Spaces with large star cardinal number

Yan-Kui Song

Abstract. In this paper, we prove the following statements:

(1) For any cardinal κ, there exists a Tychonoff star-Lindelöf space X such that $a(X) \geq \kappa$.

(2) There is a Tychonoff discretely star-Lindelöf space X such that $aa(X)$ does not exist.

(3) For any cardinal κ, there exists a Tychonoff pseudocompact σ-starcompact space X such that $\mathrm{st}\text{-}l(X) \geq \kappa$.

Keywords: star-Lindelöf number, the Aquaro number, the absolute Aquaro number, star-Lindelöf, centered-Lindelöf, discretely star-Lindelöf, absolutely discretely star-Lindelöf, σ-starcompact, pseudocompact

Classification: 54A25, 54D20

1. Introduction

By a space, we mean a topological space. Recall from [6] that a space X is *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $\mathrm{St}(F,\mathcal{U}) = X$, where $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. It is well-known that starcompactness is equivalent to countably compactness for Hausdorff spaces (see [3], [6]).

A space X is *discretely absolutely star-Lindelöf* (see [12], [13]) if for every open cover $\mathcal{U}$ of X and every dense subset D of X, there exists a countable subset F of D such that F is discrete and closed in X and $\mathrm{St}(F,\mathcal{U}) = X$.

A space X is *star-Lindelöf* (see [1], [2], [3], [4], [6] under different names) (*discretely star-Lindelöf*) (see [11], [15]) if for every open cover $\mathcal{U}$ of X, there exists a countable subset (a countable discrete closed subset, respectively) F of X such that $\mathrm{St}(F,\mathcal{U}) = X$. It is clear that every separable space is star-Lindelöf as well as every space of countable extent (in particular, every countably compact space or every Lindelöf space).

A space X is *centered-Lindelöf* (see [1], [6]) if every open cover $\mathcal{U}$ of X has a σ-centered subcover. A family of sets is *centered* if every finite subfamily has non-empty intersection and a family is *σ-centered* if it can be represented as the union of countably many centered subfamilies.

The author acknowledges the support from NSFC Project 11271036.

A space X is *σ-starcompact* (see [14]) if for every open cover $\mathcal{U}$ of X, there exists a σ-compact subset F of X such that $\mathrm{St}(F,\mathcal{U}) = X$.

From the above definitions, it is not difficult to see that every discretely absolutely star-Lindelöf space is discretely star-Lindelöf, every discretely star-Lindelöf space is star-Lindelöf, every star-Lindelöf space is centered-Lindelöf and every star-Lindelöf space is σ-starcompact.

As natural generalizations of star-Lindelöfness and discretely star-Lindelöfness, one can consider the following cardinal functions:

Definition 1.1 ([1], [6], [7]). The *star-Lindelöf number* of the space X is the cardinal number

$\mathrm{st}\text{-}l(X) = \min\{\kappa :$ for every open cover $\mathcal{U}$ of X, there exists a subset $F \subseteq X$ such that $|F| \leq \kappa$ and $\mathrm{St}(F,\mathcal{U}) = X\}$.

Definition 1.2 ([7]). The *Aquaro number* of the space X is the cardinal number

$a(X) = \min\{\kappa :$ for every open cover $\mathcal{U}$ of X, there exists a discrete closed subset $F \subseteq X$ such that $|F| \leq \kappa$ and $\mathrm{St}(F,\mathcal{U}) = X\}$.

As a natural generalization of discretely absolutely star-Lindelöfness, we can define the following cardinal function:

Definition 1.3. The *absolute Aquaro number* of the space X is the cardinal number

$aa(X) = \min\{\kappa :$ for every open cover $\mathcal{U}$ of X and for every dense subset D of X, there exists a discrete closed subset (in X) $F \subseteq D$ such that $|F| \leq \kappa$ and $\mathrm{St}(F,\mathcal{U}) = X\}$.

It is easily proved that the following inequalities hold for every space X:

$$\mathrm{st}\text{-}l(X) \leq a(X) \leq aa(X).$$

Bonanzinga-Matveev [1] and Matveev [6] asked if the $\mathrm{st}\text{-}l(X)$ of a Tychonoff centered-Lindelöf space X cannot be greater than $\mathfrak{c}$. The author [10] answered negatively the question by giving an example to show that for any cardinal κ there exists a Tychonoff centered-Lindelöf space X such that $\mathrm{st}\text{-}l(X) \geq \kappa$. In [14], the author constructed an example showing that there exists a Tychonoff σ-starcompact space that is not star-Lindelöf. However, the author's space is not pseudocompact and its star-Lindelöf number is not greater than $\mathfrak{c}$. It is natural for us to consider the following questions:

Question 1. Is it true that the Aquaro number of a Tychonoff star-Lindelöf space cannot be greater than $\mathfrak{c}$?

Question 2. Is it true that the absolute Aquaro number of a Tychonoff discretely star-Lindelöf space cannot be greater than $\mathfrak{c}$?

Question 3. Is it true that the star-Lindelöf number of a Tychonoff pseudocompact σ-starcompact space cannot be greater than $\mathfrak{c}$?

The purpose of this paper is to answer negatively the above three questions by showing the three statements stated in the abstract.

The cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal and $\mathfrak{c}$ denote the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each ordinal α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$. Other terms and symbols that we do not define will be used as in [5].

2. Spaces with large star cardinal number

In this section, we show the three statements stated in the abstract. All examples of this section are of the form

$$(X \times \alpha) \cup (Y \times \{\alpha\})$$

where X is a space, Y is a subspace of X and α is an ordinal. The first two examples use Matveev's space. We now sketch the construction of Matveev's space M defined in [8], [9]. Let κ be an infinite cardinal and $D = \{0, 1\}$ be the discrete space. For every $\alpha < \kappa$, let z_α be the point of D^κ defined by $z_\alpha(\alpha) = 1$ and $z_\alpha(\beta) = 0$ for $\beta \neq \alpha$. Put $Z = \{z_\alpha : \alpha < \kappa\}$. For a given ordinal τ, Matveev's space $M(\kappa, \tau)$ is the subspace

$$M(\kappa, \tau) = (D^\kappa \times \tau) \cup (Z \times \{\tau\})$$

of the product space $D^\kappa \times (\tau + 1)$. Then $M(\kappa, \tau)$ is Tychonoff and $Z \times \{\tau\}$ is a discrete closed set of $M(\kappa, \tau)$ with $|Z \times \{\tau\}| = \kappa$.

We need the following lemma:

Lemma 2.1 ([9], [10]). *Assume that there exists a family $\{V_\alpha : \alpha < \kappa\}$ of open sets in D^κ such that $z_\alpha \in V_\alpha$ for each $\alpha < \kappa$. Then there exists a countable set $S \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \emptyset$ for each $\alpha < \kappa$ and $\mathrm{cl}_{D^\kappa} S \cap Z = \emptyset$.*

Theorem 2.2. *For any cardinal κ, there exists a Tychonoff star-Lindelöf space X such that $a(X) \geq \kappa$.*

PROOF: Since for any cardinal κ there is a larger regular uncountable cardinal, we can assume that κ itself is a regular uncountable cardinal. Choose a regular uncountable cardinal τ such that $\tau > \kappa$ and let $X = M(\kappa, \tau)$.

First we show that X is star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of X. For every $\alpha < \kappa$, there exists an $U_\alpha \in \mathcal{U}$ such that $\langle z_\alpha, \tau \rangle \in U_\alpha$. Choose $\beta_\alpha < \tau$ and an open neighborhood V_α of z_α in D^κ such that

$$((V_\alpha \cap Z) \times \{\tau\}) \cup (V_\alpha \times (\beta_\alpha, \tau)) \subseteq U_\alpha.$$

By applying Lemma 2.1 to the family $\{V_\alpha : \alpha < \kappa\}$, then we can find a countable set $S \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Let $\beta' = \sup\{\beta_\alpha : \alpha < \kappa\}$. Then $\beta' < \tau$, since τ is regular and $\tau > \kappa$. Let $F_0 = S \times \{\beta'\}$. Then $Z \times \{\tau\} \subseteq \mathrm{St}(F_0, \mathcal{U})$, since $U_\alpha \cap F_0 \neq \emptyset$ for each $\alpha < \kappa$. On the other hand, since $D^\kappa \times \tau$ is countably compact, we can find a finite subset $F_1 \subseteq D^\kappa \times \tau$ such that $D^\kappa \times \tau \subseteq \mathrm{St}(F_1, \mathcal{U})$. If we put $F = F_0 \cup F_1$, then F is a countable subset of X such that $X = \mathrm{St}(F, \mathcal{U})$, which shows that X is star-Lindelöf.

Next we show that $a(X) \geq \kappa$. We can partition κ as $\kappa = \cup\{A_{n\gamma} : n \in \omega, \gamma < \kappa\}$ such that $|A_{n\gamma}| = n$ for each $n \in \omega$ and $\gamma < \kappa$, $A_{n\gamma} \cap A_{n'\gamma'} = \emptyset$ for $\langle n, \gamma\rangle \neq \langle n', \gamma'\rangle$. For each $\alpha < \kappa$, pick an open neighborhood U_α of $\langle z_\alpha, \tau\rangle$ such that $U_\alpha \cap (Z \times \{\tau\}) = \langle z_\alpha, \tau\rangle$, and $U_{\alpha_1} \cap U_{\alpha_2} = \emptyset$ if $\alpha_1, \alpha_2 \in A_{n\gamma}$ and $\alpha_1 \neq \alpha_2$ for each $n \in \omega$ and $\gamma < \kappa$.

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \cup \{D^\kappa \times \tau\}$$

of the space X. It remains to show that $\mathrm{St}(F, \mathcal{U}) \neq X$ for any discrete closed subset of X with $|F| < \kappa$. To show this, let F be any discrete closed subset of X with $|F| < \kappa$. Let

$$\alpha' = \sup\{\gamma : F \cap \{\langle z_\alpha, \tau\rangle : \alpha \in A_{n\gamma}\} \neq \emptyset \text{ for some } n \in \omega \text{ and some } \gamma < \kappa\}.$$

Then $\alpha' < \kappa$, since κ is regular and $|F| < \kappa$. Thus $F \cap \{\langle z_\alpha, \tau\rangle : \alpha \in A_{n\gamma}\} = \emptyset$ for each $n \in \omega$ and $\gamma > \alpha'$. On the other hand, since $D^\kappa \times \tau$ is countably compact, then $F \cap (D^\kappa \times \tau)$ is finite. Thus we choose $n_0 \in \omega$ and $\gamma_0 > \alpha'$ such that $\{\langle z_\alpha, \tau\rangle : \alpha \in A_{n_0\gamma_0}\} \cap F = \emptyset$. Therefore $\langle z_\alpha, \tau\rangle \notin \mathrm{St}(F, \mathcal{U})$ for each $\alpha \in A_{n_0\gamma_0}$, which shows $a(X) \geq \kappa$. □

For a Tychonoff space X, let βX denote the Čech-Stone compactification of the space X.

Theorem 2.3. *There is a Tychonoff discretely star-Lindelöf space X such that $aa(X)$ does not exist.*

Proof: The author [10] showed that $M(\omega_1, \omega)$ is discretely star-Lindelöf.

Let

$$X = (\beta M(\omega_1, \omega) \times \omega_1) \cup (M(\omega_1, \omega) \times \{\omega_1\})$$

be the subspace of the product space $\beta M(\omega_1, \omega) \times (\omega_1 + 1)$.

First we show that X is discretely star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of X. Since $\beta M(\omega_1, \omega) \times \omega_1$ is countably compact, we can find a finite subset $F_1 \subseteq \beta M(\omega_1, \omega) \times \omega_1$ such that

$$\beta M(\omega_1, \omega) \times \omega_1 \subseteq \mathrm{St}(F_1, \mathcal{U}).$$

On the other hand, $M(\omega_1,\omega)\times\{\omega_1\}$ is discretely star-Lindelöf, since it is homeomorphic to $M(\omega_1,\omega)$. Thus there exists a countable subset $F_2 \subseteq M(\omega_1,\omega)\times\{\omega_1\}$ such that F_2 is discrete closed in $M(\omega_1,\omega)\times\{\omega_1\}$ and

$$M(\omega_1,\omega)\times\{\omega_1\} \subseteq \mathrm{St}(F_2,\mathcal{U}).$$

Since $M(\omega_1,\omega)\times\{\omega_1\}$ is closed in X, then F_2 is closed in X. If we put $F = F_1 \cup F_2$, then F is a countable discrete closed subset of X such that $X = \mathrm{St}(F,\mathcal{U})$, which shows that X is discretely star-Lindelöf.

Next we show that $aa(X)$ does not exist. For each $\alpha<\omega_1$, let $U_\alpha = \{\langle z_\alpha,\omega\rangle\}\cup(D^{\omega_1}\times\omega)$. Since $Z\times\{\omega\}$ is relatively discrete the set U_α is an open neighborhood of $\langle z_\alpha,\omega\rangle$ such that $U_\alpha\cap(Z\times\{\omega\}) = \{\langle z_\alpha,\omega\rangle\}$.

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha\times(\alpha,\omega_1] : \alpha<\omega_1\}\cup\{\beta M(\omega_1,\omega)\times\omega_1\}$$

of the space X and the dense subset $\beta M(\omega_1,\omega)\times\omega_1$ of the space X. It remains to show that $\mathrm{St}(F,\mathcal{U})\neq X$ for any discrete closed subset F of $\beta M(\omega_1,\omega)\times\omega_1$. To show this, let F be any discrete closed subset of $\beta M(\omega_1,\omega)\times\omega_1$. Then F is finite subset of $\beta M(\omega_1,\omega)\times\omega_1$, since $\beta M(\omega_1,\omega)\times\omega_1$ is countably compact. Let $\alpha' = \sup\{\alpha : \alpha\in\pi(F)\}$, where $\pi : \beta M(\omega_1,\omega)\times\omega_1\to\omega_1$ is the projection. Then $\alpha'<\omega_1$, since F is finite. If we pick $\beta>\alpha'$, then $\langle\langle z_\beta,\omega\rangle,\omega_1\rangle\notin\mathrm{St}(F,\mathcal{U})$, since $U_\beta\times(\beta,\omega_1]$ is the only element of $\mathcal{U}$ containing $\langle\langle z_\beta,\omega\rangle,\omega_1\rangle$ and $(U_\beta\times(\beta,\omega_1])\cap F=\emptyset$, which shows that $aa(X)$ does not exist. □

Remark 2.1. The referee asked whether there is a Tychonoff star-Lindelöf space X such that $aa(X)$ does not exist. The author noticed that there is a Tychonoff countably compact (hence, starcompact, star-Lindelöf and discretely star-Lindelöf) space X such that $aa(X)$ does not exist. The construction of the example is very much simpler than the construction of the space X in Theorem 2.3. In fact, let $X=\omega_1\times(\omega_1+1)$ be the product of ω_1 and ω_1+1. Then X is Tychonoff countably compact space. Let us show that $aa(X)$ does not exist. For each $\alpha<\omega_1$, let $U_\alpha=[0,\alpha)\times(\alpha,\omega_1]$. Let us consider the open cover $\mathcal{U}=\{U_\alpha : \alpha<\omega_1\}\cup\{D\}$ and the dense subspace D of X, where $D=\omega_1\times\omega_1$. It remains to show that $\mathrm{St}(F,\mathcal{U})\neq X$ for any discrete closed subset F of D. To show this, let F be any discrete closed subset of D. Then F is finite subset of D, since D is countably compact. Let $\alpha_0=\sup\{\alpha : \alpha\in\pi(F)\}$, where $\pi:\omega_1\times(\omega_1+1)\to\omega_1+1$ is the projection. Then $\alpha_0<\omega_1$, since F is finite. If we pick $\alpha'>\alpha_0$, then $\langle\alpha',\omega_1\rangle\notin\mathrm{St}(F,\mathcal{U})$. Indeed, for every $U_\beta\in\mathcal{U}$, if $\langle\alpha',\omega_1\rangle\in U_\beta$, then $\beta>\alpha'$. Finally, for each $\beta>\alpha'$, $U_\beta\cap F=\emptyset$, which shows that $aa(X)$ does not exist.

Theorem 2.4. *For any cardinal κ, there exists a pseudocompact σ-starcompact Tychonoff space X such that* $\mathrm{st}\text{-}l(X)\geq\kappa$.

Proof: We may assume that κ is a regular uncountable cardinal, as we have done in Theorem 2.2. Let $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete space of the cardinality κ and

$$Y = (\beta D \times \omega) \cup (D \times \{\omega\})$$

be the subspace of the product space $\beta D \times (\omega + 1)$. Then Y is σ-starcompact, since $\beta D \times \omega$ is a σ-compact dense subset of Y.

Let

$$X = (\beta Y \times \kappa) \cup (Y \times \{\kappa\})$$

be the subspace of the product space $\beta Y \times (\kappa + 1)$. Clearly, X is a Tychonoff space. Since κ has uncountable cofinality, then $\beta Y \times \kappa$ is a countably compact dense subset of X, hence X is pseudocompact.

First we show that X is σ-starcompact. To this end, let $\mathcal{U}$ be an open cover of X. Since $\beta Y \times \kappa$ is countably compact, there exists a finite subset F of $\beta Y \times \kappa$ such that

$$\beta Y \times \kappa \subseteq \mathrm{St}(F, \mathcal{U}).$$

On the other hand, $Y \times \{\kappa\}$ is σ-starcompact, since it is homeomorphic to Y. Thus

$$Y \times \{\kappa\} \subseteq \mathrm{St}((\beta D \times \omega) \times \{\kappa\}, \mathcal{U}),$$

since $(\beta D \times \omega) \times \{\kappa\}$ is a σ-compact dense subset of $Y \times \{\kappa\}$. Since $Y \times \{\kappa\}$ is closed in X, then $(\beta D \times \omega) \times \{\kappa\}$ is a σ-compact subset of X. If we put

$$E = F \cup ((\beta D \times \omega) \times \{\kappa\}).$$

Then E is a σ-compact subset of X such that $X = \mathrm{St}(E, \mathcal{U})$, which shows that X is σ-starcompact.

Next we show $\mathrm{st}\text{-}l(X) \geq \kappa$. For each $\alpha < \kappa$, let $U'_\alpha = \{d_\alpha\} \times [0, \omega]$, then U'_α is a compact subset of Y, hence U'_α is a clopen subset of Y and $U'_\alpha \cap U'_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. For each $\alpha < \kappa$, let $U_\alpha = U'_\alpha \times (\kappa + 1)$, then U_α is an open subset of X and $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. For each $n \in \omega$, let $V'_n = \beta D \times \{n\}$, then V'_n is a compact subset of Y, hence V'_n is a clopen subset of Y and $V_n \cap V_m = \emptyset$ for $n \neq m$. For each $n \in \omega$, let $V_n = V'_n \times (\kappa + 1)$, then V_n is an open subset of X. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \cup \{V_n : n \in \omega\} \cup \{\beta Y \times [0, \kappa)\}$$

of X. It remains to show that $\mathrm{St}(F, \mathcal{U}) \neq X$ for any subset F of X with $|F| < \kappa$. To show this, let F be any subset of X with $|F| < \kappa$. Then there exists $\alpha_0 < \kappa$ such that $F \cap U_{\alpha_0} = \emptyset$, since κ is regular and $|F| < \kappa$. Hence $\langle\langle d_{\alpha_0}, \omega\rangle, \kappa\rangle \notin \mathrm{St}(F, \mathcal{U})$, since U_{α_0} is the only element of $\mathcal{U}$ containing $\langle\langle d_{\alpha_0}, \omega\rangle, \kappa\rangle$, which shows $\mathrm{st}\text{-}l(X) \geq \kappa$. □

For normal spaces, it is well-known that countably compactness is equivalent with pseudocompactness, and countably compact space is starcompact. Thus we have the following result.

Theorem 2.5. *For any normal space X, the following conditions are equivalent:*

(1) *X is pseudocompact σ-starcompact;*
(2) *X is star-Lindelöf.*

Remark 2.2. The author does not know if there exists an example of a σ-starcompact normal space that is not star Lindelöf.

Acknowledgments. The author would like to thank Prof. R. Li for his valuable suggestions. He would also like to thank the referee for his/her careful reading of the paper and a number of valuable suggestions which led to improvements on several places. Remark 2.1 is due to his/her suggestion.

References

[1] Bonanzinga M., *Star-Lindelöf and absolutely star-Lindelöf spaces*, Questions Answers Gen. Topology. **14** (1998), 79–104.
[2] Bonanzinga M., Matveev M.V., *Star-Lindelöfness versus centered-Lindelöfness*, Comment. Math. Univ. Carolin. **41** (2002), no. 1, 111–122.
[3] van Douwen E.K., Reed G.M., Roscoe A.W., Tree I.J., *Star covering properties*, Topology Appl. **39** (1991), 71–103.
[4] Dai M.-M., *A class of topological spaces containing Lindelöf spaces and separable spaces*, Chinese Ann. Math. Ser. **A 4** (1983), no. 5, 571–575.
[5] Engelking R., *General Topology. Revised and completed edition*, Heldermann Verlag, Berlin, 1989.
[6] Matveev M.V., *A survey on star covering properties*, Topology Atlas, preprint No. 330 (1998).
[7] Matveev M.V., *A survey on star covering properties II*, Topology Atlas, preprint No. 431 (2000).
[8] Matveev M.V., *How weak is weak extent?*, Topology Appl. **119** (2002), 229–232.
[9] Matveev M.V., *On space in countable web*, arxiv:math/0006199v1 (2000).
[10] Song Y.-K., *Spaces with large extent and large star-Lindelöf number*, Houston J. Math. **29** (2003), no. 2, 347–354.
[11] Song Y.-K., *Discretely star-Lindelöf spaces*, Tsukuba J. Math. **25** (2001), no. 2, 371–382.
[12] Song Y.-K., *Remarks on star-Lindelöf spaces*, Questions Answers Gen. Topology **20** (2002), 49–51.
[13] Song Y.-K., *Closed subsets of absolutely star-Lindelöf spaces II*, Comment. Math. Univ. Carolin. **44** (2003), no. 2, 329–334.
[14] Song Y.-K., *On σ-starcompact spaces*, Applied General. Topology **9** (2008), no. 2, 293–299.
[15] Yasui Y., Gao Z.-M., *Spaces in countable web*, Houston J. Math. **25** (1999), no. 2, 327–335.

Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing 210046, P.R. China

E-mail: songyankui@njnu.edu.cn

(*Received* October 2, 2011, *revised* March 10, 2012)