A poset of topologies on the set of real numbers

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. On the set $\mathbb{R}$ of real numbers we consider a poset $\mathcal{P}_\tau(\mathbb{R})$ (by inclusion) of topologies $\tau(A)$, where $A \subseteq \mathbb{R}$, such that $A_1 \supseteq A_2$ iff $\tau(A_1) \subseteq \tau(A_2)$. The poset has the minimal element $\tau(\mathbb{R})$, the Euclidean topology, and the maximal element $\tau(\emptyset)$, the Sorgenfrey topology. We are interested when two topologies τ_1 and τ_2 (especially, for $\tau_2 = \tau(\emptyset)$) from the poset define homeomorphic spaces $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$. In particular, we prove that for a closed subset A of $\mathbb{R}$ the space $(\mathbb{R}, \tau(A))$ is homeomorphic to the Sorgenfrey line $(\mathbb{R}, \tau(\emptyset))$ iff A is countable. We study also common properties of the spaces $(\mathbb{R}, \tau(A))$, $A \subseteq \mathbb{R}$.

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1. Introduction

The Sorgenfrey line $\mathbb{S}$ (cf. [E]) is the set $\mathbb{R}$ of real numbers with the lower limit topology. The space $\mathbb{S}$ is an important example of topological spaces. Thus it would be nice to be able to identify $\mathbb{S}$ among topological spaces. For example, it is known (cf. [M]) that any non-empty closed subset of $\mathbb{S}$ which is additionally dense in itself is homeomorphic to $\mathbb{S}$, i.e. one gets a topological copy of S by choosing a suitable subspace of $\mathbb{S}$. In this paper we are looking for topological spaces which are homeomorphic to $\mathbb{S}$ by making the lower limit topology on $\mathbb{R}$ coarser.

Let $A \subseteq \mathbb{R}$. Following [H] define the topology $\tau(A)$ on $\mathbb{R}$ as follows:

(1) for each $x \in A$, $\{(x-\epsilon, x+\epsilon) : \epsilon > 0\}$ is the neighborhood base at x,
(2) for each $x \in \mathbb{R} \setminus A$, $\{[x, x+\epsilon) : \epsilon > 0\}$ is the neighborhood base at x.

Let τ_E (resp. τ_S) be Euclidean (resp. the lower limit) topology on $\mathbb{R}$. Note that for any $A, B \subseteq \mathbb{R}$ we have $A \supseteq B$ iff $\tau(A) \subseteq \tau(B)$, in particular $\tau(\mathbb{R}) = \tau_E \subseteq \tau(A), \tau(B) \subseteq \tau(\emptyset) = \tau_S$. Put $\mathcal{P}_{top}(\mathbb{R}) = \{\tau(A) : A \subseteq \mathbb{R}\}$ and define a partial order $\leq$ on $\mathcal{P}_{top}(\mathbb{R})$ by inclusion: $\tau(A) \leq \tau(B)$ iff $\tau(A) \subseteq \tau(B)$.

We continue with the following example.

Example 1.1. Let τ_d be the discrete topology on $\mathbb{R}$. Let also $\{\mathbb{R}_i\}_{i=1}^\infty$ be a sequence of disjoint copies of $\mathbb{R}$ and τ_d^i (resp. τ_E^i or τ_S^i) the corresponding topology on the copy $\mathbb{R}_i, i \geq 1$. Consider the set $X = \oplus_{i=1}^\infty \mathbb{R}_i$ and the topology $\tau_1 = \tau_d^1 \oplus \oplus_{i=2}^\infty \tau_E^i$ (resp. $\tau_2 = \tau_d^1 \oplus \tau_S^2 \oplus \oplus_{i=3}^\infty \tau_E^i$ or $\tau_3 = \tau_d^1 \oplus \tau_d^2 \oplus_{i=3}^\infty \tau_E^i$) on X. Note that $\tau_1 \subset \tau_2 \subset \tau_3$ and $\tau_1 \neq \tau_2$, $\tau_2 \neq \tau_3$. Moreover, the spaces (X_1, τ_1) and (X_3, τ_3) are metrizable and homeomorphic to each other but the space (X_2, τ_2), containing a copy of the Sorgenfrey line as a closed subset, is not metrizable and hence it is homeomorphic neither to (X_1, τ_1) nor (X_3, τ_3).

Taking into account the previous observation it is natural to pose the following general question.

Question 1.1 (see also [H, Question 5.2])**.** For what subsets A, B of $\mathbb{R}$ are the spaces $(\mathbb{R}, \tau(A))$ and $(\mathbb{R}, \tau(B))$ homeomorphic?

In [H] it was observed the following.

(a) If $F \subset \mathbb{R}$ is finite, then the space $(\mathbb{R}, \tau(\mathbb{R} \setminus F))$ is homeomorphic to the topological sum of $|F|$-many copies of the half-open interval $[0, 1)$ and one copy of the open interval $(0, 1)$. Hence, the spaces $(\mathbb{R}, \tau(\mathbb{R} \setminus F_1))$ and $(\mathbb{R}, \tau(\mathbb{R} \setminus F_2))$, where F_1, F_2 are finite subsets of $\mathbb{R}$, are homeomorphic iff $|F_1| = |F_2|$.

(b) If A is a discrete closed subspace of $(\mathbb{R}, \tau_E)$ then $(\mathbb{R}, \tau(A))$ is homeomorphic to $(\mathbb{R}, \tau_S)$ but if a subset A of $\mathbb{R}$ has a non-empty interior in $(\mathbb{R}, \tau_E)$ then $(\mathbb{R}, \tau(A))$ is not homeomorphic to $(\mathbb{R}, \tau_S)$.

In this paper we continue to answer Question 1.1. In particular (see Theorem 2.1), we show that for a set $A \subseteq \mathbb{R}$ which is closed in $(\mathbb{R}, \tau_E)$ the space $(\mathbb{R}, \tau(A))$ is homeomorphic to $(\mathbb{R}, \tau_S)$ iff $|A| \leq \aleph_0$. Then we observe (see Proposition 2.3) that for $B \subseteq \mathbb{R}$ the space $(\mathbb{R}, \tau(B))$ has a countable base iff $|\mathbb{R} \setminus B| \leq \aleph_0$. Moreover, if $\mathbb{R} \setminus B$ is countable and dense in the space $(\mathbb{R}, \tau_E)$ then the space $(\mathbb{R}, \tau(B))$ is additionally zero-dimensional and nowhere locally compact. We study also common properties of the spaces $(\mathbb{R}, \tau), \tau \in \mathcal{P}_{top}(\mathbb{R})$.

For notions and notations we refer to [E].

2. Answers to Question 1.1

Lemma 2.1. *Let $A \subseteq \mathbb{R}$ and $B \subseteq A$ and $C \subseteq \mathbb{R} \setminus A$. Then*

(1) $\tau(A)|_B = \tau_E|_B$, *and*

(2) $\tau(A)|_C = \tau_S|_C$.

Proof: (1). Note that for any $x \in A$ (resp. $x \in \mathbb{R} \setminus A$) and any $\epsilon > 0$ we have $(x - \epsilon, x + \epsilon) \cap B \in \tau_E|_B$ (resp. $[x, x + \epsilon) \cap B = (x, x + \epsilon) \cap B \in \tau_E|_B$). Hence, $\tau(A)|_B \subset \tau_E|_B$. Since $\tau(A) \supseteq \tau_E$, the opposite inclusion is evident.

(2). Note that $\tau(A)|_C \subseteq \tau_S|_C$. Consider $x < y$. If $x \in \mathbb{R} \setminus A$ then $[x, y) \in \tau(A)$ and hence $[x, y) \cap C \in \tau(A)|_C$. If $x \in A$ then $(x, y) \in \tau(A)$ and $[x, y) \cap C = (x, y) \cap C \in \tau(A)|_C$. Hence, $\tau(A)|_C \supseteq \tau_S|_C$. □

Proposition 2.1. *Let $A \subseteq \mathbb{R}$ and A contain an uncountable subset B which is compact in $(\mathbb{R}, \tau_E)$. Then B is compact in $(\mathbb{R}, \tau(A))$ and hence the space $(\mathbb{R}, \tau(A))$ is not homeomorphic to $(\mathbb{R}, \tau_S)$.*

Proof: By Lemma 2.1 we have $\tau(A)|_B = \tau_E|_B$. Hence the space $(B, \tau(A)|_B)$ is compact by the assumption. Recall ([E-J]) that each compact subspace of the Sorgenfrey line $(\mathbb{R}, \tau_S)$ is countable. This implies the statement. □

Let $\mathbb{P}$ be the set of irrational numbers. Recall (cf. [vM]) that a set $A \subseteq \mathbb{R}$ is analytic if the space $(A, \tau_E|_A)$ is a continuous image of $(\mathbb{P}, \tau_E|_{\mathbb{P}})$. In particular, the set $\mathbb{P}$ is analytic as well as any set which is G_δ (for example, closed) in $(\mathbb{R}, \tau_E)$.

The following statement answers in negative [H, Question 5.5].

Corollary 2.1. *Let A be analytic and uncountable. Then $(\mathbb{R}, \tau(A))$ is not homeomorphic to $(\mathbb{R}, \tau_S)$.*

Proof: Let us only remind (cf. [vM]) that A contains an uncountable set which is compact in $(\mathbb{R}, \tau_E)$. □

Let X be a space and X^d be the derived set of X. Recall that for an ordinal number α the Cantor-Bendixson derivative $X^{(\alpha)}$ is defined as follows:

$$X^{(\alpha)} = \begin{cases} X, & \text{if } \alpha = 0, \\ (X^{(\alpha-1)})^d, & \text{if } \alpha \text{ is nonlimit}, \\ \bigcap_{\beta<\alpha} X^{(\beta)}, & \text{if } \alpha \text{ is limit and } \geq \omega_0. \end{cases}$$

Since $X^{(\alpha)} \supseteq X^{(\beta)}$ for $\alpha < \beta$ we have a minimal ordinal α such that $X^{(\alpha)} = X^{(\alpha+1)}$. This ordinal α denoted by $\mathrm{ht}(X)$, is called *the Cantor-Bendixson rank, or the scattered height of X*.

The following statement essentially generalizes the first part of the point (b) from the Introduction.

Proposition 2.2. *Let A be countable and closed in $(\mathbb{R}, \tau_E)$. Then $(\mathbb{R}, \tau(A))$ is homeomorphic to $(\mathbb{R}, \tau_S)$.*

Proof: Let us consider a sequence $\{a_i\}_{i=-\infty}^{\infty}$ of real numbers such that

(a) $a_i < a_{i+1}$,
(b) $\lim_{i\to\infty} a_i = \infty$ and $\lim_{i\to-\infty} a_i = -\infty$,
(c) $a_i \notin A$ for each i.

Note that for each i the set $A_i = [a_i, a_{i+1}] \cap A = [a_i, a_{i+1}) \cap A$ is compact in the space $(\mathbb{R}, \tau_E)$ and the set $[a_i, a_{i+1})$ is clopen in the space $(\mathbb{R}, \tau(A))$. It is enough to show that for each i the space $([a_i, a_{i+1}), \tau(A)|_{[a_i,a_{i+1})}) = ([a_i, a_{i+1}), \tau(A_i)|_{[a_i,a_{i+1})})$ is homeomorphic to $(\mathbb{R}, \tau_S)$. For that we will prove the following statement.

Claim 2.1. Let $[a, b)$ be a half-open non-empty bounded interval of $\mathbb{R}$ and B a countable subset of $[a, b)$ such that $a \notin B$ and B is compact in $(\mathbb{R}, \tau_E)$. Then the space $([a, b), \tau(B)|_{[a,b)})$ is homeomorphic to $(\mathbb{R}, \tau_S)$.

Proof: Let us notice that for each compact countable subspace B of $(\mathbb{R}, \tau_E)$ the Cantor-Bendixson rank $\mathrm{ht}(B)$ is an isolated countable ordinal ≥ 1 and $X^{(\mathrm{ht}(B))} = \emptyset$.

Apply induction on $\mathrm{ht}(B) \geq 1$. If $\mathrm{ht}(B) = 1$ then B is finite. One can easily show that $([a, b), \tau(B)|_{[a,b)})$ is homeomorphic to $(\mathbb{R}, \tau_S)$. But for readers convenience let us suggest a proof by an argument similar to [H, Proposition 4.12 (2)]. At first, we assume that B is a singleton. Let $B = \{c\}$. Put $a_1 = a$, $b_1 = b$ and let $\{a_1, a_2, \dots\}$ be a strictly increasing sequence in (a, c) converging to c, and $\{b_1, b_2, \dots\}$ be a strictly decreasing sequence in (c, b) converging to c. For each $n \geq 1$ we put $A_n = [a_n, a_{n+1})$ and $B_n = [b_{n+1}, b_n)$. Then for each n let $f_n : A_n \to ([\frac{1}{2n}, \frac{1}{2n-1}), \tau_S|_{[\frac{1}{2n}, \frac{1}{2n-1})})$ and $g_n : B_n \to ([\frac{1}{2n+1}, \frac{1}{2n}), \tau_S|_{[\frac{1}{2n+1}, \frac{1}{2n})})$ are homeomorphisms. Now, we can define a mapping $h : ([a, b), \tau(B)|_{[a,b)}) \to ([0, 1), \tau_S|_{[0,1)})$ such as

(i) $h|_{A_n} = f_n$ for each $n = 1, 2, \dots$,
(ii) $h|_{B_n} = g_n$ for each $n = 1, 2, \dots$ and,
(iii) $h(c) = 0$.

It is easy to show that h is a homeomorphism, and $([0, 1), \tau_S|_{[0,1)})$ is homeomorphic to $(\mathbb{R}, \tau_S)$. Hence, $([a, b), \tau(B)|_{[a,b)})$ is homeomorphic to $(\mathbb{R}, \tau_S)$ in this case. Now, we suppose that $B = \{c_1, \dots, c_k\}$ and $k > 1$. We take points $d_1, \dots, d_k, d_{k+1} \in (a, b)$ such that $a = d_1 < c_1 < d_2 < c_2 < \cdots < d_k < c_k < d_{k+1} = b$. Note that $([a, b), \tau(B)|_{[a,b)})$ is the topological sum $\oplus_{i=1}^{k}([d_i, d_{i+1}), \tau(\{c_i\})|_{[d_i,d_{i+1})})$. By the argument above, all spaces of the sum are homeomorphic to $(\mathbb{R}, \tau_S)$. Thus, $([a, b), \tau(B)|_{[a,b)})$ is also homeomorphic to $(\mathbb{R}, \tau_S)$.

Assume now that the statement is valid for all countable ordinals $\leq \alpha$.

Let $\mathrm{ht}(B) = \alpha + 1$. Hence $B^{(\alpha)}$ is finite. As we showed above, it is enough to check the case $|B^{(\alpha)}| = 1$. Let $B^{(\alpha)} = \{c\}$. Then we consider a strictly increasing sequence $\{l_i\}_{i=1}^{\infty}$ and a strictly decreasing sequence $\{r_i\}_{i=1}^{\infty}$ in $[a, b)$ such that $l_1 = a$, $r_1 = b$, $\{l_i\}_{i=1}^{\infty}$ and $\{r_i\}_{i=1}^{\infty}$ converge to c w.r.t. τ_E, and $\{l_i\}_{i=1}^{\infty} \cap B = \{r_i\}_{i=1}^{\infty} \cap B = \emptyset$. Note that for each interval $[l_i, l_{i+1})$ (resp. $[r_{i+1}, r_i)$) the set $B_{l,i} = B \cap [l_i, l_{i+1})$ (resp. $B_{r,i} = B \cap [r_{i+1}, r_i)$) is compact in $(\mathbb{R}, \tau_E)$ and $\mathrm{ht}(B_{l,i}) \leq \alpha$ (resp. $\mathrm{ht}(B_{r,i}) \leq \alpha$). By the inductive assumption the space $([l_i, l_{i+1}), \tau(B)|_{[l_i,l_{i+1})}) = ([l_i, l_{i+1}), \tau(B_{l,i})|_{[l_i,l_{i+1})})$ (resp. $([r_{i+1}, r_i), \tau(B)|_{[r_{i+1},r_i)}) = ([r_{i+1}, r_i), \tau(B_{r,i})|_{[r_{i+1},r_i)})$) is homeomorphic to the space $([l_i, l_{i+1}), \tau_S|_{[l_i,l_{i+1})})$ (resp. $([r_{i+1}, r_i), \tau_S|_{[r_{i+1},r_i)})$) for each i. Then, by a similar argument as above, for the case $|B| = 1$, the space $([a, b), \tau(B)|_{[a,b)})$ is also homeomorphic to $(\mathbb{R}, \tau_S)$. □

Summarizing Corollary 2.1 and Proposition 2.2, we get

Theorem 2.1. *Let A be a closed set in $(\mathbb{R}, \tau_E)$. Then the space $(\mathbb{R}, \tau(A))$ is homeomorphic to $(\mathbb{R}, \tau_S)$ iff $|A| \leq \aleph_0$.*

Question 2.1. Let A be a countable non-closed set in $(\mathbb{R}, \tau_E)$. Is $(\mathbb{R}, \tau(A))$ homeomorphic to $(\mathbb{R}, \tau_S)$?

(Especially, we are interested in the cases when A is dense in the space $(\mathbb{R}, \tau_E)$ and when A has a countable closure in the space $(\mathbb{R}, \tau_E)$.)

Proposition 2.3. *Let $A \subseteq \mathbb{R}$. Then the space $(\mathbb{R}, \tau(A))$ has a countable base iff $|\mathbb{R} \setminus A| \leq \aleph_0$. Moreover, if $\mathbb{R} \setminus A$ is countable and dense in the space $(\mathbb{R}, \tau_E)$ then the space $(\mathbb{R}, \tau(A))$ (in particular, the space $(\mathbb{R}, \tau(\mathbb{P}))$) is additionally zero-dimensional and nowhere locally compact, i.e. no open non-empty subset of $(\mathbb{R}, \tau(A))$ has a compact closure.*

Proof: Sufficiency. Let $|\mathbb{R} \setminus A| \leq \aleph_0$. Consider a countable set $B \subset A$ which is dense in the space $(\mathbb{R}, \tau_E)$. Note that the family $\mathcal{B} = \{[x, x + \frac{1}{n},) : x \in \mathbb{R} \setminus A, n =$

$1, 2, \dots\} \cup \{(x - \frac{1}{n}, x + \frac{1}{n}) : x \in B, n = 1, 2, \dots\}$ is a countable base for the topology $\tau(A)$. Necessity. Let $|\mathbb{R} \setminus A| > \aleph_0$. Note that each uncountable subspace of $(\mathbb{R}, \tau_S)$ has weight $> \aleph_0$. Apply now Lemma 2.1.

Assume now that $\mathbb{R} \setminus A$ is countable and dense in the space $(\mathbb{R}, \tau_E)$. Note that the family $\mathcal{B} = \{[a, b) : a < b; a, b \in \mathbb{R} \setminus A\}$ is a base for the space $(\mathbb{R}, \tau(A))$ consisting of clopen sets. So $\operatorname{ind}(\mathbb{R}, \tau(A)) = 0$. Observe also that for any $a, b \in \mathbb{R} \setminus A$, such that $a < b$, the clopen set $[a, b)$ of the space $(\mathbb{R}, \tau(A))$ can be written as the disjoint union $\oplus_{i=1}^{\infty} [a_i, a_{i+1})$ of clopen sets there, where $a_1 = a < a_2 < \cdots < b$, $\lim_{i \to \infty} a_i = b$ and $a_i \in \mathbb{R} \setminus A$. This implies that no open non-empty subset of $(\mathbb{R}, \tau(A))$ has a compact closure. □

The next statement is evident.

Corollary 2.2. *Let $A, B \subseteq \mathbb{R}$ such that $|\mathbb{R} \setminus A| > \aleph_0$ and $|\mathbb{R} \setminus B| \leq \aleph_0$. Then the space $(\mathbb{R}, \tau(A))$ cannot be embedded into the space $(\mathbb{R}, \tau(B))$ (in particular, the space $(\mathbb{R}, \tau(A))$ is not homeomorphic to the space $(\mathbb{R}, \tau(B))$).*

Remark 2.1. We have the following complement to the previous discussion. Recall (cf. [M]) that a subset of the Sorgenfrey line which is closed and dense in itself (in particular, the Cantor set with the Sorgenfrey topology after the isolated points have been removed) is homeomorphic to the Sorgenfrey line. So if $\mathbb{R} \setminus A$ is analytic and uncountable then the space $(\mathbb{R}, \tau(A))$ (in particular, the space $(\mathbb{R}, \tau(\mathbb{Q}))$, where $\mathbb{Q}$ is the set of rational numbers) contains a copy of the Sorgenfrey line.

Taking into account the point (a) from the Introduction we may ask the following question.

Question 2.2. Let $A \subset \mathbb{R}$ such that $|\mathbb{R} \setminus A| = \aleph_0$. What is the space $(\mathbb{R}, \tau(A))$?

(Especially, we are interested in the cases when the set $\mathbb{R} \setminus A$ is dense in the space $(\mathbb{R}, \tau_E)$ and when $\mathbb{R} \setminus A$ is closed in the space $(\mathbb{R}, \tau_E)$).

3. Common properties of $(\mathbb{R}, \tau(A)), A \subseteq \mathbb{R}$

Let τ_1, τ_2 be topologies on a set X. Following [ChN] we say that the topology τ_2 on X is an admissible extension of τ_1 if

(i) $\tau_1 \subseteq \tau_2$; and
(ii) τ_1 is a π-base for τ_2, i.e. for each non-empty element O of τ_2 there is a non-empty element V of τ_1 which is a subset of O.

Let us denote the closure (resp. the interior) of a subset A of the set X in the space (X, τ_i) by $\mathrm{Cl}_{\tau_i} A$ (resp. $\mathrm{Int}_{\tau_i} A$), where $i = 1, 2$.

Lemma 3.1. *Let X be a set and τ_1, τ_2 topologies on X such that τ_2 is an admissible extension of τ_1.*

(a) *If O is a non-empty element of τ_2 then O is a semi-open set of (X, τ_1), i.e. there is an element V of τ_1 such that $V \subseteq O \subseteq \mathrm{Cl}_{\tau_1} V$ ([L]).*
(b) *If $Y \subseteq X$ then $\mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_1} Y = \emptyset$ iff $\mathrm{Int}_{\tau_2} \mathrm{Cl}_{\tau_2} Y = \emptyset$.*

(c) *If* (X, τ_1) *is a Baire space then the space* (X, τ_2) *is also Baire.* (*Moreover, if the Tychonoff product* $\prod_{\gamma\in\Gamma}(X_\gamma, \tau^1_\gamma)$ *of spaces* $(X_\gamma, \tau^1_\gamma)$, $\gamma \in \Gamma$, *is a Baire space and for each* $\gamma \in \Gamma$ *the topology* τ^2_γ *is an admissible extension of the topology* τ^1_γ *then the Tychonoff product* $\prod_{\alpha\in A}(X_\gamma, \tau^2_\gamma)$ *of spaces* $(X_\gamma, \tau^2_\gamma)$, $\gamma \in \Gamma$, *is also a Baire space.*)

Proof: (a) Put $V = \mathrm{Int}_{\tau_1} O$ and note that $V \neq \emptyset$. We will show that $\mathrm{Cl}_{\tau_1} V \supseteq O$. In fact, assume that $W = O \setminus \mathrm{Cl}_{\tau_1} V \neq \emptyset$. Since τ_2 is an admissible extension of τ_1 then $W \in \tau_2$ and there is $\emptyset \neq U \in \tau_1$ such that $U \subseteq W \subseteq O$. It is easy to see that U must be a subset of V. We have a contradiction which proves the statement.

(b) Put $O_1 = \mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_1} Y$ and $O_2 = O_1 \setminus \mathrm{Cl}_{\tau_2} Y$. Assume that $O_2 \neq \emptyset$ and note that $O_2 \in \tau_2$. Then there is $\emptyset \neq O_3 \in \tau_1$ such that $O_3 \subseteq O_2$. Since $O_3 \subseteq \mathrm{Cl}_{\tau_1} Y$ we have $O_3 \cap Y \neq \emptyset$. This is a contradiction. So $O_1 \subseteq \mathrm{Cl}_{\tau_2} Y$. This implies that $O_1 \subseteq \mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_2} Y \subseteq \mathrm{Int}_{\tau_2} \mathrm{Cl}_{\tau_2} Y$. Hence if $\mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_1} Y \neq \emptyset$ then $\mathrm{Int}_{\tau_2} \mathrm{Cl}_{\tau_2} Y \neq \emptyset$.

Assume now that $O_2 = \mathrm{Int}_{\tau_2} \mathrm{Cl}_{\tau_2} Y \neq \emptyset$. Note that there is $\emptyset \neq O_1 \in \tau_1$ such that $O_1 \subseteq O_2$. Since $O_2 \subseteq \mathrm{Cl}_{\tau_2} Y \subseteq \mathrm{Cl}_{\tau_1} Y$ we have that $O_1 \subseteq \mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_1} Y \neq \emptyset$. The equivalence is proved.

(c) Let $Y = \bigcup_{i=1}^\infty Y_i$, each Y_i be closed in the space (X, τ_2) and $\mathrm{Int}_{\tau_2} Y_i = \emptyset$. By (b) we have that $\mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_1} Y_i = \emptyset$ for each i. Since the space (X, τ_1) is Baire, we have $\mathrm{Int}_{\tau_1}(\bigcup_{i=1}^\infty \mathrm{Cl}_{\tau_1} Y_i) = \emptyset$. Assume that $O_2 = \mathrm{Int}_{\tau_2}(\bigcup_{i=1}^\infty Y_i) \neq \emptyset$. Note that there is $\emptyset \neq O_1 \in \tau_1$ such that $O_1 \subseteq O_2$. Hence, $O_1 \subseteq \mathrm{Int}_{\tau_1}(\bigcup_{i=1}^\infty Y_i) \neq \emptyset$. Since $\emptyset \neq \mathrm{Int}_{\tau_1}(\bigcup_{i=1}^\infty Y_i) \subseteq \mathrm{Int}_{\tau_1}(\bigcup_{i=1}^\infty \mathrm{Cl}_{\tau_1} Y_i) = \emptyset$, we have a contradiction which proves (c). □

Proposition 3.1. *Let* $A \subseteq \mathbb{R}$. *Then*

(a) $\tau(A)$ *is an admissible extension of* τ_E,
(b) *each element of* $\tau(A)$ *is a semi-open set of* $(\mathbb{R}, \tau_E)$,
(c) *the space* $(\mathbb{R}, \tau(A))$ *is regular, hereditarily Lindelöf* (*hence, it is hereditarily paracompact*) *and hereditarily separable,*
(d) *the space* $(\mathbb{R}, \tau(A))$ *is Baire* (*moreover, any Tychonoff product* $\prod_{\gamma\in\Gamma}(\mathbb{R}, \tau(A_\gamma))$ *of spaces* $(\mathbb{R}, \tau(A_\gamma))$, *where* $A_\gamma \subseteq \mathbb{R}$ *and* $\gamma \in \Gamma$, *is a Baire space*).

Proof: (a) is evident. (b) follows from (a) and Lemma 3.1(a). (c) Note that $(\mathbb{R}, \tau(A))$ is evidently regular. Since $\mathbb{R} = A \cup (\mathbb{R} \setminus A)$ and $\tau(A)|_A = \tau_E|_A$, $\tau(A)|_{\mathbb{R}\setminus A} = \tau_S|_{\mathbb{R}\setminus A}$, we have that the space $(\mathbb{R}, \tau(A))$ is hereditarily Lindelöf and hereditarily separable. (d) Since the space $(\mathbb{R}, \tau_E)$ is Baire and the topology $\tau(A)$ is an admissible extension of τ_R, it follows from Lemma 3.1(c) that the space $(\mathbb{R}, \tau(A))$ is also Baire. □

Corollary 3.1. *Let* $A \subseteq \mathbb{R}$ *be such that* $\mathbb{R} \setminus A$ *is countable and dense in the space* $(\mathbb{R}, \tau_E)$. *Then the space* $(\mathbb{R}, \tau(A))$ *is nowhere locally* σ*-compact* (*i.e. no non-empty open set is* σ*-compact*).

Proof: Assume that there is an open non-empty subset O of $(\mathbb{R}, \tau(A))$ which is σ-compact, i.e. $O = \bigcup_{i=1}^\infty K_i$, where every K_i is compact in $(\mathbb{R}, \tau(A))$. Since the subspace O of the space $(\mathbb{R}, \tau(A))$ is Baire, the interior V of some K_i in the space $(\mathbb{R}, \tau(A))$ is non-empty. Recall that V contains the set $[a, b)$ for some points a, b from $\mathbb{R} \setminus A$ which is clopen and noncompact in the space $(\mathbb{R}, \tau(A))$ (see the proof of Proposition 2.3). Since the set $[a, b)$ is closed in the compactum K_i, we have a contradiction. □

It is well known that the real line (in our notations the space $(\mathbb{R}, \tau(\mathbb{R}))$) is topologically complete but the Sorgenfrey line (in our notations the space $(\mathbb{R}, \tau(\emptyset))$) is not topologically complete (cf. [T]).

Question 3.1. For what $A \subseteq \mathbb{R}$ is the space $(\mathbb{R}, \tau(A))$ topologically complete?

(Since the space of irrational numbers in the realm of separable metrizable spaces is the topologically unique non-empty, topologically complete, nowhere locally compact and zero-dimensional space (cf. [vM]), we are especially interested in the case when the set $\mathbb{R} \setminus A$ is dense in the space $(\mathbb{R}, \tau_E)$ and countable.)

Recall ([AL]) that a space is almost complete if it contains a dense topologically complete subspace. Note that if the set $\mathbb{R}\setminus A$ is dense in the real line and countable then the set $\mathbb{R} \setminus A$ (resp. A) with the Sorgenfrey (resp. the real line) topology is homeomorphic to the space of rational (resp. irrational) numbers. Hence the space $(\mathbb{R}, \tau(A))$ contains a dense subset which is homeomorphic to the space of irrational (resp. rational) numbers and so it is almost complete.

We continue with the following examples.

Example 3.1. Let J be an interval on the real line $\mathbb{R}$. Denote by $P(J)$ the set of irrational numbers of J and by $P^Q(J)$ any countable dense subset of $P(J)$. Note that the space $P(J)$ and its subspace $P(J)\setminus P^Q(J)$ are homeomorphic to the space of irrational numbers of the real line, and the space $P^Q(J)$ is homeomorphic to the space of rational numbers of the real line. Moreover, the set $P(J) \setminus P^Q(J)$ is dense in the space $P(J)$.

Let us consider the following subspaces in the real plane $\mathbb{R}^2$

$$X = (P^Q([0,1]) \times \{0\}) \cup \bigcup_{i=0}^{\infty} (\bigcup \{\{\frac{j}{2^i}\} \times P([0, \frac{1}{2^i}]) : j \text{ is odd and } 0 < j < 2^i\}),$$

$$Y = (P^Q([0,1]) \times \{0\}) \cup \bigcup_{i=0}^{\infty} (\bigcup \{\{\frac{j}{2^i}\} \times P^Q([0, \frac{1}{2^i}]) : j \text{ is odd and } 0 < j < 2^i\})$$

and $Z = X \setminus Y$.

Note that the sets Y, Z are dense in X, the space Z (resp. Y) is homeomorphic to the space of irrational (resp. rational) numbers of the real line, and the space X is almost complete, non topologically complete, zero-dimensional and nowhere locally σ-compact.

It is interesting to know what conditions on an almost complete separable metrizable space imply the topological completeness. Let us remind (cf. [CP]) that the Sorgenfrey line is not even almost complete. So one can also ask for what $A \subseteq \mathbb{R}$ the space $(\mathbb{R}, \tau(A))$ is almost complete.

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