. Thus, we introduce some additional property. We consider the compactum FX as a convex subset of a locally convex linear space.

Recall that for any subset A of a linear space L the notation $\mathrm{aff}(A)$ stands for the affine hull of A, that is, the set $\mathrm{aff}(A) = \{ta + (1-t)b \mid a, b \in A,\ t \in \mathbb{R}\}$.

Definition 2. A convex functor $F : \mathsf{Comp} \to \mathsf{Comp}$ is called *strongly convex* if for each compactum X, each closed subset $A \subset X$ we have $(FX \setminus FA) \cap \mathrm{aff}\, FA = \emptyset$.

Proposition 8. *Each convex subfunctor F of the functor V is strongly convex.*

Proof: By Lemma 1 any element from $\mathrm{aff}\, FA$ takes the same value at any two functions from $C(X)$ which coincide on A, which is not true for functionals from $FX \setminus FA$. □

Proposition 9. *Let F be a strongly convex functor. Then $F_\beta Y$ is contained in a σZ-set in FX.*

Proof: Take any $y \in X\setminus Y$. Then $F_\beta Y \subset F_\beta(X\setminus\{y\})$, and $X\setminus\{y\}$ can be represented as a countable union of its compact subsets A_n with the property that $A_n \subset \mathrm{int} A_{n+1}$, hence, $F_\beta(X\setminus\{y\}) = \bigcup_{n\in\mathbb{N}} F(A_n)$. Let us show that all $F(A_n)$ are Z-sets in FX. Take any $\nu \in FX \setminus F_\beta(X \setminus \{y\})$ and the set $Z = \{t\nu + (1-t)\mu \mid t \in (0,1], \mu \in F_\beta(X \setminus \{y\})\}$. Since F is strongly convex, we have $Z \cap F_\beta(X \setminus \{y\}) = \emptyset$. Since Z is a convex and dense subset of FX, there exists a homotopy $H : FX \times [0,1] \to FX$ such that $H(FX \times (0,1]) \subset Z$ (see, for example, Example 12, 13 to Section 1.2 in [4]). □

Now, we are going to obtain the complete topological classification of the pair $(FX, F_\beta Y)$ where X is a metrizable compactum and Y its proper dense G_δ-subset. We need some characterization theorems.

Theorem A ([8]). *Let C be an infinite-dimensional dense convex subspace of a convex metrizable compactum K, C is contained in a σZ-set of K and additionally let C be a countable union of its finite-dimensional compact subspaces. Then the pair (K, C) is homeomorphic to (Q, σ).*

Theorem B ([7]). *Let K be a convex metrizable compactum, and let $C \subset K$ be its proper dense convex σ-compact subspace that contains an infinite-dimensional convex compactum and is contained in a σZ-set of K. Then the pair (K, C) is homeomorphic to the pair (Q, Σ).*

The following theorem follows from 5.3.6, 5.2.6, 3.1.10 in [4].

Theorem C. *Let K be a convex compact subset of a locally convex linear metric space, and let $C \subset K$ be its proper dense convex $F_{\sigma\delta}$ subspace such that C is contained in a σZ-set of K, $(K \setminus C) \cap \operatorname{aff} C = \emptyset$, and additionally there exists a continuous embedding $h : Q \to K$ such that $h^{-1}(C) = \Sigma^\omega$. Then the pair (K, C) is homeomorphic to the pair (Q, Σ^ω).*

Theorem 2. *Let F be a strongly convex functor, X is a metrizable compactum and Y is its proper dense G_δ-subset. The pair $(FX, F_\beta Y)$ is homeomorphic to*

(1) *(Q, σ), if Y is a discrete subspace of X and $F(n)$ is finite-dimensional for each $n \in \mathbb{N}$;*
(2) *(Q, Σ), if Y is a discrete subspace of X and $F(n)$ is infinite-dimensional for some $n \in \mathbb{N}$ or Y is a locally compact non-discrete subspace of X;*
(3) *(Q, Σ^ω), if Y is not locally compact.*

Proof: It is easy to see that $F_\beta Y$ is a convex subset of FX.

We prove the first assertion. Since X is metrizable, Y is countable. We can represent $Y = \bigcup_{n=1}^\infty Y_n$ where $|Y_n| = n$. Then $F_\beta Y = \bigcup_{n=1}^\infty FY_n$. Since PY_n could be considered as an $(n-1)$-dimensional subspace of FY_n, the space $F_\beta Y$ is infinite-dimensional. Moreover, $F_\beta Y$ is a σZ-set by Proposition 9. Since each FY_n is a finite-dimensional compactum, we can apply Theorem A.

We prove the second assertion. In the case when Y is discrete, FY_n is an infinite-dimensional convex compactum for some n. When Y is not discrete, it contains an infinite compactum Y' and FY' is an infinite-dimensional convex compactum. We apply Proposition 9 and Theorem B.

For the third assertion, note that the pair $(PX, P_\beta Y)$ is homeomorphic to (Q, Σ^ω) [2]. Since F is strongly convex, we have $(FX \setminus F_\beta Y) \cap \operatorname{aff} F_\beta Y = \emptyset$. We apply Lemma 2, Proposition 9 and Theorem C. □

Corollary 2. *Suppose that F is a strongly convex functor. Then for any separable metrizable space X*

(1) *$X \cong \mathbb{N}$ implies $F_\beta(X) \cong Q_f$ in case $F(n)$ is finite-dimensional for any $n \in \mathbb{N}$ or $F_\beta(X) \cong \Sigma$ otherwise;*
(2) *if X is locally compact non-discrete and non-compact then $F_\beta(X) \cong \Sigma$;*
(3) *if X is topologically complete not locally compact then $F_\beta(X) \cong \Sigma^\omega$.*

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(*Received* June 4, 2011, *revised* February 2, 2012)