On some convexity properties in the Besicovitch-Musielak-Orlicz space of almost periodic functions with Luxemburg norm

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Abstract. In this article, it is shown that geometrical properties such as local uniform convexity, mid point local uniform convexity, H-property and uniform convexity in every direction are equivalent in the Besicovitch-Musielak-Orlicz space of almost periodic functions ($\widetilde{B}^{\varphi}a.p.$) endowed with the Luxemburg norm.

Keywords: local uniform convexity, uniform convexity in every direction, mid point locally uniform, H-property, strict convexity, approximation, Besicovitch-Musielak-Orlicz space, almost periodic function

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1. Introduction and preliminaries

This article is a continuation of the investigations concerning the geometrical properties in the space of Besicovitch-Orlicz of almost periodic functions (see [1]). Here we are interested in such properties as local uniform convexity, Kadec-Klee property, mid point local uniform convexity and uniform convexity in every direction in the widest class of Besicovitch-Musielak-Orlicz space of almost periodic functions $\widetilde{B}^{\varphi}a.p.$. We are finding criteria for these properties. An approximation property in $\widetilde{B}^{\varphi}a.p.$ is also presented.

Now, we recall the needed definitions and notations.

We say that a Banach space $(X, \|\cdot\|)$ is locally uniformly convex LUC (see [10]) if for each $\varepsilon > 0$ and each $y \in S(X)$ there is a $\delta_X(\varepsilon, y) > 0$ such that if $x \in S(X)$ and $\|x - y\| \geq \varepsilon$, then $\|\frac{1}{2}(x + y)\| \leq 1 - \delta_X(\varepsilon, y)$, where as usual, the notations $S(X)$ and $B(X)$ are used for the unit sphere and unit ball of X respectively.

There are also sequential characterizations of LUC (see [10]): the space $(X, \|\cdot\|)$ is LUC if and only if for each $x \in S(X)$ and every sequence (y_n) in $S(X)$ (or $B(X)$) for which $\|\frac{1}{2}(x + y_n)\| \to 1$, we have $\|y_n - x\| \to 0$.

Let $x \in S(X)$. If $x_n \in X$, $x_n \to x$ weakly ($x_n \overset{w}{\to} x$) and $\|x_n\| \to \|x\| = 1$ imply $x_n \to x$ in norm, then we call x an H-point of $B(X)$. If every point in $S(X)$ is an H-point of $B(X)$, then we say that X has the H-property (or satisfy the Kadec-Klee property also called the Radon Riesz property) (see [5]).

The space X is called mid point locally uniformly convex (in short MLUC) when every point $x \in S(X)$ is strongly extreme, i.e., for each sequence (x_n) in X, the conditions $\|x + x_n\| \to 1$ and $\|x - x_n\| \to 1$ implies $\|x_n\| \to 0$.

Now we present the class of Banach spaces introduced by A.G. Garkari, the so-called uniformly convex in every direction (see [4], [15]). We mention that these spaces (among others) are important in approximation theory since they are exactly those Banach spaces in which every bounded set has at most one Cebyshev center. If K is a subset of Banach space X then the Cebyshev centers of K are the elements c in K with the property that

$$\sup_{k\in K} \|c-k\| = \inf_{s\in X} \sup_{t\in K} \|s-t\| .$$

The Banach space X is said to be uniformly convex in every direction (in short UCED) if the following property holds: for every nonzero z in X and $\varepsilon > 0$ there exists $\delta(z, \varepsilon) > 0$ such that $|a| < \varepsilon$ if $\|x\| = \|y\| = 1$, $x - y = az$, and $\|x + y\| = 2[1 - \delta(z, \varepsilon)]$. We mention the following characterization of UCED Banach spaces in terms of sequences: for any $z \in X$, and every sequence (x_n) in X, the conditions $\|x_n\| \to 1$, $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$ imply $z = 0$.

Let us note that the implications LUC $\Rightarrow$ MLUC $\Rightarrow$ SC (strict convexity), LUC $\Rightarrow$ H-property and UCED $\Rightarrow$ SC hold in general Banach spaces (see e.g. [10]).

In the case of Musielak-Orlicz spaces, these geometrical properties are well characterized in [11], [7].

The most important geometrical properties of the space $\widetilde{B}^{\varphi}a.p.$ with respect to the Luxemburg norm are characterized in [8] and [9]. The authors have obtained the following results (see Theorem 3.1 of [9] and Theorem 1 of [8] respectively):

Theorem 1. *The space $\widetilde{B}^{\varphi}a.p.$ endowed with the Luxemburg norm is uniformly convex if and only if φ is uniformly convex and satisfies the $\Delta_2^{B^1}$-condition.*

Theorem 2. *The space $\widetilde{B}^{\varphi}a.p.$ endowed with the Luxemburg norm is strictly convex if and only if φ is strictly convex and satisfies the $\Delta_2^{B^1}$-condition.*

Now, we introduce some notions joined with Besicovitch-Musielak-Orlicz spaces of almost periodic functions. In what follows, let us denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the natural, real and complex numbers respectively.

Let $\varphi : \mathbb{R} \times [0, +\infty[\longrightarrow [0, +\infty[$ be a continuous function on $\mathbb{R} \times [0, +\infty[$ satisfying:

(1) $\forall t \in \mathbb{R}$, $\varphi(t, u) = 0 \quad \text{iff} \quad u = 0$,
(2) $\forall t \in \mathbb{R}$, $\varphi(t, u)$ is convex with respect to $u \in [0, +\infty[$,
(3) $\forall u \in [0, +\infty[$, $\varphi(t, u)$ is periodic with respect to $t \in \mathbb{R}$, the period τ being fixed and independent of $u \in [0, +\infty[$. Without loss of generality we may suppose that $\tau = 1$.

As a consequence of these assumptions, we get that the function $\phi(\alpha) = \inf_{t\in\mathbb{R}}\{\varphi(t, \alpha)\}$ is strictly positive and convex. This fact will be very useful in our computations.

We denote by $L^\varphi_{\mathrm{loc}}(\mathbb{R})$ the subspace of φ-locally integrable functions, i.e. the subspace of all Lebesgue measurable functions on $\mathbb{R}$ such that for each compact $K \subset \mathbb{R}$, there exists $\lambda_K > 0$ for which $\int_K \varphi(t, \lambda_K |f(t)|)\, dt < +\infty$. The functional

$$\begin{array}{rcl} \rho_{B^\varphi} : L^\varphi_{\mathrm{loc}}(\mathbb{R}) & \longrightarrow & [0, +\infty] \\ f & \longrightarrow & \rho_{B^\varphi}(f) = \limsup\limits_{T \longrightarrow +\infty} \dfrac{1}{2T} \displaystyle\int_{-T}^{+T} \varphi(t, |f(t)|)\, dt, \end{array} \tag{1.1}$$

is a convex pseudomodular (see [12]).

We define the Besicovitch-Musielak-Orlicz space associated to this pseudomodular by

$$\begin{array}{rcl} B^\varphi(\mathbb{R}) & = & \{f \in L^\varphi_{\mathrm{loc}}(\mathbb{R}) : \lim\limits_{\alpha \to 0} \rho_{B^\varphi}(\alpha f) = 0\}, \\ & = & \{f \in L^\varphi_{\mathrm{loc}}(\mathbb{R}) : \rho_{B^\varphi}(\alpha f) < 0, \quad \text{for some} \quad \alpha > 0\}. \end{array}$$

The space $B^\varphi(\mathbb{R})$ is naturally endowed with the Luxemburg (pseudo)norm

$$\|f\|_{B^\varphi} = \inf\{k > 0 : \rho_{B^\varphi}(\frac{f}{k}) \le 1\}, \qquad f \in B^\varphi(\mathbb{R}).$$

Under the Luxemburg norm, $B^\varphi(\mathbb{R})$ is a Banach space.

Let $\mathcal{A}$ be the set of all generalized trigonometric polynomials, i.e.,

$$\mathcal{A} = \{P_n(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t},\ a_j \in \mathbb{C},\ \lambda_j \in \mathbb{R},\ n \in \mathbb{N}\}.$$

The Besicovitch-Musielak-Orlicz space of almost periodic functions, denoted $B^\varphi a.p.$, is the closure of the set $\mathcal{A}$ in $B^\varphi(\mathbb{R})$ with respect to the (pseudo)norm $\|\cdot\|_{B^\varphi}$:

$$\begin{array}{rcl} B^\varphi a.p. & = & \{f \in B^\varphi(\mathbb{R}) : \exists f_n \in \mathcal{A}, \forall k > 0, \lim\limits_{n \to +\infty} \rho_{B^\varphi}(k(f_n - f)) = 0\}, \\ & = & \{f \in B^\varphi(\mathbb{R}) : \exists f_n \in \mathcal{A}, \lim\limits_{n \to +\infty} \|f_n - f\|_{B^\varphi} = 0\}. \end{array}$$

We shall also be concerned with the space

$$\widetilde{B}^\varphi a.p. = \{f \in B^\varphi(\mathbb{R}) : \exists f_n \in \mathcal{A}, \exists k_0 > 0, \lim_{n \to +\infty} \rho_{B^\varphi}(k_0(f_n - f)) = 0\},$$

which is defined as the closure of the set $\mathcal{A}$ in $B^\varphi(\mathbb{R})$ with respect to the (pseudo)-modular $\rho_{B^\varphi}(\cdot)$.

Some topological properties (reflexivity and duality properties) of these spaces are considered in [3]. Clearly, we have the following inclusions

$$B^\varphi a.p. \subseteq \widetilde{B}^\varphi a.p. \subseteq B^\varphi(\mathbb{R}).$$

When $\varphi(t, \cdot) = |\cdot|$, we denote by $B^1(\mathbb{R})$ and $B^1 a.p.$ the respective spaces. The notation ρ_1 is used for the associated pseudomodular.

If in addition the Musielak-Orlicz function satisfies the condition that for every $u_0 > 0$ there is a $c > 0$ for which $\frac{\varphi(t,u)}{u} \geq c$ for $u \geq u_0$ and $t \in \mathbb{R}$ (see [12, p. 91, Theorem 13.18]), we get the inclusion $B^{\varphi}a.p. \subseteq B^1 a.p.$. So, to every f in $B^{\varphi}a.p.$ we can associate a formal Fourier series. Questions concerning the convergence of the Fourier series are not considered.

Remark 1. To each function $f \in B^{\varphi}a.p.$, one can associate a Bochner-Fejèr polynomial σ^f as follows:

$$\sigma^f(x) = M(f(x+\cdot)K_B(\cdot)) = \lim_{T\to\infty} \int_{-T}^{+T} f(x+t)K_B(t)\,dt,$$

where $K_B(\cdot)$ is the Bochner-Fejèr kernel (see e.g. [6]). An important question is the approximation property of Bochner-Fejèr, that is, for any $f \in B^{\varphi}a.p.$ and for each $\varepsilon > 0$, can one find a Bochner-Fejèr polynomial σ_ε^f such that $\|f - \sigma_\varepsilon^f\|_{B^\phi} \leq \varepsilon$? It is still an open problem whether this approximation property is true or not for Besicovitch-Musielak-Orlicz spaces of almost periodic functions $\widetilde{B}^{\varphi}a.p.$. The only trouble is that, for $f \in \widetilde{B}^{\varphi}a.p.$ and the associated Bochner-Fejèr's polynomial σ^f, one cannot prove the inequality

$$\rho_{B^\varphi}\left(\sigma^f\right) \leq \rho_{B^\varphi}(f)$$

for any Musielak-Orlicz function φ.

Another fundamental result concerning the functions in $B^{\varphi}a.p.$ is the fact that if $f \in B^{\varphi}a.p.$ then $\varphi(\cdot, |f(\cdot)|) \in B^1 a.p.$ (see [8]). This property guaranties the existence of the limit in (1.1).

We say that φ satisfies the $\Delta_2^{B^1}$-condition ($\varphi \in \Delta_2^{B^1}$) if there exists $k > 1$ and a measurable nonnegative function h such that $\rho_1(h) < +\infty$ and $\varphi(t, 2u) \leq k\varphi(t,u) + h(t)$ for almost all $t \in \mathbb{R}$ and all $u \geq 0$.

We say that φ satisfies the $\nabla_2^{B^1}$-condition ($\varphi \in \nabla_2^{B^1}$) if its conjugate ψ given by the formula

$$\psi(t,u) = \sup_{v\geq 0}\{uv - \varphi(t,v)\}, \quad \text{for } \ t \in \mathbb{R} \ \text{ and } \ u \geq 0$$

satisfies the $\Delta_2^{B^1}$-condition.

Let us mention the following important fact (see [8]): φ satisfies the $\Delta_2^{B^1}$-condition if and only if φ satisfies the $\Delta_2^{L^1}$-condition, that is, there exist $k > 0$ and a positive function h with $\int_0^1 h(t)dt < +\infty$ such that

$$\varphi(t, 2u) \leq k\varphi(t,u) + h(t), \quad \text{for almost all } \ t \in [0,1] \ \text{ and } \ u \geq 0.$$

2. Auxiliary results

Let $P(\mathbb{R})$ be the family of subsets of $\mathbb{R}$ and $\Sigma(\mathbb{R})$ the Σ-algebra of its Lebesgue measurable sets. We define the set function

$$\overline{\mu}(A) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{+T} \chi_A(t)\,dt = \overline{\lim_{T\to\infty}} \frac{1}{2T} \mu(A \cap [-T,+T])$$

where χ_A denotes the characteristic function of $A \in \Sigma(\mathbb{R})$.

It is easily seen that the set function $\overline{\mu}$ is not σ-additive.
A sequence $\{f_n\} \subset B^{\varphi}(\mathbb{R})$ is said to be $\overline{\mu}$-convergent to some $f \in B^{\varphi}(\mathbb{R})$ when, for every $\alpha > 0$, we have

$$\lim_{n\to\infty} \overline{\mu}\{x \in \mathbb{R} : |f_n(x) - f(x)| > \alpha\} = 0.$$

This convergence concept satisfies the following property:
If $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ are two sequences of Σ-measurable functions $\overline{\mu}$-convergent to f and g respectively, then for all real α and β the sequence $\{\alpha f_n + \beta g_n\}$ is $\overline{\mu}$-convergent to $\alpha f + \beta g$.

Remark 2. We can also see that $\overline{\mu}$ does not satisfy the extraction property. Indeed, let us consider the sequence $(f_n)_n$ of $B^{\phi}(\mathbb{R})$ defined by

$$f_n(t) = \chi_{[-n,n]}(t).$$

It is not difficult to see that f_n is $\overline{\mu}$-convergent to $f \equiv 0$ in $B^{\phi}(\mathbb{R})$. Nevertheless, there is no subsequence which converges $\overline{\mu}$ almost everywhere ($\overline{\mu}$ a.e.) to $f \equiv 0$. More exactly, for any bijection $\theta : \mathbb{N} \longrightarrow \mathbb{N}$, the sequence $(f_{\theta(n)})_n$ converges to 1 with respect to the $\overline{\mu}$ a.e. convergence on $\mathbb{R}$.

We give here some technical results that are the key arguments in proof of the main theorems. First we need the following results (see [8] and [9]):

Lemma 1 ([8], [9])**.** *Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^{\varphi}(\mathbb{R})$. Then:*

(i) *if $\{f_n\}_{n\geq 1}$ is modular convergent to $f \in B^{\varphi}a.p.$ it is also $\overline{\mu}$-convergent to f;*

(ii) *if $\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to $f \in B^1a.p.$ and there exists $g \in B^1a.p.$ satisfying $\max(|f_n|,|f|) \leq g$, then*

$$\lim_{n\to\infty} \rho_1(f_n) = \rho_1(f).$$

Lemma 2 ([8], [9])**.** *Let $f \in B^{\varphi}a.p.$. Then*

(1) *$\|f\|_{B^{\varphi}} \leq 1$ if and only if $\rho_{B^{\varphi}}(f) \leq 1$,*

(2) *$\|f\|_{B^{\varphi}} = 1$ if and only if $\rho_{B^{\varphi}}(f) = 1$.*

Lemma 3 ([8])**.** *Let $\{f_n\}, \{g_n\}$ be sequences in $B^{\varphi}a.p.$ such that $\rho_{B^{\varphi}}(f_n) \leq 1$, $\rho_{B^{\varphi}}(g_n) \leq 1$ and $\lim_{n\to\infty} \rho_{B^{\varphi}}(\frac{1}{2}(f_n + g_n)) = 1$. Suppose that φ is strictly convex. Then, the sequence $\{f_n - g_n\}_n$ is $\overline{\mu}$-convergent to zero.*

In the following we denote by $\mathcal{M}(\mathbb{R})$ the set of Lebesgue measurable functions on $\mathbb{R}$, and $L^\varphi([0,1])$ the usual Musielak-Orlicz class

$$L^\varphi([0,1]) = \{f \in \mathcal{M}(\mathbb{R}) : \exists\lambda > 0, \int_0^1 \varphi(t, \lambda|f(t)|)\, dt < +\infty\}.$$

Proposition 1 ([8], [9]). *Let $f \in L^\varphi([0,1])$. Then,*

(1) *if $\widetilde{f}$ is the periodic extension of f to the whole $\mathbb{R}$ (with period $\tau = 1$), we have $\widetilde{f} \in \widetilde{B}^\varphi a.p.$.*
(2) *The injection map $i : L^\varphi([0,1]) \hookrightarrow \widetilde{B}^\varphi a.p.$, $i(f) = \widetilde{f}$ is an isometry with respect to the modulars and for the respective Luxemburg norms.*

We are ready now to present our results.

Lemma 4. *Let $f \in B^\varphi(\mathbb{R})$. Then $\lim_{n\to+\infty} \overline{\mu}\{t \in \mathbb{R},\ |f(t)| \geq n\} = 0$.*

Proof: For f being in $B^\varphi(\mathbb{R})$ there exists $\alpha > 0$ for which $\rho_{B^\varphi}(\alpha f) < \infty$. For an integer N, let f_N be the truncation of f, i.e.,

$$f_N(t) = \begin{cases} f(t) & \text{if } |f(t)| \leq N, \\ N & \text{if } |f(t)| > N. \end{cases}$$

Putting $E_N = \{t \in \mathbb{R}, |f(t)| \geq N\}$ and taking into account the convexity of ϕ we will have for each $N \in \mathbb{N}$,

$$\begin{aligned} \rho_{B^\varphi}(\alpha f) &\geq \rho_{B^\varphi}(\alpha f_N) \\ &\geq \rho_{B^\varphi}(\alpha f_N \chi_{E_N}) \\ &= \rho_{B^\varphi}(\alpha N \chi_{E_N}) \\ &\geq \phi(\alpha N)\overline{\mu}(E_N). \end{aligned}$$

Then, letting N tend to infinity, it follows directly that $\lim_{N\to\infty} \overline{\mu}(E_N) = 0$. □

Lemma 5. *Let $f \in B^\varphi a.p.$. Then the following equivalence holds:*

$$\rho_{B^\varphi}(f) = 0 \ \text{ iff } \ f = 0 \quad \overline{\mu} \ a.e.$$

Proof: The assertion that $\rho_{B^\varphi}(f) = 0$ implies $f = 0$ $\overline{\mu}$ *a.e.* is a direct consequence of (i) in Lemma 1.

Let us show that if $\rho_{B^\varphi}(f) > 0$ then there exist real numbers $\alpha, \theta > 0$ such that

$$\overline{\mu}\{t \in \mathbb{R}, |f(t)| \geq \alpha\} > \theta.$$

In the contrary case, we will have for all $n \geq 1$

$$\overline{\mu}\{G_n\} \leq \frac{1}{n}$$

with $G_n = \{t \in \mathbb{R}, |f(t)| \geq \frac{1}{n}\}$. We will denote by G_n^c its complement.

Since $\lim_{n\to\infty}\overline{\mu}\{G_n\}=0$, by using Lemma 4 in [8], we get

$$\lim_{n\to\infty}\rho_{B^{\varphi}}\left(f\chi_{G_n}\right)=0.$$

On the other hand,

$$\rho_{B^{\varphi}}\left(f\chi_{G_n^c}\right)\le\sup_{t\in\mathbb{R}}\varphi\left(t,\frac{1}{n}\right)\overline{\mu}\left(G_n^c\right)\le\sup_{t\in\mathbb{R}}\varphi\left(t,\frac{1}{n}\right). \tag{2.1}$$

Letting n tend to infinity in (2.1), it follows

$$\lim_{n\to+\infty}\rho_{B^{\varphi}}\left(f\chi_{G_n^c}\right)=0.$$

Otherwise, we have for all $n\ge 1$

$$\rho_{B^{\varphi}}(f)\le\rho_{B^{\varphi}}\left(f\chi_{G_n}\right)+\rho_{B^{\varphi}}\left(f\chi_{G_n^c}\right). \tag{2.2}$$

Finally, by choosing n sufficiently large, the last term of inequality (2.2) can be made smaller than any $\varepsilon>0$ from which we get $\rho_{B^{\varphi}}(f)=0$. This is a contradiction, which finishes the proof. □

Lemma 6. *Let $\{f_n\}$ and f be in $B^{\varphi}(\mathbb{R})$ such that f_n is $\overline{\mu}$-convergent to f, then the sequence $(\varphi(\cdot,|f_n(\cdot)|))_n$ is $\overline{\mu}$-convergent to $\varphi(\cdot,|f(\cdot)|)$ in $B^1(\mathbb{R})$.*

Proof: Let us mention that the continuity of φ is sufficient to show the desired result. The method developed here is influenced by the proof of Proposition 1 in [8]. In view of Lemma 4, for each $\theta\in]0,1[$ there is an $M>0$ such that

$$\overline{\mu}\{t\in\mathbb{R},|f(t)|\ge M\}<\theta.$$

Let now $\varepsilon>0$. We define the set

$$G_n=\{t\in\mathbb{R},|f(t)|\ge M\}\cup\{t\in\mathbb{R},|f_n(t)-f(t)|\ge\varepsilon\}.$$

The function φ being continuous on $\mathbb{R}\times[0,+\infty[$ is also uniformly continuous on $[0,1]\times[0,M+\varepsilon]$. Moreover, using the periodicity of $\varphi(t,u)$ with respect to $t\in\mathbb{R}$, it follows that φ is uniformly continuous on $\mathbb{R}\times[0,M+\varepsilon]$.
Then, there exists $\eta>0$ for which the following implication holds:

$$|\varphi(t,|f_n(t)|)-\varphi(t,|f(t)|)|>\varepsilon\Rightarrow|f_n(t)-f(t)|>\eta,\ \forall t\in G_n^c.$$

On the other hand, since $\{f_n\}$ is $\overline{\mu}$-convergent to f, we have

$$\lim_{n\to+\infty}\overline{\mu}\left\{t\in G_n^c,|\varphi(t,|f_n(t)|)-\varphi(t,|f(t)|)|>\varepsilon\right\}=0 \tag{2.3}$$

and then

$$\begin{aligned}
& \overline{\mu}\left\{t \in \mathbb{R}, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\} \\
\leq \ & \overline{\mu}\left\{t \in G_n, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\} \\
& + \overline{\mu}\left\{t \in G_n^c, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\} \\
\leq \ & \overline{\mu}\left(G_n\right) + \overline{\mu}\left\{t \in G_n^c, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\} \\
\leq \ & \overline{\mu}\left\{t \in \mathbb{R}, \left|f\left(t\right)\right| \geq M\right\} + \overline{\mu}\left\{t \in \mathbb{R}, \left|f_n\left(t\right) - f\left(t\right)\right| \geq \varepsilon\right\} \\
& + \overline{\mu}\left\{t \in G_n^c, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\}.
\end{aligned}$$

Now, letting n tend to infinity and in view of (2.3) we get:

$$\lim_{n\to+\infty} \overline{\mu}\left\{t \in \mathbb{R}, \left|\varphi\left(t, |f_n(t)|\right) - \varphi\left(t, |f(t)|\right)\right| \geq \varepsilon\right\} \leq \theta.$$

Since θ is arbitrary, it follows that the sequence $\{\varphi(\cdot, |f_n|)\}_n$ is $\overline{\mu}$-convergent to $\varphi(\cdot, |f|)$. $\square$

Corollary 1. *If $\{f_n\}_{n\geq 1} \subset B^{\varphi}(\mathbb{R})$ is $\overline{\mu}$-convergent to $f \in B^{\varphi}a.p.$ and there exists $g \in B^{\varphi}a.p.$ satisfying $\max(|f_n|, |f|) \leq g$, then*

$$\lim_{n\to\infty} \rho_{B^{\varphi}}(f_n) = \rho_{B^{\varphi}}(f).$$

Proof: First, remark that in the proof of (ii) of Lemma 1 (see Lemma 4 of [8] and Lemma 2.6. of [9]) we can assume that $\{f_n\}_{n\geq 1}$ and f are in $B^1(\mathbb{R})$ instead of $B^1a.p.$.

Now, let us show the corollary. Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^{\varphi}(\mathbb{R})$ *convergent to* f in the sense of $\overline{\mu}$-convergence. Then in view of Lemma 6, we get that the sequence $\varphi(\cdot, f_n(\cdot))$ is $\overline{\mu}$-convergent to $\varphi(., f(\cdot)) \in B^1(\mathbb{R})$ and satisfies the following fact:

$$\max\left(\varphi\left(., |f_n(\cdot)|\right), \varphi\left(., |f(\cdot)|\right)\right) \leq \varphi\left(., |g(\cdot)|\right) \in B^1 a.p.$$

Consequently, using Lemma 1, we deduce that

$$\lim_{n\to\infty} \rho_1(\varphi(\cdot, |f_n(\cdot)|)) = \rho_1(\varphi\left(\cdot, |f(\cdot)|\right)),$$

which means that

$$\lim_{n\to\infty} \rho_{B^{\varphi}}\left(f_n\right) = \rho_{B^{\varphi}}(f). \qquad \square$$

We now give an adapted version of Fatou's Lemma in $B^{\varphi}a.p.$.

Lemma 7. *Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^{\varphi}(\mathbb{R})$ $\overline{\mu}$-convergent to $f \in B^{\varphi}a.p.$, then we have*

$$\underline{\lim}_{n\to+\infty} \rho_{B^{\varphi}}\left(f_n\right) \geq \rho_{B^{\varphi}}(f).$$

PROOF: Consider the following sequence

$$g_n(t) = f(t)\chi_{E_n}(t) + f_n(t)\chi_{E_n^c}(t), \quad t \in \mathbb{R}$$

where $E_n = \{t \in \mathbb{R}, |f_n(t)| > |f(t)|\}$ and E_n^c is its complement. It is clear that for each $n \in \mathbb{N}$, g_n belongs to $B^{\varphi}(\mathbb{R})$ and satisfies

$$|g_n(t) - f(t)| = \begin{cases} 0 & \text{if } \ |f_n(t)| > |f(t)|, \\ |f_n(t) - f(t)| & \text{if } \ |f_n(t)| \leq |f(t)|. \end{cases}$$

It follows that $|g_n(t) - f(t)| \leq |f_n(t) - f(t)|$ and consequently the sequence $\{g_n\}_n$ is $\overline{\mu}$-convergent to f.

Now, since $|g_n(t)| \leq |f(t)|$ and $f \in B^{\varphi}a.p.$, using Corollary 1 we deduce that $\lim_{n\to+\infty} \rho_{B^{\varphi}}(g_n) = \rho_{B^{\varphi}}(f)$. Hence,

$$\rho_{B^{\varphi}}(f) = \lim_{n\to+\infty} \rho_{B^{\varphi}}(g_n) \leq \varliminf_{n\to+\infty} \rho_{B^{\varphi}}(f_n). \qquad \square$$

Lemma 8. *Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^{\varphi}a.p.$. Suppose that $\{f_n\}$ is $\overline{\mu}$-convergent to $f \in B^{\varphi}(\mathbb{R})$ and* $\lim_{n\to+\infty} \rho_{B^{\varphi}}(f_n) = \rho_{B^{\varphi}}(f)$. *Then,*

$$\lim_{n\longrightarrow+\infty} \rho_{B^{\varphi}}\left(\frac{f_n - f}{2}\right) = 0.$$

If in addition, $\varphi \in \Delta_2^{B^1}$ then $\lim_{n\longrightarrow+\infty} \|f_n - f\|_{B^{\varphi}} = 0$.

PROOF: In view of Lemma 6, we deduce that $\{\varphi(\cdot, \frac{|f_n - f|}{2})\}_n$ is $\overline{\mu}$-convergent to 0 and consequently the sequence $g_n = \frac{\varphi(\cdot,|f_n|)+\varphi(\cdot,|f|)}{2} - \varphi(\cdot, \frac{|f_n-f|}{2})$ is also $\overline{\mu}$-convergent to $g = \varphi(\cdot, |f|)$. Then, by using Lemma 7, we get that

$$\varliminf_{n\to+\infty} \rho_1(g_n) \geq \rho_1(g).$$

Consequently, in virtue of the existence of the limit in the expression of $\rho_1(\cdot)$, we obtain

$$\begin{aligned} \rho_{\varphi}(f) &= \rho_1(g) \\ &\leq \varliminf_{n\to+\infty} \rho_1\left(\frac{\varphi(|f_n|) + \varphi(|f|)}{2} - \varphi\left(\frac{|f_n - f|}{2}\right)\right) \\ &\leq \varliminf_{n\to+\infty} \left\{\frac{1}{2}\rho_{B^{\varphi}}(f_n) + \frac{1}{2}\rho_{B^{\varphi}}(f) - \rho_{B^{\varphi}}\left(\frac{f_n - f}{2}\right)\right\} \\ &\leq \rho_{B^{\varphi}}(f) - \varlimsup_{n\to+\infty} \rho_{B^{\varphi}}\left(\frac{f_n - f}{2}\right). \end{aligned}$$

Finally, we get $\lim_{n\to+\infty} \rho_{B^{\varphi}}(\frac{f_n - f}{2}) = 0$. $\square$

3. Main results

Theorem 3. *The following properties are equivalent to each other:*

(1) $\widetilde{B}^{\varphi}a.p.$ *is LUC,*
(2) $\widetilde{B}^{\varphi}a.p.$ *has the H-property,*
(3) φ *is strictly convex and* φ *satisfies the* $\Delta_2^{B^1}$*-condition.*

Proof: We will show the following implications: (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Observe that the implication (1) $\Longrightarrow$ (2) holds in general Banach spaces.

To prove (3) $\Longrightarrow$ (1), let f_n, f be in $\widetilde{B}^{\varphi}a.p.$ such that

$$\|f_n\|_{B^{\varphi}} = \|f\|_{B^{\varphi}} = 1 \ \text{ and } \ \left\|\frac{f+f_n}{2}\right\|_{B^{\varphi}} \to 1 \ \text{ as } \ n \to +\infty.$$

Recall that since φ satisfies the $\Delta_2^{B^1}$-condition, we have $B^{\varphi}a.p. = \widetilde{B}^{\varphi}a.p.$ and from Lemma 2, we have $\rho_{B^{\varphi}}(f_n) = \rho_{B^{\varphi}}(f) = 1$. Following analogous arguments to those of [14, Lemma 2], it is possible to show the following assertion:

$$\rho_{B^{\varphi}}\left(\frac{f+f_n}{2}\right) \to 1 \ \text{ as } \ n \to +\infty$$

whenever

$$\left\|\frac{f+f_n}{2}\right\|_{B^{\varphi}} \to 1 \ \text{ as } \ n \to +\infty.$$

Indeed, suppose the assertion is false. Then, there exists $\varepsilon > 0$ such that the following inequalities hold for all $n \geq 1$: $\rho_{B^{\varphi}}(\frac{f+f_n}{2}) \leq 1-\varepsilon$ or $\rho_{B^{\varphi}}(\frac{f+f_n}{2}) \geq 1+\varepsilon$. In both cases, we will obtain a contradiction. In the first case, by using the $\Delta_2^{B^1}$-condition, we get $\sup_n \rho_{B^{\varphi}}(f+f_n) < \infty$, and consequently

$$\begin{aligned}
1 \;&=\; \rho_{B^{\varphi}}\left(\frac{f+f_n}{\|f+f_n\|_{B^{\varphi}}}\right) = \rho_{B^{\varphi}}\left(\left(\frac{2}{\|f+f_n\|_{B^{\varphi}}} - 1\right)(f+f_n)\right. \\
&\quad + \left.\left(2 - \frac{2}{\|f+f_n\|_{B^{\varphi}}}\right)\left(\frac{f+f_n}{2}\right)\right) \\
&\leq \left(\frac{2}{\|f+f_n\|_{B^{\varphi}}} - 1\right)\rho_{B^{\varphi}}\left(f+f_n\right) + \left(2 - \frac{2}{\|f+f_n\|_{B^{\varphi}}}\right)\rho_{B^{\varphi}}\left(\frac{f+f_n}{2}\right) \\
&\leq \left(\frac{2}{\|f+f_n\|_{B^{\varphi}}} - 1\right)\sup_n \rho_{B^{\varphi}}\left(f+f_n\right) + \left(2 - \frac{2}{\|f+f_n\|_{B^{\varphi}}}\right)(1-\varepsilon)\,.
\end{aligned}$$

Passing to the limit for $n \to +\infty$, we obtain $1 \leq 1-\varepsilon$, that is, a contradiction.

If $\rho_{B^\varphi}(\frac{f+f_n}{2}) \geq 1+\varepsilon$, the $\Delta_2^{B^1}$-condition implies that $\sup_n \rho_{B^\varphi}(2\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}) < \infty$, and then

$$\begin{aligned} 1+\varepsilon \quad &\leq \quad \rho_{B^\varphi}\left(\frac{f+f_n}{2}\right) = \rho_{B^\varphi}\left(\left(2-\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}\right)\left(\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right)\right. \\ &\quad + \left.\left(\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}-1\right)\left(2\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right)\right) \\ &\leq \quad \left(2-\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}\right)\rho_{B^\varphi}\left(\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right) \\ &\quad + \left(\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}-1\right)\rho_{B^\varphi}\left(2\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right) \\ &\leq \quad \left(2-\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}\right)+\left(\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}-1\right)\sup_n \rho_{B^\varphi}\left(2\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right). \end{aligned}$$

Letting n tend to infinity, we get $1+\varepsilon \leq 1$, a contradiction. This completes the proof of the previous assertion.

Hence, in view of Lemma 3, it follows that the sequence $\{f_n\}_n$ is $\overline{\mu}$-convergent to f. Then using Lemma 8 and the $\Delta_2^{B^1}$-condition on φ, we conclude that

$$\|f_n - f\|_{B^\varphi} \to 0 \quad \text{as} \quad n \to +\infty.$$

(2) $\Longrightarrow$ (3): Suppose that $\widetilde{B}^\varphi a.p.$ has the H-property. Using Proposition 1 and the same techniques as in [1] (see the proof of Theorem 1) we will show that the Musielak-Orlicz space $L^\varphi([0,1])$ has also the H-property. We repeat this justification for the clarity of the proof. Let $\{f_n\}$ be a sequence in $L^\varphi([0,1])$ such that:

- $\{f_n\}$ converge weakly to some f in $L^\varphi([0,1])$,
- $\|f_n\|_\varphi \longrightarrow \|f\|_\varphi$ (here, the notation $\|\cdot\|_\varphi$ is used to designate the Luxemburg norm associated to the Musielak-Orlicz space $L^\varphi([0,1])$).

Then, for each G in the dual space $(\widetilde{B}^\varphi a.p.)^*$, we have $G \circ i \in (L^\varphi([0,1]))^*$. Moreover, since $f_n \longrightarrow f$ weakly in $L^\varphi([0,1])$, we get

$$G \circ i\,(f_n) \longrightarrow G \circ i(f)$$

or equivalently $G(\widetilde{f_n}) \longrightarrow G(\widetilde{f})$. Thus $\widetilde{f_n} \longrightarrow \widetilde{f}$ weakly in $\widetilde{B}^\varphi a.p.$.

It is clear that $\|\widetilde{f_n}\|_{B^\varphi} \longrightarrow \|\widetilde{f}\|_{B^\varphi}$ and since $\widetilde{B}^\varphi a.p.$ has the H-property, we can write $\|\widetilde{f_n - f}\|_{B^\varphi} \longrightarrow 0$ and finally $\|f_n - f\|_\varphi \longrightarrow 0$. This means that the Musielak-Orlicz space $L^\varphi([0,1])$ has the H-property.

It follows from [11] that φ is strictly convex and satisfies the $\Delta_2^{L^1}$-condition. Since it satisfies also the $\Delta_2^{B^1}$-condition, the proof is finished. □

Theorem 4. *The following properties are equivalent to each other:*

(1) $\widetilde{B}^\varphi a.p.$ *is UCED;*

(2) *φ is strictly convex and φ satisfies the $\Delta_2^{B^1}$-condition.*

Proof: Since $\widetilde{B}^{\varphi}a.p.$ is a pseudonormed space, we will adapt the definition of UCED property to this space as follows: for any $g \in \widetilde{B}^{\varphi}a.p.$, and every sequence (f_n) in $\widetilde{B}^{\varphi}a.p.$, the conditions $\|f_n\| \to 1$, $\|f_n + g\| \to 1$ and $\|2f_n + g\| \to 2$ imply $\|g\| = 0$. Remark that this definition is equivalent to that of UCED property of a normed space.

(2) $\Longrightarrow$ (1): Let $\|f_n\|_{B^\varphi} \to 1$, $\|f_n + g\|_{B^\varphi} \to 1$ and $\|2f_n + g\|_{B^\varphi} \to 2$. Assume that φ is strictly convex and φ satisfies the $\Delta_2^{B^1}$-condition. Then, we have also $\rho_{B^\varphi}(f_n) \to 1$, $\rho_{B^\varphi}(f_n + g) \to 1$ and $\rho_{B^\varphi}(\frac{2f_n+g}{2}) \to 1$. Now, applying Lemma 3 to the sequences $(f_n)_n$ and $(f_n + g)_n$, we get that $g = 0$ $\overline{\mu}$ *a.e.* and in view of Lemma 5 we deduce that $\rho_{B^\varphi}(g) = 0$ and using again the $\Delta_2^{B^1}$-condition it follows that $\|g\|_{B^\varphi} = 0$.

(1) $\Longrightarrow$ (2): Using Proposition 1, and since the UCED property of $\widetilde{B}^{\varphi}a.p.$ implies the UCED property of $L^\varphi([0,1])$, we get the necessity of the strict convexity of φ and the $\Delta_2^{L^1}$-condition (see [7]) and then the necessity of the $\Delta_2^{B^1}$-condition. ☐

Corollary 2. *The following properties are equivalent to each other:*

(1) *$\tilde{B}^{\varphi}a.p.$ is LUC;*
(2) *$\tilde{B}^{\varphi}a.p.$ is MLUC;*
(3) *$\tilde{B}^{\varphi}a.p.$ has the H-property;*
(4) *$\tilde{B}^{\varphi}a.p.$ is UCED;*
(5) *$\tilde{B}^{\varphi}a.p.$ is SC;*
(6) *φ is strictly convex and φ satisfies the $\Delta_2^{B^1}$-condition.*

Now, we apply the previous results to give an application in best approximation.

Let $(X, \|\cdot\|_X)$ be a Banach space, C be a subset of X and $x \in X$. Let us consider the metric projection

$$P_C : x \to d(x, C) = \inf\left\{\|x - y\|_X, y \in C\right\}.$$

In the paper [3], the authors have shown that, under the additional conditions on φ:

$$\forall t \in \mathbb{R}, \quad \lim_{u\to\infty} \frac{\varphi(t,u)}{u} = +\infty, \quad \lim_{u\to 0} \frac{\varphi(t,u)}{u} = 0, \tag{3.1}$$

the space $\widetilde{B}^{\varphi}a.p.$ is reflexive if and only if $\varphi \in \Delta_2^{B^1} \cap \nabla_2^{B^1}$.

Since reflexive strictly convex Besicovitch-Musielak-Orlicz spaces of almost periodic functions are LUC, and so they have the H-property, we get the following corollary which is a generalization of Doob Theorem:

Corollary 3. *Assume that φ is strictly convex, $\varphi \in \Delta_2^{B^1} \cap \nabla_2^{B^1}$ and φ satisfies the conditions* (3.1)*, then for any closed convex sets $C_1 \supset C_2 \supset \cdots \supset C_\infty = \overline{\cap_n C_n}$*

in $\widetilde{B}^{\varphi}a.p.$ *and any* $x \in \widetilde{B}^{\varphi}a.p.$,

$$\|P_{C_n}(x) - P_{C_\infty}(x)\| \to \infty, \quad as \;\; n \to \infty.$$

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