A global uniqueness result for fractional order implicit differential equations

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Abstract. In this paper we investigate the global existence and uniqueness of solutions for the initial value problems (IVP for short), for a class of implicit hyperbolic fractional order differential equations by using a nonlinear alternative of Leray-Schauder type for contraction maps on Fréchet spaces.

Keywords: partial hyperbolic differential equation, fractional order, left-sided mixed Riemann-Liouville integral, mixed regularized derivative, solution, Fréchet space, fixed point

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1. Introduction

The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [13], [17], [18]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [5], Kilbas *et al.* [14], Miller and Ross [16], Samko *et al.* [19], and Podlubny [18], the papers of Abbas and Benchohra [2], [3], Abbas *et al.* [1], [4], Belarbi *et al.* [7], Benchohra *et al.* [8], [9], [10], Kilbas and Marzan [15], Vityuk [20], Vityuk and Golushkov [21], Vityuk and Mykhailenko [22], and the references therein. In [23] Vityuk and Mykhailenko used iterative method to obtain solution to problem (1)–(2) on bounded domain. By means of the Banach contraction principle and the nonlinear alternative of Leray-Schauder type Abbas *et al.* [6] present some existence as well as uniqueness results for problem (1)–(2) on bounded domain.

In this paper we are concerned with the global existence and uniqueness of solutions to fractional order IVP for the system

$$(1)\qquad \overline{D}^r_\theta u(x,y) = f(x,y,u(x,y),\overline{D}^r_\theta u(x,y));\ \text{ if }\ (x,y)\in J := [0,\infty)\times[0,\infty),$$

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$$
\begin{cases} u(x,0) = \varphi(x), \ u(0,y) = \psi(y); \ x, y \in [0,\infty), \\ \varphi(0) = \psi(0), \end{cases} \tag{2}
$$

where $\theta = (0,0)$, $\overline{D}_\theta^r$ is the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function and $\varphi, \psi : [0,\infty) \to \mathbb{R}^n$ are absolutely continuous functions. We make use of the nonlinear alternative of Leray-Schauder type for contraction maps on Fréchet spaces [11]. Many properties of solutions for differential equations and inclusions, such as stability or oscillation, require global properties of solutions. This is the main motivation to look for sufficient conditions that ensure global existence of solutions for IVP (1)–(2). This paper initiates the study on unbounded domain of IVP (1)–(2).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $p \in \mathbb{N}$ and $J_0 = [0,p] \times [0,p]$. By $C(J_0)$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$
\|w\|_\infty = \sup_{(x,y) \in J_0} \|w(x,y)\|,
$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

As usual, by $AC(J_0)$ we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$ and $L^1(J_0)$ is the space of Lebesgue-integrable functions $w : J_0 \to \mathbb{R}^n$ with the norm

$$
\|w\|_1 = \int_0^p \int_0^p \|w(x,y)\| \, dy \, dx.
$$

Definition 2.1 ([14], [19]). Let $\alpha \in (0,\infty)$ and $u \in L^1(J_0)$. The partial Riemann-Liouville integral of order α of $u(x,y)$ with respect to x is defined by the expression

$$
I_{0,x}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s,y) \, ds,
$$

for almost all $x \in [0,p]$ and almost all $y \in [0,p]$.

Analogously, we define the integral

$$
I_{0,y}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-s)^{\alpha-1} u(x,s) \, ds,
$$

for almost all $x \in [0,p]$ and almost all $y \in [0,p]$.

Definition 2.2 ([14], [19]). Let $\alpha \in (0,1]$ and $u \in L^1(J_0)$. The Riemann-Liouville fractional derivative of order α of $u(x,y)$ with respect to x is defined by

$$
(D_{0,x}^\alpha u)(x,y) = \frac{\partial}{\partial x} I_{0,x}^{1-\alpha} u(x,y),
$$

for almost all $x \in [0, p]$ and almost all $y \in [0, p]$.

Analogously, we define the derivative

$$(D_{0,y}^{\alpha} u)(x, y) = \frac{\partial}{\partial y} I_{0,y}^{1-\alpha} u(x, y),$$

for almost all $x \in [0, p]$ and almost all $y \in [0, p]$.

Definition 2.3 ([14], [19])**.** Let $\alpha \in (0, 1]$ and $u \in L^1(J_0)$. The Caputo fractional derivative of order α of $u(x, y)$ with respect to x is defined by the expression

$$^{c}D_{0,x}^{\alpha} u(x, y) = I_{0,x}^{1-\alpha} \frac{\partial}{\partial x} u(x, y),$$

for almost all $x \in [0, p]$ and almost all $y \in [0, p]$.

Analogously, we define the derivative

$$^{c}D_{0,y}^{\alpha} u(x, y) = I_{0,y}^{1-\alpha} \frac{\partial}{\partial y} u(x, y),$$

for almost all $x \in [0, p]$ and almost all $y \in [0, p]$.

Definition 2.4 ([21])**.** Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1(J_0)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_{\theta}^{r} u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} u(s, t)\, dt\, ds.$$

In particular,

$$(I_{\theta}^{\theta} u)(x, y) = u(x, y), \;\; (I_{\theta}^{\sigma} u)(x, y) = \int_0^x \int_0^y u(s, t)\, dt\, ds;$$

for almost all $(x, y) \in J_0$, where $\sigma = (1, 1)$.

For instance, $I_{\theta}^{r} u$ exists for all $r_1, r_2 \in (0, \infty)$, when $u \in L^1(J_0)$. Note also that when $u \in C(J_0)$, then $(I_{\theta}^{r} u) \in C(J_0)$, moreover

$$(I_{\theta}^{r} u)(x, 0) = (I_{\theta}^{r} u)(0, y) = 0; \;\; x, y \in [0, p].$$

Example 2.5. *Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then*

$$I_{\theta}^{r} x^{\lambda} y^{\omega} = \frac{\Gamma(1 + \lambda)\Gamma(1 + \omega)}{\Gamma(1 + \lambda + r_1)\Gamma(1 + \omega + r_2)}\, x^{\lambda + r_1} y^{\omega + r_2}, \quad \textit{for all } \; (x, y) \in J_0.$$

By $1 - r$ we mean $(1 - r_1, 1 - r_2) \in [0, 1) \times [0, 1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$ the mixed second order partial derivative.

Definition 2.6 ([21]). Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J_0)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression $D_\theta^r u(x,y) = (D_{xy}^2 I_\theta^{1-r} u)(x,y)$ and the Caputo fractional-order derivative of order r of u is defined by the expression ${}^c D_\theta^r u(x,y) = (I_\theta^{1-r} D_{xy}^2 u)(x,y)$.

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(x,y) = ({}^c D_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y), \quad \text{for almost all } \ (x,y) \in J_0.$$

Example 2.7. *Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then*

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} \, x^{\lambda-r_1} y^{\omega-r_2}, \quad \textit{for almost all } \ (x,y) \in J_0.$$

Definition 2.8 ([23]). For a function $u : J_0 \to \mathbb{R}^n$, we set

$$q(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$ of a function $u(x,y)$, we name the function

$$\overline{D}_\theta^r u(x,y) = D_\theta^r q(x,y).$$

The function

$$\overline{D}_{0,x}^{r_1} u(x,y) = D_{0,x}^{r_1}[u(x,y) - u(0,y)],$$

is called the partial r_1–order regularized derivative of the function $u(x,y) : J_0 \to \mathbb{R}^n$ with respect to the variable x. Analogously, we define the derivative

$$\overline{D}_{0,y}^{r_2} u(x,y) = D_{0,y}^{r_2}[u(x,y) - u(x,0)].$$

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ satisfies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \dots \quad \text{for every } \ x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \quad \text{for all } \ y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote by $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote by $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote by $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [11].

Definition 2.9. Let X be a Fréchet space. A function $N : X \longrightarrow X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in [0,1)$ such that

$$\|N(u) - N(v)\|_n \leq k_n \|u - v\|_n \quad \text{for all} \quad u, v \in X.$$

Theorem 2.10 ([11])**.** *Let X be a Fréchet space and $Y \subset X$ a closed subset in X. Let $N : Y \longrightarrow X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

(a) *the operator N has a unique fixed point;*
(b) *there exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $u \in \partial_n Y^n$ such that $\|u - \lambda N(u)\|_n = 0$.*

For each $p \in \mathbb{N}$ we define in $C(J)$ the semi-norms by:

$$\|u\|_p = \sup_{(x,y)\in J_0} \|u(x,y)\|.$$

Then $C(J)$ is a Fréchet space with the family of semi-norms $\{\|u\|_p\}_{p\in\mathbb{N}}$.

3. Existence of solutions

Let us start by defining what we mean by a solution of the problem (1)–(2).

Definition 3.1. A function $u \in C(J)$ such that $\overline{D}_{0,x}^{r_1} u(x,y)$, $\overline{D}_{0,y}^{r_2} u(x,y)$, $\overline{D}_{\theta}^{r} u(x,y)$ are continuous for $(x,y) \in J$ and $I_\theta^{1-r} u(x,y) \in AC(J)$ is said to be a solution of (1)–(2) if u satisfies the equation (1) and the conditions (2) on J.

For the existence of solutions for the problem (1)–(2) we need the following lemma.

Lemma 3.2 ([23])**.** *Let a function $f(x,y,u,z) : J_0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem (1)–(2) is equivalent to the problem of the solution of the equation*

$$g(x,y) = f(x,y,\mu(x,y) + I_\theta^r g(x,y), g(x,y)), \tag{3}$$

and if $g \in C(J_0)$ is the solution of (6), then $u(x,y) = \mu(x,y) + I_\theta^r g(x,y)$, where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

In the sequel we use the following version of Gronwall's Lemma for two independent variables and singular kernel.

Lemma 3.3 ([12])**.** *Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(\cdot,\cdot)$ be a non-negative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that,*

$$\upsilon(x,y) \leq \omega(x,y) + c \int_0^x \int_0^y \frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\, dt\, ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that,

$$\upsilon(x,y) \leq \omega(x,y) + \delta c \int_0^x \int_0^y \frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\, dt\, ds,$$

for every $(x,y) \in J$.

We are now in the position to give conditions for the existence and uniqueness of a solution of problem (1)–(2).

Theorem 3.4. *Assume*

(H_1) *the function* $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous;*

(H_2) *for each* $p \in \mathbb{N}$, *there exists constants* $k_p > 0$ *and* $0 < l_p < 1$ *such that for each* $(x,y) \in J_0$

$$\|f(x,y,u,z) - f(x,y,v,w)\| \le k_p\|u-v\| + l_p\|z-w\|,$$

for each $u, v, w, z \in \mathbb{R}^n$.

If

$$\frac{k_p p^{r_1+r_2}}{(1-l_p)\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{4}$$

then there exists a unique solution for IVP (1)–(2) *on* $[0,\infty) \times [0,\infty)$.

Proof: Transform the problem (1)–(2) into a fixed point problem. Consider the operator $N : C(J) \to C(J)$ defined by,

$$N(u)(x,y) = \mu(x,y) + I_\theta^r g(x,y), \tag{5}$$

where $g \in C(J)$ such that $g(x,y) = f(x,y,u(x,y),g(x,y))$. The operator N is well defined, that is, for each $u \in C(J)$ there exists a unique $g \in C(J)$ such that

$$g(x,y) = f(x,y,u(x,y),g(x,y)) \ \text{ for each } \ (x,y) \in J. \tag{6}$$

Indeed, assume that for each $u \in C(J)$ there exist $g_1, g_2 \in C(J)$ satisfying (6). Then using (H_2) we get for each $(x,y) \in J$

$$(1-l_p)\|g_1(x,y) - g_2(x,y)\| \le 0,$$

which implies that

$$g_1(x,y) = g_2(x,y) \ \text{ for each } \ (x,y) \in J.$$

Let u be a possible solution of the problem $u = \lambda N(u)$ for some $0 < \lambda < 1$. This implies that for each $(x,y) \in J_0$, we have

$$u(x,y) = \lambda\mu(x,y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,dt\,ds. \tag{7}$$

By (H_2) we get

$$\|g(x,y)\| \le f^* + k_p\|u(x,y)\| + l_p\|g(x,y)\|,$$

where

$$f^* = \sup_{(x,y)\in J_0} \|f(x,y,0,0)\|.$$

Then

$$\|g(x,y)\| \quad \leq \quad \frac{f^* + k_p\|u(x,y)\|}{1-l_p}\,.$$

Thus, (7) implies that

$$\begin{aligned}
&\|u(x,y)\| \leq \|\mu(x,y)\| \\
&+ \frac{1}{(1-l_p)\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\Big(f^* + k_p\|u(s,t)\|\Big)\,dt\,ds \\
&\leq \|\mu\|_p + \frac{f^* p^{r_1+r_2}}{(1-l_p)\Gamma(1+r_1)\Gamma(1+r_2)} \\
&+ \frac{k_p}{(1-l_p)\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|u(s,t)\|\,dt\,ds.
\end{aligned}$$

Set

$$w = \|\mu\|_p + \frac{f^* p^{r_1+r_2}}{(1-l_p)\Gamma(1+r_1)\Gamma(1+r_2)}\,.$$

Lemma 3.3 implies that there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\begin{aligned}
\|u(x,y)\| \quad &\leq \quad w\Big(1 + \frac{\delta k_p}{(1-l_p)\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\,dt\,ds\Big) \\
&\leq \quad w\Big(1 + \frac{\delta k_p p^{r_1+r_2}}{(1-l_p)\Gamma(1+r_1)\Gamma(1+r_2)}\Big) := M_p.
\end{aligned}$$

Then for every $p \in \mathbb{N}$ we have $\|u\|_p \leq M_p$. Set

$$U = \{u \in C(J) : \|u\|_p \leq M_p + 1 \;\text{ for all }\; p \in \mathbb{N}\}.$$

We shall show that $N : U \longrightarrow C(J_0)$ is a contraction map. Indeed, consider $v, w \in C(J_0)$. Then, for $(x,y) \in J_0$, we have

$$\begin{aligned}
&\|N(v)(x,y) - N(w)(x,y)\| \\
&\quad \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t) - h(s,t)\|\,dt\,ds,
\end{aligned} \tag{8}$$

where $g, h \in C(J_0)$ such that

$$g(x,y) = f(x,y,v(x,y),g(x,y))$$

and

$$h(x,y) = f(x,y,w(x,y),h(x,y)).$$

By (H_2), we get

$$\|g(x,y)-h(x,y)\| \leq k_p\|v(x,y)-w(x,y)\| + l_p\|g(x,y)-h(x,y)\|.$$

Then

$$\|g(x,y)-h(x,y)\| \leq \frac{k_p}{1-l_p}\|v-w\|_p.$$

Thus, (8) implies that

$$\begin{aligned}
&\|N(v)-N(w)\|_p \\
&\quad \leq \frac{k_p}{(1-l_p)\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|v-w\|_p\,dt\,ds \\
&\quad \leq \frac{k_p p^{r_1+r_2}}{(1-l_p)\Gamma(1+r_1)\Gamma(1+r_2)}\|v-w\|_p.
\end{aligned}$$

Hence

$$\|N(v)-N(w)\|_p \leq \frac{k_p p^{r_1+r_2}}{(1-l_p)\Gamma(1+r_1)\Gamma(1+r_2)}\ \|v-w\|_p.$$

By (4), $N : U \longrightarrow C(J_0)$ is a contraction. By the choice of U, there is no $u \in \partial_n U^n$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 2.10, we deduce that N has a unique fixed point u in U which is a solution to problem (1)–(2). □

4. An example

As an application of our results we consider the following implicit partial hyperbolic fractional order differential equation of the form

(9)
$$\overline{D}_\theta^r u(x,y) = \frac{1}{7e^{x+y+2}(1+c_p|u(x,y)|+|\overline{D}_\theta^r u(x,y)|)};\quad \text{if}\quad (x,y)\in[0,\infty)\times[0,\infty),$$

$$u(x,0)=x,\ \ u(0,y)=y^2;\ x,y\in[0,\infty), \tag{10}$$

where

$$c_p = \frac{\Gamma(1+r_1)\Gamma(1+r_2)}{p^{r_1+r_2}};\ p\in\mathbb{N}^* := \{1,2,3,\ldots\}.$$

Define f by

$$f(x,y,u,v) = \frac{1}{7e^{x+y+2}(1+c_p|u|+|v|)};\ (x,y)\in[0,\infty)\times[0,\infty) \text{ and } u,v\in\mathbb{R}.$$

Clearly, the function f is continuous. For each $p\in\mathbb{N}^*$ and $(x,y)\in J_0$ we have

$$|f(x,y,u(x,y),v(x,y)) - f(x,y,\overline{u}(x,y),\overline{v}(x,y))| \leq \frac{1}{7e^2}(c_p\|u-\overline{u}\| + \|v-\overline{v}\|),$$

for each $u, v, \overline{u}, \overline{v} \in \mathbb{R}$.

Hence condition (H_2) is satisfied with $k_p = \frac{c_p}{7e^2}$ and $l_p = \frac{1}{7e^2}$. We shall show that condition (4) holds for all $p \in \mathbb{N}^*$. Indeed

$$\frac{k_p p^{r_1+r_2}}{(1-l_p)\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{c_p p^{r_1+r_2}}{(7e^2-1)\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{7e^2-1} < 1.$$

Consequently, Theorem 3.4 implies that problem (9)–(10) has a unique solution defined on $[0,\infty) \times [0,\infty)$.

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