Moufang loops of order 243

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Abstract. We present a computer-assisted determination of the 72 non-isomorphic, non-associative Moufang loops of order 243. Some of their properties and distinguishing features are discussed.

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1. Introduction

A quasigroup Q is an algebraic structure with a binary operation $\cdot$ such that for any two elements $a, b \in Q$, there are two unique elements x and y in Q that satisfy the equations $a \cdot x = b$ and $y \cdot a = b$. A quasigroup with an identity element is called a loop, and a loop is Moufang if it satisfies the identity $xy \cdot zx = x(yz \cdot x)$. Every group is a Moufang loop, but there are also non-associative Moufang loops.

The classification of non-associative Moufang loops started in earnest with the work of Orin Chein and Hala Pflugfelder [3]. They demonstrated that there exists a unique smallest non-associative Moufang loop of order 12. In 1974 Chein [2] continued this work in discovering all non-associative Moufang loops of orders less than 32. This research also introduced a construction method for Moufang loops that involved extending nonabelian groups by order 2 cyclic groups. A few years later Chein and Pflugfelder [4] published another article that included all non-associative Moufang loops of order less than 64.

It was not until three decades later that Gábor Nagy and Petr Vojtěchovský [8] continued Chein and Pflugfelder's work by discovering all non-isomorphic non-associative Moufang loops of order 64 and 81. The most innovative approach of Nagy and Vojtěchovský was their use of computers to assist in the numerous complex computations. The computational discrete algebra software program GAP [5] and the LOOPS package, written by Nagy and Vojtěchovský [7], was utilized to store known small Moufang loops to speed up fundamental computations, computationally extend loops, and verify that the list of possible non-associative Moufang loops were in fact non-isomorphic. Together the mathematical theory and the computational techniques enabled Nagy and Vojtěchovský to identify all non-isomorphic non-associative Moufang loops of order 64 and 81.

Our computation used techniques developed for Moufang loops of order 81 in [8]. That is, every Moufang loop of order 243 can be constructed as a central extension of a Moufang loop of order 81. Consequently, we computed the central extensions of the 15 groups and the 5 non-associative Moufang loops of order 81. Removing the groups with generating sets of size less than three we were left with 10 loops of order 81 to extend. By extending these and removing isomorphic copies, we found 72 non-associative Moufang loops of order 243.

2. Groups and loops of order 81

It will be convenient to easily refer to the groups and non-associative Moufang loops of order 81. Our notation is chosen to match the GAP libraries, so G(81,k) refers to the group created by the GAP function call SmallGroup(81,k). Similarly, L(81,k) denotes MoufangLoop(81,k) from the Loops package. We include the pairing between our notation and that of [1] for the non-associative cases.

notation	Chee notation	description
G(81,11)	–	abelian type [3,3,9]
G(81,12)	–	(extra-special, exponent 3) $\times Z_3$
G(81,13)	–	(extra-special, exponent 9) $\times Z_3$
G(81,14)	–	central product of an extra-special group of order 27 with Z_9
G(81,15)	–	elementary abelian
L(81,1)	M_{81}(3,0,0,0)	exponent 3, commutative
L(81,2)	M_{81}(9,0,0,0)	exponent 9, commutative
L(81,3)	M_{81}(3,1,0,0)	exponent 3, non-commutative
L(81,4)	M_{81}(9,0,0,1)	exponent 9, non-commutative has subloop order 27, exponent 3
L(81,5)	M_{81}(9,1,0,0)	exponent 9, non-commutative no subloop order 27, exponent 3

3. The computation

As shown in Corollary 13 of [8], any Moufang loop of order p^n, p prime, has a non-trivial center. Consequently, any Moufang loop of order 243 can be found as an extension of a loop of order 81 by a central element of order 3.

An attempt to compute all these extensions in one large GAP session ran out of memory, so we split the computation apart to take advantage of a grid of computers we had access to. Each computer was given one of the loops of order 81 and used cohomology classes to compute a set of central extensions of order 243 (as described in [8]). Then, on that same computer, those particular extensions were tested for isomorphisms between any pair (again, as in [8]). At the same time, any associative loops were discarded. Consequently, each of the 10 computers returned a list of representatives of the isomorphism classes of the non-associative loops found as central extensions of the loop of order 81 that it was given. Our extension results are summarized in the following table.

Starting loop	# of extensions	# of isomorphism classes
G(81,11)	12	10
G(81,12)	20	17
G(81,13)	20	15
G(81,14)	26	13
G(81,15)	19	17
L(81,1)	5	5
L(81,2)	15	11
L(81,3)	16	12
L(81,4)	44	24
L(81,5)	63	29

This process left us with a list of 153 loops guaranteed to contain at least one representative of each isomorphism class of non-associative Moufang loops of order 243.

Since it is possible that a loop might appear as an extension of more than one structure of order 81, the next step was to look for such duplication in our list. However, we encountered a lucky accident. In each of the 153 loops in our list, the associator subloop was order 3 and contained in the center of the loop. Hence, every one of these loops could be produced as a central extension of a *group* of order 81. Furthermore, each was an extension of a unique group of order 81. Consequently, without any further isomorphism testing, we could see that the 72 loops found by extending the groups of order 81 constitute a complete list of isomorphism types of non-associative Moufang loops of order 243.

Theorem 3.1. *There are* 72 *non-associative Moufang loops of order* 243. *Each of these has an associator subloop of order* 3, *which is contained in the center of the loop.*

It is interesting to note that the property of having the associator subloop contained in the center was important in the classification of Moufang loops of order p^5 for $p > 3$ in [6]. It is surprising to find the same property for $p = 3$.

4. Isomorphism testing

When testing loops for isomorphism, it is useful to have an isomorphism invariant which tends to distinguish non-isomorphic loops. For our work we used a slight variation of the discriminator provided in the Loops package. Our discriminator is a multi-set of sextuples for any Moufang loop of order 243 computed as follows.

For any element x of the Moufang loop L, we define a "commuting profile of x" which is a list of the number of elements of L of various possible orders that commute with x. That is, the list $[a_0, a_1, \dots]$ records that x commutes with a_i elements of order 3^i (of course, a_0 is always 1).

For x in L, the "element profile of x" is the sextuple: the order of x, the commuting profile of x, the number of y in L such that $y^3 = x$, the boolean value of 'x is central in L', the boolean value of 'x is in the associator subloop of L', and the order of the normal closure of the subloop generated by x in L.

The discriminator of L is the multi-set of these element profiles for all of the elements of L.

Some measure of the effectiveness of this invariant can be found by grouping the non-associative Moufang loops of order 243 into classes that have the same discriminator value. In this case we find 39 classes of size one, 8 of size two, 3 of size three, and 2 of size four.

5. Properties of Moufang loops of order 243

Our list of loops exhibits the following.

Theorem 5.1. *There are* 6 *commutative non-associative Moufang loops of order* 243.

These loops can be distinguished by their discriminator values. In particular, the properties listed in the following table characterize each of these commutative loops. Each line describes a single loop. The "–" entries are not needed to distinguish the corresponding loop. For example, knowing a loop in this set has exponent 3 determines the loop uniquely, but knowing it has exponent 9 does not.

exponent	# of elements of order 3	exponent of center	size of normal closure of element of order 9
3	–	–	–
9	26	–	–
9	80	3	9
9	80	3	27
9	80	9	–
27	–	–	–

In [1], Chee introduces a concept of a minimal non-associative Moufang loop that requires no proper non-associative subloops and no proper non-associative quotients. For loops of order 81, every non-associative loop is minimal, but for order 243, the matter is more interesting. Based on a brute-force computation of possible subloops and quotients, 17 of our loops contain a non-associative subloop of order 81. In fact, each of these loops have 40 distinct subloops of order 81, 27 of which are non-associative. We also find that 47 of the loops have a non-associative quotient of order 81. Those counts include 5 loops that have both a proper subloop and proper quotient that are non-associative. Consequently, we see that there are 13 minimal non-associative loops of order 243.

Theorem 5.2. *There are exactly* 13 *non-associative Moufang loops of order* 243 *that have no proper non-associative subloops or quotients.*

These loops may be distinguished from other non-associative Moufang loops of order 243 by their discriminator values which are given in the listing below.

We use a short-hand notation for the discriminator to list the values of the minimal non-associative Moufang loops of order 243. Each loop is described by a string of symbols representing the corresponding multiset of sextuples. Each element type in the multiset is described by a multiplicity followed by an asterisk followed by the item. For instance, we would write $\{a, a, a, b, c, c\}$ as $\{3*a, 1*b, 2*c\}$. In each case, the corresponding minimal non-associative Moufang loop is the only non-associative Moufang loop of order 243 with that discriminant. Here are the discriminants of the 13 minimal non-associative Moufang loops of order 243.

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{1*1[1,26,216]27TT1, 2*3[1,26,216]108TT3, 24*3[1,26,54]0FF9,
      216*9[1,8,18]0FF27}
{1*1[1,26,216]27TT1, 2*3[1,26,216]108TT3, 6*3[1,26,216]0FF9,
      18*3[1,26,54]0FF9, 54*9[1,26,54]0FF27, 162*9[1,8,18]0FF27}
{1*1[1,26,54,162]27TT1, 2*3[1,26,54,162]27TT3, 6*3[1,26,54,162]0FF9,
      6*9[1,26,54,162]27TF9, 12*9[1,26,54,162]0FF9,
      18*3[1,26,54,0]0FF9, 36*9[1,26,54,0]0FF9, 162*27[1,8,18,54]0FF27}
{1*1[1,26,54,162]27TT1, 2*3[1,26,54,162]27TT3, 6*9[1,26,54,162]27TF9,
      18*27[1,26,54,162]0FF27, 24*3[1,8,18,54]0FF9,
      48*9[1,8,18,54]0FF9, 144*27[1,8,18,54]0FF27}
{1*1[1,26,54,162]27TT1, 2*3[1,26,54,162]27TT3, 6*9[1,26,54,162]27TF9,
      24*3[1,26,54,162]0FF9, 48*9[1,26,54,162]0FF9,
      162*27[1,26,54,162]0FF27}
{1*1[1,80,162]81TT1, 2*3[1,80,162]81TT3, 6*3[1,26,54]0FF9,
      18*3[1,26,0]0FF27, 18*9[1,26,54]0FF9, 54*3[1,8,18]0FF27,
      144*9[1,8,18]0FF27}
{1*1[1,80,162]81TT1, 2*3[1,80,162]81TT3, 6*3[1,26,54]0FF9,
      6*9[1,80,162]0FF9, 12*9[1,26,54]0FF9, 18*3[1,26,54]0FF27,
      36*9[1,26,54]0FF27, 54*3[1,8,18]0FF27, 108*9[1,8,18]0FF27}
{1*1[1,80,162]81TT1, 2*3[1,80,162]81TT3, 6*3[1,80,0]0FF9,
      18*3[1,26,54]0FF9, 54*3[1,26,0]0FF27, 162*9[1,8,18]0FF27}
{1*1[1,80,162]81TT1, 2*3[1,80,162]81TT3, 6*3[1,80,162]0FF9,
      18*3[1,26,54]0FF9, 54*3[1,26,0]0FF27, 54*9[1,26,54]0FF27,
      108*9[1,8,18]0FF27}
{1*1[1,134,108]135TT1, 2*3[1,134,108]54TT3, 12*3[1,26,54]0FF9,
      12*3[1,80,0]0FF9, 108*3[1,26,0]0FF27, 108*9[1,8,18]0FF27}
{1*1[1,134,108]135TT1, 2*3[1,134,108]54TT3, 6*3[1,134,108]0FF9,
      18*3[1,80,0]0FF9, 54*3[1,26,0]0FF27, 54*3[1,80,0]0FF27,
      108*9[1,8,18]0FF27}
{1*1[1,188,54]189TT1, 2*3[1,188,54]27TT3, 6*3[1,188,54]0FF9,
      18*3[1,26,54]0FF9, 54*9[1,26,54]0FF27, 162*3[1,26,0]0FF27}
{1*1[1,188,54]189TT1, 2*3[1,188,54]27TT3, 6*3[1,26,54]0FF9,
      18*3[1,80,0]0FF9, 54*9[1,8,18]0FF27, 162*3[1,26,0]0FF27}
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In addition, we discovered other interesting statistics regarding order 243 Moufang loops.

Theorem 5.3. *All non-associative Moufang loops of order* 243 *have a nucleus of order* 9.

Theorem 5.4. *The non-commutative Moufang loops of order* 243 *have commutant size* 27 *for* 27 *of the loops,* 9 *for* 29 *of the loops, and* 3 *for the remaining* 10 *loops.*

Theorem 5.5. *In each non-associative Moufang loop of order* 243*, the commutator subloop is normal with orders* 1, 3*, and* 9 *occurring.*

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