Comment.Math.Univ.Carolin. 53,2 (2012) 307–318

Metrization of function spaces with the Fell topology

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Abstract. For a Tychonoff space X, let $\downarrow \mathrm{C}_F(X)$ be the family of hypographs of all continuous maps from X to $[0,1]$ endowed with the Fell topology. It is proved that X has a dense separable metrizable locally compact open subset if $\downarrow \mathrm{C}_F(X)$ is metrizable. Moreover, for a first-countable space X, $\downarrow \mathrm{C}_F(X)$ is metrizable if and only if X itself is a locally compact separable metrizable space. There exists a Tychonoff space X such that $\downarrow \mathrm{C}_F(X)$ is metrizable but X is not first-countable.

Keywords: space of continuous maps, Fell topology, hyperspace, metrizable, hypograph, separable, first-countable

Classification: 54C35, 54E45, 54B20

1. Introduction and main results

For a topological space X, let $\mathrm{C}(X)$ denote the set of all continuous maps from X to the unit closed interval $\mathbf{I} = [0,1]$ with the usual topology. Then we can endow $\mathrm{C}(X)$ with various topologies. For example, the topology of uniform convergence, the topology of pointwise convergence and the compact-open topology are well known. In [4]–[10], $\mathrm{C}(X)$ is endowed with other natural topologies inherited from the spaces $\mathrm{Cld}(X \times \mathbf{I})$ of nonempty closed sets in $X \times \mathbf{I}$.

For a space Y, let $\mathrm{Cld}(Y)$ be the set of all nonempty closed sets in Y. For an open set U in Y, let

$$U^- = \{A \in \mathrm{Cld}(Y) : A \cap U \neq \emptyset\} \text{ and } U^+ = \{A \in \mathrm{Cld}(Y) : A \subset U\}.$$

The most well-known topology of $\mathrm{Cld}(Y)$, called the *Vietoris topology*, is generated by

$$\{U^-, U^+ : U \text{ is open in } Y\}.$$

In this paper, we consider the *Fell topology* of $\mathrm{Cld}(Y)$, which is generated by

$$\{U^-, (Y \setminus K)^+ : U \text{ is open and } K \text{ is compact in } Y\}.$$

The hyperspaces $\mathrm{Cld}(Y)$ with the above two topologies are denoted by $\mathrm{Cld}_V(Y)$ and $\mathrm{Cld}_F(Y)$, respectively. It is well-known that $\mathrm{Cld}_V(Y)$ (resp. $\mathrm{Cld}_F(Y)$) is metrizable if and only if Y is a compact (resp. locally compact and separable) metrizable space. Obviously, when Y is compact, the Fell topology of $\mathrm{Cld}(Y)$ is equal to the Vietoris topology.

For every $f \in \mathrm{C}(X)$, let

$$\downarrow f = \{(x, s) \in X \times \mathbf{I} : s \leq f(x)\} \in \mathrm{Cld}(X \times \mathbf{I}),$$

which is called the *hypograph* of f. By identifying each $f \in \mathrm{C}(X)$ with $\downarrow f \in \mathrm{Cld}_V(X \times \mathbf{I})$, we can regard $\mathrm{C}(X)$ as the subset

$$\downarrow\mathrm{C}(X) = \{\downarrow f : f \in \mathrm{C}(X)\} \subset \mathrm{Cld}(X \times \mathbf{I}).$$

Let $\downarrow \mathrm{C}_V(X)$ and $\downarrow \mathrm{C}_F(X)$ be the spaces with the topologies inherited from $\mathrm{Cld}_V(X \times \mathbf{I})$ and $\mathrm{Cld}_F(X \times \mathbf{I})$, respectively. These topologies are different from the three topologies mentioned previously (see [4, Corollary 1]). In [9, Theorem 1], it was proved that, for a Tychonoff space X, $\downarrow\mathrm{C}_V(X)$ is metrizable if and only if $\downarrow\mathrm{C}_V(X)$ is second-countable if and only if X is compact and metrizable. The following theorem is our main result.

Theorem 1. *For a Tychonoff space X, the following conditions are equivalent:*

(a) *$\downarrow\mathrm{C}_F(X)$ is separable metrizable;*
(b) *$\downarrow\mathrm{C}_F(X)$ is metrizable.*

In case X is first-countable, the above two conditions are equivalent to

(c) *X is a locally compact and separable metrizable space.*

We also prove the following theorem.

Theorem 2. *Let $\bigoplus_{s\in S} Y_s$ be the topological sum of Tychonoff spaces Y_s, $s \in S$, and $a_s \in Y_s$ a non-isolated point for every $s \in S$. Let, further, Y be the quotient space of $\bigoplus_{s\in S} Y_s$ with the set $\{a_s : s \in S\}$ identified to a point. Then $\downarrow\mathrm{C}_F(Y)$ is homeomorphic to a subspace of the product space $\prod_{s\in S} \downarrow\mathrm{C}_F(Y_s)$.*

Applying this theorem, we show the following.

Corollary 1. *There exists a Tychonoff space X such that $\downarrow\mathrm{C}_F(X)$ is separable metrizable but X is not first-countable.*

The above corollary shows that the first-countability of X is essential for the equivalence between (a) and (c) in Theorem 1. The following Theorem 3 tells us that, the non-compact case is very different from the compact one.

Theorem 3. *There exists a countable Tychonoff space X such that $\downarrow\mathrm{C}_F(X)$ is Hausdorff and second-countable but not regular.*

In [1, 5.1.2 Proposition], it was proved that, for a Tychonoff space X, the following conditions are equivalent: (a) $\mathrm{Cld}_F(X)$ is Hausdorff, (b) $\mathrm{Cld}_F(X)$ is regular, (c) $\mathrm{Cld}_F(X)$ is Tychonoff, and (d) X is locally compact. Theorem 3 shows that we cannot replace $\mathrm{Cld}_F(X)$ by $\downarrow\mathrm{C}_F(X)$ in [1, 5.1.2 Proposition].

The following Theorem 4 states that, even for a compact space X, the regularity and the first-countability of $\downarrow\mathrm{C}_F(X)$ do not imply the metrizability of it.

Theorem 4. *There exists a compact space X such that $\downarrow\mathrm{C}_F(X)$ is Tychonoff, separable and first-countable but not metrizable.*

Finally, we will give a necessary condition for the metrizability of $\downarrow\mathrm{C}_F(X)$.

Theorem 5. *For a Tychonoff space X, if $\downarrow\mathrm{C}_F(X)$ is metrizable, then there exists a dense, locally compact, open and separable metrizable subspace of X. But the converse is not true.*

2. Preparatory results

In the following, we always assume that X is a Tychonoff space and $p : X \times \mathbf{I} \to X$ is the projection. For $s \in \mathbf{I}$, we use $\underline{s}$ to denote the constant function from X to $\mathbf{I}$ which maps all elements to s. By $\mathbb{R}$ and $\mathbb{Q}$, we denote the sets of all real numbers and of all rational numbers, respectively. Let cl_Y and int_Y be the closure-operator and the interior-operator in a space Y. If $Y = X$, the subscript in the above operators will be omitted. And, for a closed set F in Y, let

$$F^* = (Y \setminus F)^+ = \{A \in \mathrm{Cld}(Y) : A \cap F = \emptyset\}.$$

By the definition, the topology of $\downarrow\mathrm{C}_F(X)$ is generated, as a base, by the following sets:

$$\bigcap_{i=1}^{n} G_i^- \cap K^* \cap \downarrow\mathrm{C}(X),$$

where $G_1, G_2, \cdots, G_n$ are open sets in $X \times (0,1]$ and K is a compact set in $X \times (0,1]$. In particular,

$$\left\{ \bigcap_{i=1}^{n} G_i^- \cap \downarrow\mathrm{C}(X) : G_1, \cdots, G_n \text{ are nonempty open in } X \times (0,1] \right\}$$
$$\text{and } \{K^* \cap \downarrow\mathrm{C}(X) : K \text{ is compact in } X \times (0,1]\}$$

are neighborhood bases at $\downarrow\underline{1}$ and $\downarrow\underline{0}$ in $\downarrow\mathrm{C}_F(X)$, respectively.

To prove our theorems, we need some lemmas. At first, we show the following lemma.

Lemma 1. *For a space X, the following hold:*

(1) *$\downarrow\mathrm{C}_F(X)$ is T_1;*
(2) *$\downarrow\mathrm{C}_F(X)$ is Hausdorff if and only if there exists a dense open subset U of X which is locally compact.*

Proof: (1): Let $f \neq g \in \mathrm{C}(X)$. We may assume that $f(x_0) < g(x_0)$ for some $x_0 \in X$. Then x_0 has an open neighborhood W such that $f(x) < a < g(x)$ for every $x \in W$, where $a = \frac{f(x_0)+g(x_0)}{2}$. Thus $\downarrow f \in (\{x_0\} \times [a,1])^* \not\ni \downarrow g$ and $\downarrow g \in (W \times (a,1])^- \not\ni \downarrow f$.

(2): The "if" part: Take $f, g \in \mathrm{C}(X)$, $x_0 \in W$ and $a \in \mathbf{I}$ as the same as in (1). Since f and g are continuous, we assume that $x_0 \in U$. Because U is locally compact, we have an open set V in X such that $x_0 \in V \subset \mathrm{cl}\, V \subset U \cap W$ and $\mathrm{cl}\, V$ is compact. Since $f(x) < a < g(x)$ for $x \in \mathrm{cl}\, V$, $(\mathrm{cl}\, V \times [a,1])^* \cap \downarrow\mathrm{C}(X)$ and $(V \times (a,1])^- \cap \downarrow\mathrm{C}(X)$ are disjoint neighborhoods of $\downarrow f$ and $\downarrow g$, respectively.

The "only if" part: We define an open set

$$U = \bigcup\{\operatorname{int} K : K \text{ is compact in } X\} \subset X.$$

Then U is locally compact. We show that U is dense in X. Assume that U is not dense in X. Then there exists a nonempty open set V in X such that the interior of every compact subset of V is empty. Because X is Tychonoff, we can choose $f \in \mathrm{C}(X)$ such that $f(X \setminus V) \subset \{1\}$ and $f(x_0) = 0$ for some $x_0 \in V$. Since $\downarrow \mathrm{C}_F(X)$ is Hausdorff, there exist disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ in $\downarrow \mathrm{C}_F(X)$ such that $\downarrow \underline{1} \in \mathcal{U}$ and $\downarrow f \in \mathcal{V}$. Then we can find nonempty open sets $G_1, G_2, \cdots, G_n, \cdots, G_m \subset X \times (0,1]$ and a compact set $K \subset X \times (0,1]$ such that

$$\downarrow \underline{1} \in G_1^- \cap G_2^- \cap \cdots \cap G_n^- \cap \downarrow \mathrm{C}(X) \subset \mathcal{U} \text{ and}$$
$$\downarrow f \in G_{n+1}^- \cap \cdots \cap G_m^- \cap K^* \cap \downarrow \mathrm{C}(X) \subset \mathcal{V}.$$

Since $f(X \setminus V) \subset \{1\}$, it follows that $p(K) \subset V$, which implies that $\operatorname{int} p(K) = \emptyset$. For every $i \le m$, $p(G_i) \setminus p(K) \neq \emptyset$ since $p(G_i)$ is a nonempty open set in X. Take $x_i \in p(G_i) \setminus p(K)$. Because X is Tychonoff, we have $g \in \mathrm{C}(X)$ satisfying

$$g(x_i) = 1 \text{ for } i \le m \text{ and } g(p(K)) = \{0\}.$$

Then $\downarrow g \in \mathcal{U} \cap \mathcal{V}$, which contradicts that $\mathcal{U} \cap \mathcal{V} = \emptyset$. □

Lemma 2. *If $\downarrow \mathrm{C}_F(X)$ is first-countable, then there exist compact sets $C_1 \subset C_2 \subset \cdots$ in X such that every compact set in X is contained in some C_n. In particular, $X = \bigcup_{n=1}^{\infty} C_n$.*

Proof: Because $\downarrow \mathrm{C}_F(X)$ is first-countable, we can find compact sets $K_1 \subset K_2 \subset \cdots$ in $X \times (0,1]$ such that $\{K_n^* \cap \downarrow \mathrm{C}(X) : n = 1, 2, \dots\}$ is a neighborhood base of $\downarrow \underline{0}$ in $\downarrow \mathrm{C}_F(X)$. Then $C_n = p(K_n)$, $n = 1, 2, \dots$, are the desired compact sets in X. We verify that every compact set C in X is contained in some C_n. Otherwise, for every n, we can choose $x_n \in C \setminus C_n$ and define $f_n \in \mathrm{C}(X)$ such that $f_n(x_n) = 1$ and $f_n(C_n) = \{0\}$. Then $\downarrow f_n \in K_n^*$ for every n and hence $\downarrow f_n \to \downarrow \underline{0}$ in $\downarrow \mathrm{C}_F(X)$. But every $\downarrow f_n$ is not contained in the neighborhood $(C \times \{1\})^*$ of $\downarrow \underline{0}$, which is a contradiction. □

Lemma 3. *If X and $\downarrow \mathrm{C}_F(X)$ are first-countable, then X is locally compact.*

Proof: Suppose there exists $x_0 \in X$, which has no compact neighborhood. Because X is first-countable, x_0 has a countable open neighborhood base $\{U_n : n = 1, 2, \dots\}$, where $U_n \supset U_{n+1}$ for every n. Since $\downarrow \mathrm{C}_F(X)$ is also first-countable, we can find compact sets $K_1 \subset K_2 \subset \cdots$ in $X \times (0,1]$ such that $\{K_n^* \cap \downarrow \mathrm{C}(X) : n = 1, 2, \dots\}$ is a neighborhood base at $\downarrow \underline{0}$ in $\downarrow \mathrm{C}(X)$. By the assumption, $p(K_n) \not\supset U_n$ for every $n = 1, 2, \dots$, hence we can take $x_n \in U_n \setminus p(K_n)$. Then $x_n \to x_0$ in X. Since X is Tychonoff, we have $f_n \in \mathrm{C}(X)$ such that

$$f_n(x_n) = 1 \text{ and } f_n(p(K_n) \cup (X \setminus U_n)) = \{0\}.$$

Then $\downarrow f_n \in K_n^*$ and hence $\downarrow f_n \to \downarrow \underline{0}$. On the contrary,

$$(\{x_n : n = 0, 1, 2, \cdots\} \times \{1\})^* \cap \downarrow \mathrm{C}(X)$$

is a neighborhood of $\downarrow \underline{0}$ in $\downarrow \mathrm{C}_F(X)$ which does not contain any $\downarrow f_n$. □

When X is locally compact and non-compact, let $\alpha X = X \cup \{\infty\}$ be the one-point compactification of X. Using Lemmas 2 and 3, we may prove the following

Proposition 1. *If X and $\downarrow \mathrm{C}_F(X)$ are first-countable, then*

(1) *X is locally compact and αX is also first-countable;*
(2) *$\downarrow \mathrm{C}_F(\alpha X)$ is first-countable;*
(3) *$\downarrow \mathrm{C}_F(\alpha X)$ is second-countable if $\downarrow \mathrm{C}_F(X)$ is second-countable.*

Proof: The assertion (1) directly follows from Lemmas 2 and 3. To show (2) and (3), we only consider the case that X is not compact. Let $\{U_n : n = 1, 2, \ldots\}$ be a countable open neighborhood base at ∞ in αX, and let $\phi : \mathrm{C}(\alpha X) \to \mathrm{C}(X)$ be the restriction, that is,

$$\phi(f) = f|X \quad \text{for every} \quad f \in \mathrm{C}(\alpha X).$$

Then it is not hard to verify that $\downarrow \phi : \downarrow \mathrm{C}_F(\alpha X) \to \downarrow \mathrm{C}_F(X)$ is a continuous injection. Unfortunately, it is not an embedding. However, the following $\mathcal{S}$ is a subbase of $\downarrow \mathrm{C}_F(\alpha X)$:

$$\mathcal{S} = \{(\downarrow \phi)^{-1}(G) : G \in \mathcal{G}\} \\ \cup \{(\mathrm{cl}_{\alpha X} U_n \times [r, 1])^* \cap \downarrow \mathrm{C}(\alpha X) : r \in \mathbb{Q} \cap (0, 1], n = 1, 2, \ldots\},$$

where $\mathcal{G}$ is an open base for $\downarrow \mathrm{C}_F(X)$. Obviously, $\mathcal{S}$ is a subfamily of the topology of $\downarrow \mathrm{C}_F(\alpha X)$. For every open set V in $\alpha X \times \mathbf{I}$, $V \cap (X \times \mathbf{I})$ is open in $X \times \mathbf{I}$ and

$$V^- \cap \downarrow \mathrm{C}(\alpha X) = (\downarrow \phi)^{-1}((V \cap (X \times \mathbf{I}))^- \cap \downarrow \mathrm{C}(\alpha X)).$$

For every compact set K in $\alpha X \times (0, 1]$, if $K \cap (\{\infty\} \times \mathbf{I}) = \emptyset$, then K is also compact in $X \times \mathbf{I}$ and

$$K^* \cap \downarrow \mathrm{C}(\alpha X) = (\downarrow \phi)^{-1}(K^* \cap \downarrow \mathrm{C}(X)).$$

If $K \cap (\{\infty\} \times \mathbf{I}) \neq \emptyset$, then for every $\downarrow f \in K^* \cap \downarrow \mathrm{C}(\alpha X)$, using the Wallace's Theorem, there exist n and a rational number $r \in (0, 1]$ such that

$$(\mathrm{cl}_{\alpha X} U_n \times [r, 1]) \cap \downarrow f = \emptyset \quad \text{and} \\ K \cap (\mathrm{cl}_{\alpha X} U_n \times \mathbf{I}) \subset \mathrm{cl}_{\alpha X} U_n \times [r, 1].$$

Let

$$K_1 = (K \cap ((\alpha X \setminus U_n) \times \mathbf{I})) \cup (\mathrm{cl}_{\alpha X} U_n \times [r, 1]).$$

Then K_1 is compact in $\alpha X \times (0,1]$, $K_1 \supset K$ and $K_1 \cap \downarrow f = \emptyset$. Thus, $\downarrow f \in K_1^* \subset K^*$. Note that

$$K_1^* \cap \downarrow \mathrm{C}_F(\alpha X) = (\downarrow\phi)^{-1}((K \cap ((\alpha X \setminus U_n) \times \mathbf{I}))^*) \\ \cap (\mathrm{cl}(U_n) \times [r,1])^* \cap \downarrow \mathrm{C}_F(\alpha X),$$

that is, $K_1^* \cap \downarrow \mathrm{C}_F(\alpha X)$ is an intersection of two elements of $\mathcal{S}$.

As a conclusion, $\mathcal{S}$ is a subbase for $\downarrow \mathrm{C}_F(\alpha X)$. Therefore, $\downarrow \mathrm{C}_F(\alpha X)$ is first-countable. Moreover, $\downarrow \mathrm{C}_F(\alpha X)$ is second-countable if $\downarrow \mathrm{C}_F(X)$ is second-countable. Hence (2) and (3) hold. □

Lemma 4. *We consider the following statements.*

(a) $\downarrow \mathrm{C}_F(X)$ *is first-countable.*
(b) $\downarrow \mathrm{C}_F(X)$ *has a countable neighborhood base at* $\downarrow \underline{1}$.
(c) *There exists a countable family* $\mathcal{U}$ *of nonempty open sets in* X *such that every nonempty open set in* X *includes an element of* $\mathcal{U}$, *that is,* $\mathcal{U}$ *is a countable* π*-base for* X.
(d) $\downarrow \mathrm{C}_F(X)$ *is separable.*

Then the implications (a)⇒(b)⇒(c)⇒(d) *hold.*

Furthermore, when X *is compact, the implication* (c)⇒(a) *holds and hence* (a), (b) *and* (c) *are equivalent.*

Proof: The implication (a)⇒(b) is trivial.

(b)⇒(c): We may assume that

$$\{(G_1^n)^- \cap (G_2^n)^- \cap \cdots \cap (G_{k(n)}^n)^- \cap \downarrow \mathrm{C}(X) : n = 1, 2, \ldots\}$$

is a countable neighborhood base at $\downarrow \underline{1}$ in $\downarrow \mathrm{C}_F(X)$. Let

$$\mathcal{U} = \{p(G_i^n) : i = 1, 2, \ldots, k(n), n = 1, 2, \ldots\}.$$

Then $\mathcal{U}$ is a countable family of nonempty open sets in X. We show that every nonempty open set U in X includes an element of $\mathcal{U}$. Take $f \in \mathrm{C}(X)$ such that $f(X \setminus U) \subset \{1\}$ and $f(x_0) = 0$ for some point $x_0 \in U$. Because $\downarrow \mathrm{C}_F(X)$ is T_1 by Lemma 1(1), $\downarrow f \notin \bigcap_{i=1}^{k(n)} (G_i^n)^-$ for some n, hence $\downarrow f \notin (G_i^n)^-$ for some $i \leq k(n)$. Then $\downarrow f \cap G_i^n = \emptyset$. Since $f(X \setminus U) \subset \{1\}$, we have $U \supset p(G_i^n)$, as required.

(c)⇒(d): Let $\mathcal{U}$ be a countable π-base for X. For every $U \in \mathcal{U}$ and $r \in \mathbb{Q} \cap (0,1]$, we can take a continuous map $f_{(U,r)} : X \to [0,r]$ such that $f_{(U,r)}(X \setminus U) \subset \{0\}$ and $f_{(U,r)}(x) = r$ for some $x \in U$. Let

$$D = \{\max\{f_{(U,r)} : U \in \mathcal{F}, r \in F\} : \mathcal{F} \text{ and } F \text{ are} \\ \text{finite subsets of } \mathcal{U} \text{ and } \mathbb{Q} \cap (0,1], \text{ resp.}\}.$$

Then $\downarrow D = \{\downarrow f : f \in D\}$ is a countable subset of $\downarrow \mathrm{C}(X)$. It remains to verify that $\downarrow D$ is dense in $\downarrow \mathrm{C}_F(X)$. Let $f \in \mathrm{C}(X)$, K be compact in $X \times (0,1]$ and G_i,

$i \leq k$, open in $X \times (0,1]$, such that

$$\downarrow f \in G_1^- \cap G_2^- \cap \cdots \cap G_k^- \cap K^* \cap \downarrow \mathrm{C}(X).$$

We have $x_1, \cdots, x_k \in$ X such that $\{x_i\} \times [0, f(x_i)] \cap G_i \neq \emptyset$ for each $i \leq k$. Because $\{x_i\} \times [0, f(x_i)] \cap K = \emptyset$, we have an open neighborhood W_i of x_i in X and $s_i < t_i$ such that $W_i \times (s_i, t_i) \subset G_i$ and $W_i \times [0, t_i] \cap K = \emptyset$. Thus, by (c), choose $r_i \in \mathbb{Q} \cap (s_i, t_i)$ and $U_i \in \mathcal{U}$ such that $U_i \subset W_i$. Then $\downarrow f_{(U_i, r_i)} \in G_i^- \cap K^*$ and hence

$$\downarrow \max\{f_{(U_i, r_i)} : i \leq k\} \in \downarrow D \cap G_1^- \cap G_2^- \cap \cdots \cap G_k^- \cap K^*.$$

Now, we show (c)⇒(a) under the assumption that X is compact. Let $\mathcal{U}$ be a countable π-base of X. Then, $X \times \mathbf{I}$ has the following countable π-base:

$$\mathcal{G} = \{U \times (s,t) : U \in \mathcal{U}, s < t \in \mathbb{Q} \cap (0,1)\}.$$

For every $f \in \mathrm{C}(X)$ and $n = 1, 2, \ldots$, let

$$\mathcal{G}(f) = \{G \in \mathcal{G} : \downarrow f \in G^-\}, \quad K_n(f) = \{(x,t) \in X \times \mathbf{I} : t \geq f(x) + n^{-1}\}.$$

For every open set H in $X \times (0,1]$ with $H^- \ni \downarrow f$, there exists $x_0 \in X$ such that $\{x_0\} \times [0, f(x_0)] \cap H \neq \emptyset$. Since $f(x_0) > 0$, we can find an open neighborhood V of x_0 in X and $s < t \in \mathbb{Q} \times (0,1)$ such that $s < f(x_0)$, $V \times (s,t) \subset H$ and $s < f(x)$ for every $x \in V$. Since $\mathcal{U}$ is a π-base for X, V contains some $U \in \mathcal{U}$. Then we have $G = U \times (s,t) \in \mathcal{G}$ and $\downarrow f \in G^- \subset H^-$. Moreover, for every compact set K in $X \times \mathbf{I}$ with $K^* \ni \downarrow f$, by the compactness of X, there exists n such that $K_n(f) \supset K$ and hence $\downarrow f \in K_n(f)^* \subset K^*$. Therefore,

$$\{G_1^- \cap \cdots \cap G_k^- \cap K_n(f)^* \cap \downarrow \mathrm{C}(X) : G_i \in \mathcal{G}(f) \text{ for } i \leq k, k, n = 1, 2, \ldots\}$$

is a countable neighborhood base at $\downarrow f$ in $\downarrow \mathrm{C}_F(X)$. □

As a consequence of Lemma 4, we have the equivalence between (a) and (b) in Theorem 1, that is,

Proposition 2. *The space $\downarrow \mathrm{C}_F(X)$ is metrizable if and only if it is separable metrizable.* □

We need the following two lemmas which were proved in [8], [9], respectively.

Lemma 5. *If V is open in X such that $\mathrm{cl}\, V$ is compact, then the restriction $\phi : \downarrow \mathrm{C}_F(X) \to \downarrow \mathrm{C}_F(\mathrm{cl}\, V)$ defined by $\phi(\downarrow f) = \downarrow f | \mathrm{cl}\, V$ is a continuous open surjection.* □

Lemma 6. *If X is compact and $\downarrow \mathrm{C}_F(X) = \downarrow \mathrm{C}_V(X)$ is second-countable, then X is metrizable.* □

3. Proofs of main results

In this section, we show our main results.

Proof of Theorem 1: The equivalence between (a) and (b) is Proposition 2. If X is first-countable, then X is locally compact by Proposition 1(1). Using Proposition 1(3), the condition (b) implies that $\downarrow\mathrm{C}(\alpha X)$ is second-countable. It follows from Lemma 6 that αX is metrizable. Hence the condition (c) holds. That is, the implication (b)⇒(c) holds under the assumption that X is first-countable. The condition (c) implies that $\mathrm{Cld}_F(X \times \mathbf{I})$ is metrizable ([1, 5.1.5 Theorem]), hence so is $\downarrow\mathrm{C}_F(X)$, i.e., (b) holds. Therefore, the implication (c)⇒(b) holds. □

Proof of Theorem 2: We may think that every Y_s is a subspace of Y. Define $\phi : \mathrm{C}(Y) \to \prod_{s\in S} \mathrm{C}(Y_s)$ by

$$\phi(f) = (f|Y_s)_{s\in S} \quad \text{for each} \quad f \in \mathrm{C}(Y).$$

Evidently, ϕ is an injection and its image is

$$\phi(\mathrm{C}(Y)) = \left\{ g \in \prod_{s\in S} \mathrm{C}(Y_s) : g(s)(a_s) = g(s')(a_{s'}) \text{ for } s, s' \in S \right\}.$$

Now we show that $\downarrow\phi : \downarrow\mathrm{C}_F(Y) \to \prod_{s\in S} \downarrow\mathrm{C}_F(Y_s)$ is an embedding. Let $p_s : \prod_{s\in S} \downarrow\mathrm{C}_F(Y_s) \to \downarrow\mathrm{C}_F(Y_s)$ be the projection.

To show the continuity of $\downarrow\phi$, it is sufficient to verify that $p_s\circ \downarrow\phi$ is continuous for every $s \in S$. For every open set G in $Y_s \times (0,1]$, $G \setminus (\{a_s\} \times \mathbf{I})$ is open in $Y \times (0,1]$. Since a_s is a non-isolated point in Y_s,

$$(p_s\circ \downarrow\phi)^{-1}(G^- \cap \downarrow\mathrm{C}(Y_s)) = (G \setminus (\{a_s\} \times \mathbf{I}))^- \cap \downarrow\mathrm{C}(Y).$$

For each compact set K in $Y_s \times (0,1]$,

$$(p_s\circ \downarrow\phi)^{-1}(K^* \cap \downarrow\mathrm{C}(Y_s)) = K^* \cap \downarrow\mathrm{C}(Y).$$

Hence, $p_s\circ \downarrow\phi : \downarrow\mathrm{C}_F(Y) \to \downarrow\mathrm{C}_F(Y_s)$ is continuous for every $s \in S$.

Moreover, for every open set H in $Y \times (0,1]$, if $\downarrow f \in H^- \downarrow\mathrm{C}_F(Y)$, then there exists $s \in S$ such that $\downarrow f|Y_s \in (H \cap (Y_s \times \mathbf{I}))^-$. Hence

$$\downarrow\phi(H^- \cap \downarrow\mathrm{C}_F(Y)) = \bigcup_{s\in S} \left((H \cap (Y_s \times \mathbf{I}))^- \times \prod_{t\in S\setminus\{s\}} \downarrow\mathrm{C}(Y_t) \right) \cap \downarrow\phi(\downarrow(\mathrm{C}(Y))).$$

It shows that $\downarrow\phi(H^- \cap \downarrow\mathrm{C}_F(Y))$ is open in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y)))$. For every compact set K in $Y \times (0,1]$, there exists a finite subset S_0 of S such that $K \subset \bigcup_{s\in S_0} Y_s \times (0,1]$. Then $K \cap Y_s \times (0,1]$ is compact for every $s \in S_0$ and

$$\downarrow\phi(K^* \cap \downarrow\mathrm{C}(Y)) = \left(\prod_{s\in S_0} (K \cap Y_s \times (0,1])^* \times \prod_{s\in S\setminus S_0} \downarrow\mathrm{C}(Y_s) \right) \cap \downarrow\phi(\downarrow\mathrm{C}(Y)).$$

It follows that $\downarrow\phi(K^*\cap \downarrow\mathrm{C}(Y))$ is open in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y)))$. Since ϕ is one-to-one, we have that $\downarrow\phi$ maps every open set in $\downarrow\mathrm{C}_F(Y)$ to an open set in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y)))$.

Therefore, $\downarrow\phi : \downarrow\mathrm{C}_F(Y) \to \prod_{s\in S} \downarrow\mathrm{C}_F(Y_s)$ is an embedding. □

Remark 1. Even for a set S of two points, if a_s is an isolated point in Y_s for some s, the map $\downarrow\phi$ defined in the above proof needs not be continuous. For example, let $Y_1 = \{1\} \times (\{0\} \cup [1,2]), Y_2 = \{2\} \times \mathbf{I}$ as subspaces of $\mathbb{R}^2$. If we think that $a_1 = (1,0), a_2 = (2,0)$, then $p_1 \circ \downarrow\phi : \downarrow\mathrm{C}(Y) \to \downarrow\mathrm{C}(Y_1)$ is not continuous. In fact, choose $f_n \in \mathrm{C}(Y)$ such that $f_n(2,0) = f_n(1,0) = 0$ and $f_n(x) = 1$ for every $x \in Y \setminus (\{2\} \times [0, n^{-1}])$. Then $\downarrow f_n \to \downarrow\underline{1}$ but $(p_1 \circ \downarrow\phi)(\downarrow f_n) \not\to (p_1 \circ \downarrow\phi)(\downarrow\underline{1})$.

Proof of Corollary 1: Let $\{Y_n : n = 1, 2, \dots\}$ be a family of pairwise disjoint locally compact separable metrizable spaces Y_n with a non-isolated point a_n. Then, by Theorems 1 and 2, the space Y defined in Theorem 2 is as required. □

Proof of Theorem 3: Let $\beta\omega$ be the Čech-Stone compactification of the discrete space ω of non-negative integers and $q \in \beta\omega \setminus \omega$. Then the subspace $X = \omega \cup \{q\}$ of $\beta\omega$ satisfies the conditions in Theorem 3. By Lemma 1(2), $\downarrow\mathrm{C}_F(X)$ is Hausdorff.

Before showing that $\downarrow\mathrm{C}_F(X)$ is second-countable but not regular, we verify that every compact subset of X is finite. In fact, let C be an infinite compact subset of X. Then $q \in C$. Write $C = A \cup B \cup \{q\}$ such that A and B are disjoint infinite subsets of ω. Define a continuous map $f : \omega \to \{0,1\}$ as $f^{-1}(0) = A$. Then there exists a continuous extension $\overline{f} : X \to \{0,1\}$ since X is a subspace of $\beta\omega$. If $\overline{f}(q) = 0$, then B is closed in X and hence is compact. But it is impossible since B is infinite discrete. If $\overline{f}(q) = 1$, then A is closed in X and hence is compact. It is also impossible since A is also infinite discrete.

Now, we define a product space $Y = \prod_{x\in X} \mathbf{I}_x$, where $\mathbf{I}_x$ is a copy of the unit interval $[0,1]$ with the usual topology for $x \in \omega$ and $\mathbf{I}_q$ is $[0,1]$ with the topology generated by $\{[0,r) : r \in [0,1] \cap \mathbb{Q}\} \cup \{[0,1]\}$. Then Y is second-countable. We may regard $\downarrow\mathrm{C}(X) \subset Y$ by identifying $\downarrow f$ with $(f(x))_{x\in X}$ for every $f \in \mathrm{C}(X)$. To show that $\downarrow\mathrm{C}_F(X)$ is second-countable, it suffices to verify that $\downarrow\mathrm{C}_F(X)$ is the subspace of the space Y. It is easy to see that for each $x \in X$, the map $p_x : \downarrow\mathrm{C}_F(Y) \to \mathbf{I}_x$ defined by $p_x(\downarrow f) = f(x)$ is continuous. Hence the subspace topology is coarser than the Fell topology on $\downarrow\mathrm{C}(X)$. Conversely, take a compact set $K \subset X \times (0,1]$ and $f \in \mathrm{C}(X)$. Then $p(K)$ is compact in X. Then $p(K)$ is a finite set in X and $\downarrow f \cap K = \emptyset$ if and only if $f(x) < m(x) = \min\{s : (x,s) \in K\}$ for every $x \in p(X)$. Hence we can identify

$$K^*\cap \downarrow\mathrm{C}(X) = \left(\prod_{x\in p(K)} [0, m_x) \times \prod_{x\in X\setminus p(K)} \mathbf{I}_x \right) \cap \downarrow\mathrm{C}(X)$$

is open in the subspace topology of Y. For every open set G in $X \times (0,1]$ and

$f \in \mathrm{C}(X)$, $\downarrow f \cap G \neq \emptyset$ if and only if $\downarrow f \cap G \setminus (\{q\} \times \mathbf{I}) \neq \emptyset$ if and only if $f(n) > s_n$ for some $n \in p(G) \cap \omega$, where $s_n = \inf\{s : (n,s) \in G\}$. Hence

$$G^- \cap \downarrow\mathrm{C}(X) = \left(\bigcup_{n \in p(G)\cap\omega} p_n^{-1}(s_n, 1] \right) \cap \downarrow\mathrm{C}(X),$$

where $p_n : Y \to \mathbf{I}_n$ is the projection, is open in the subspace topology of Y. Therefore, $\downarrow\mathrm{C}_F(X)$ is the subspace of Y.

To show that $\downarrow\mathrm{C}_F(X)$ is not regular, we consider an open neighborhood $\mathcal{U} = (\{q\} \times [\frac{1}{2}, 1])^* \cap \downarrow\mathrm{C}(X)$ of $\downarrow 0$. For every compact set K in $X \times (0,1]$, $p(K)$ is finite. Define $f \in \mathrm{C}(X)$ such that $f^{-1}(0) = p(K) \cap \omega$ and $f^{-1}(1) = X \setminus (p(K) \cap \omega)$. Then $\downarrow f \in \mathrm{cl}_{\downarrow\mathrm{C}_F(X)}(K^* \cap \downarrow\mathrm{C}_F(X)) \setminus \mathcal{U}$. In fact, every neighborhood of $\downarrow f$ in $\downarrow\mathrm{C}_F(Y)$ contains the following neighborhood of $\downarrow f$:

$$\mathcal{G} = G_1^- \cap \cdots \cap G_k^- \cap G^- \cap C^* \cap \downarrow\mathrm{C}_F(X),$$

where $G_i = \{n_i\} \times (s_i, t_i)$ for $1 \leq i \leq k$ and $G = (A \cup \{q\}) \times (s,t)$ are open and C is compact in $X \times (0,1]$. Then A is an infinite subset of ω and hence we may choose $n_0 \in A \setminus p(K \cup C)$. Now, define $g \in \mathrm{C}(X)$ as

$$g(x) = \begin{cases} 0 & \text{if } \ x \in A \cup \{q\} \setminus \{n_i : 0 \leq i \leq k\}; \\ 1 & \text{if } \ x = n_0; \\ f(x) & \text{otherwise.} \end{cases}$$

Then it is easy to verify that $\downarrow g \in \mathcal{G} \cap K^*$. This shows that $\downarrow f \in \mathrm{cl}_{\downarrow\mathrm{C}_F(X)}(K^* \cap \downarrow\mathrm{C}_F(X))$. Because $f(q) = 1$, we have $\downarrow f \notin \mathcal{U}$. Hence, $\mathrm{cl}_{\downarrow\mathrm{C}_F(X)}(K^* \cap \downarrow\mathrm{C}(X)) \not\subset \mathcal{U}$ for any compact K in $X \times (0,1]$. Note that the family of all of such $K^* \cap \downarrow\mathrm{C}_F(X)$ is a neighborhood base at $\downarrow\underline{0}$ in $\downarrow\mathrm{C}_F(X)$. Therefore, $\downarrow\mathrm{C}_F(X)$ is not regular. □

Proof of Theorem 4: Choose a compact Hausdorff non-metrizable space X satisfying (c) in Lemma 4, for example, $\beta\omega$ or Helly space (see [2, Problem 5.M]). Then, by Lemma 4, $\downarrow\mathrm{C}_F(X)$ is separable and first-countable. By [3] (cf. [1, 5.1.2 Proposition]), $\mathrm{Cld}_F(X \times \mathbf{I}) = \mathrm{Cld}_V(X \times \mathbf{I})$ is Tychonoff and hence so is $\downarrow\mathrm{C}_F(X)$. Since X is compact and non-metrizable, $\downarrow\mathrm{C}_F(X)$ is not second-countable because of Lemma 6. According to Proposition 2, if $\downarrow\mathrm{C}_F(X)$ is metrizable, then $\downarrow\mathrm{C}_F(X)$ is separable metrizable, hence second-countable. Therefore, $\downarrow\mathrm{C}_F(X)$ is not metrizable. □

Proof of Theorem 5: Assume that $\downarrow\mathrm{C}_F(X)$ is metrizable, which means that $\downarrow\mathrm{C}_F(X)$ is separable metrizable by Proposition 2. Then $\downarrow\mathrm{C}_F(X)$ is second-countable. By Lemma 1(2), there exists a dense open set U in X such that U is locally compact. To complete the proof, it remains to verify that U is separable metrizable. By Lemma 2, there exists a countable family $\mathcal{C} = \{C_1, C_2, \cdots\}$ of compact sets in X such that every compact set in X is contained in some C_n. For each n, let $U_n = \mathrm{int}(U \cap C_n)$. Then, $\mathrm{cl}\, U_n$ is compact because $\mathrm{cl}\, U_n \subset C_n$. By

Lemma 5, there exists a continuous open surjection from $\downarrow\mathrm{C}_F(X)$ onto $\downarrow\mathrm{C}_F(\mathrm{cl}\, U_n)$. Therefore, $\downarrow\mathrm{C}_F(\mathrm{cl}\, U_n)$ is second-countable, hence $\mathrm{cl}\, U_n$ is compact and metrizable by Lemma 6. Thus every U_n is also separable metrizable, hence it is second-countable. Moreover, for every $x \in U$, there exists an open set V such that $x \in V$, $\mathrm{cl}\, V$ is compact and $\mathrm{cl}\, V \subset U$. Hence there exists n such that $\mathrm{cl}\, V \subset C_n$. Then, $x \in V \subset \mathrm{int}(U \cap C_n) = U_n$. It follows that $U = \bigcup_{n=1}^{\infty} U_n$. Therefore, U is second-countable, hence it is separable metrizable.

As mentioned in proof of Theorem 4, $\beta\omega$ is a compact space and $\downarrow\mathrm{C}_F(\beta\omega)$ is not metrizable but ω is a dense, locally compact, open and separable metrizable subspace of $\beta\omega$. Namely, the converse is not true. □

Remark 2. The referee pointed out that McCoy and Ntantu [11] obtained analogous results in 1992. For example, Theorem 4.12 in [11] is similar to our Theorem 1. Our Theorem 3 for $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is true for $\uparrow\mathrm{C}_F(X, \mathbb{R})$ using Theorems 3.5, 3.7, 4.11 and Example 3.3 in [11], where $\uparrow\mathrm{C}_F(X, \mathbb{R})$ is the subspace of $\mathrm{Cld}_F(X \times \mathbb{R})$ consisting of the epigraphs

$$\uparrow f = \{(x, s) \in X \times \mathbb{R} : f(x) \leq s\} \in \mathrm{Cld}(X \times \mathbb{R}),$$

of all $f \in \mathrm{C}(X, \mathbb{R})$. However our arguments are quite different from their arguments in [11].

Acknowledgment. The author is so grateful to Professors K. Sakai and K. Mine for their guidance of revising the original manuscript. He also appreciates the referee's comments on the results of McCoy and Ntantu in [11].

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(*Received* April 3, 2012, *revised* May 8, 2012)