The dual space of precompact groups

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. For any topological group G the dual object $\widehat{G}$ is defined as the set of equivalence classes of irreducible unitary representations of G equipped with the Fell topology. If G is compact, $\widehat{G}$ is discrete. In an earlier paper we proved that $\widehat{G}$ is discrete for every metrizable precompact group, i.e. a dense subgroup of a compact metrizable group. We generalize this result to the case when G is an almost metrizable precompact group.

Keywords: compact group, precompact group, representation, Pontryagin–van Kampen duality, compact-open topology, Fell dual space, Fell topology, Kazhdan property (T)

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1. Introduction

For a topological group G let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of G. The set $\widehat{G}$ can be equipped with a natural topology, the so-called Fell topology (see Section 2 for a definition).

A topological group G is *precompact* if it is isomorphic (as a topological group) to a subgroup of a compact group H (we may assume that G is dense in H). If H is compact, then $\widehat{H}$ is discrete. If G is a dense subgroup of H, the natural mapping $\widehat{H} \to \widehat{G}$ is a bijection but in general need not be a homeomorphism. Moreover, for every countable non-metrizable precompact group G the space $\widehat{G}$ is not discrete [12, Theorem 5.1], and every non-metrizable compact group H has a dense subgroup G such that $\widehat{G}$ is not discrete [12, Theorem 5.2]. (The Abelian case was considered in [5, 6, 14]). On the other hand, if G is a precompact metrizable group, then $\widehat{G}$ is discrete [12, Theorem 4.1]. (The Abelian case was considered in [2], [4]). The aim of the present paper is to generalize this result to the almost metrizable case: $\widehat{G}$ is discrete for every almost metrizable precompact topological group G. A topological group G is *almost metrizable* if it has a compact subgroup K such that the quotient space G/K is metrizable. According to Pasynkov's theorem [1, 4.3.20], a topological group is almost metrizable if and only if it is feathered in the sense of Arhangel'skii.

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We reduce the almost metrizable case to the metrizable case considered in [12, Theorem 4.1].

2. Preliminaries: Fell topologies

All topological spaces and groups that we consider are assumed to be Hausdorff. For a (complex) Hilbert space $\mathcal{H}$ the unitary group $U(\mathcal{H})$ of all linear isometries of $\mathcal{H}$ is equipped with the strong operator topology (this is the topology of pointwise convergence). With this topology, $U(\mathcal{H})$ is a topological group.

A *unitary representation* ρ of the topological group G is a continuous homomorphism $G \to U(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space. A closed linear subspace $E \subseteq \mathcal{H}$ is an *invariant* subspace for $\mathcal{S} \subseteq U(\mathcal{H})$ if $ME \subseteq E$ for all $M \in \mathcal{S}$. If there is a closed subspace E with $\{0\} \subsetneq E \subsetneq \mathcal{H}$ which is invariant for $\mathcal{S}$, then $\mathcal{S}$ is called *reducible*; otherwise $\mathcal{S}$ is *irreducible*. An *irreducible representation* of G is a unitary representation ρ such that $\rho(G)$ is irreducible.

If $\mathcal{H} = \mathbb{C}^n$, we identify $U(\mathcal{H})$ with the *unitary group of order* n, that is, the compact Lie group of all complex $n \times n$ matrices M for which $M^{-1} = M^*$. We denote this group by $\mathbb{U}(n)$.

Two unitary representations $\rho : G \to U(\mathcal{H}_1)$ and $\psi : G \to U(\mathcal{H}_2)$ are *equivalent* if there exists a Hilbert space isomorphism $M : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\rho(x) = M^{-1}\psi(x)M$ for all $x \in G$. The *dual object* of a topological group G is the set $\widehat{G}$ of equivalence classes of irreducible unitary representations of G.

If G is a precompact group, the Peter-Weyl Theorem (see [15]) implies that all irreducible unitary representation of G are finite-dimensional and determine an embedding of G into the product of unitary groups $\mathbb{U}(n)$.

If $\rho : G \to U(\mathcal{H})$ is a unitary representation, a complex-valued function f on G is called a *function of positive type* (or *positive-definite function*) *associated with* ρ if there exists a vector $v \in \mathcal{H}$ such that $f(g) = (\rho(g)v, v)$ (here $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}$). We denote by P'_ρ the set of all functions of positive type associated with ρ. Let P_ρ be the convex cone generated by P'_ρ, that is, the set of sums of elements of P'_ρ.

Let G be a topological group, $\mathcal{R}$ a set of equivalence classes of unitary representations of G. The *Fell topology* on $\mathcal{R}$ is defined as follows: a typical neighborhood of $[\rho] \in \mathcal{R}$ has the form

$$W(f_1, \cdots, f_n, C, \epsilon) = \{[\sigma] \in \mathcal{R} : \exists g_1, \cdots, g_n \in P_\sigma \ \forall x \in C \ |f_i(x) - g_i(x)| < \epsilon\},$$

where $f_1, \cdots, f_n \in P_\rho$ (or $\in P'_\rho$), C is a compact subspace of G, and $\epsilon > 0$. In particular, the Fell topology is defined on the dual object $\widehat{G}$. If G is locally compact, the Fell topology on $\widehat{G}$ can be derived from the Jacobson topology on the primitive ideal space of $C^*(G)$, the C^*-algebra of G [7, Section 18], [3, Remark F.4.5].

Every onto homomorphism $f : G \to H$ of topological groups gives rise to a continuous injective dual map $\hat{f} : \widehat{H} \to \widehat{G}$. A mapping $h : X \to Y$ between topological spaces is *compact-covering* if for every compact set $L \subset Y$ there exists a compact set $K \subset X$ such that $h(K) = L$.

Lemma 2.1. *If $f : G \to H$ is a compact-covering onto homomorphism of topological groups, the dual map $\hat{f} : \widehat{H} \to \widehat{G}$ is a homeomorphic embedding.*

PROOF: This easily follows from the definition of Fell topology. □

Let π be a unitary representation of a topological group G on a Hilbert space $\mathcal{H}$. Let $F \subseteq G$ and $\epsilon > 0$. A unit vector $v \in \mathcal{H}$ is called (F, ϵ)-*invariant* if $\|\pi(g)v - v\| < \epsilon$ for every $g \in F$.

A topological group G has *property* (T) if and only if there exists a pair (Q, ϵ) (called a *Kazhdan pair*), where Q is a compact subset of G and $\epsilon > 0$, such that for every unitary representation ρ having a unit (Q, ϵ)-invariant vector there exists a non-zero invariant vector. Equivalently, G has property (T) if and only if the trivial representation 1_G is isolated in $\mathcal{R} \cup \{1_G\}$ for every set $\mathcal{R}$ of equivalence classes of unitary representations of G without non-zero invariant vectors [3, Proposition 1.2.3].

Compact groups have property (T) [3, Proposition 1.1.5], but countable Abelian precompact groups do not have property (T) [12, Theorem 6.1].

We refer to Fell's papers [9], [10], the classical text by Dixmier [7] and the recent monographs by de la Harpe and Valette [13], and Bekka, de la Harpe and Valette [3] for basic definitions and results concerning Fell topologies and property (T).

3. Almost metrizable groups

If A is a subset of a topological space X, the *character* $\chi(A, X)$ of A in X is the least cardinality of a base of neighborhoods of A in X. (If this definition leads to a finite value of $\chi(A, X)$, we replace it by ω, the first infinite cardinal, and similarly for other cardinal invariants.) If A is a closed subset of a compact space X, the character $\chi(A, X)$ equals the *pseudocharacter* $\psi(A, X)$ — the least cardinality of a family γ of open subsets of X such that $\bigcap \gamma = A$. In particular, if A is a closed G_δ-subset of a compact space X, then $\chi(A, X) = \omega$.

If K is a compact subgroup of a topological group, then G/K is metrizable if and only if $\chi(K, G) = \omega$ [1, Lemma 4.3.19]. Let G be an almost metrizable topological group, $\mathcal{K}$ the collection of all compact subgroups $K \subset G$ such that $\chi(K, G) = \omega$. Then for every neighborhood O of the neutral element there is $K \in \mathcal{K}$ such that $K \subset O$ [1, Proposition 4.3.11]. We now show that if G is additionally ω-narrow, then K can be chosen normal (in the algebraic sense). Recall that a topological group G is ω-*narrow* [1] if for every neighborhood U of the neutral element there exists a countable set $A \subset G$ such that $AU = G$.

Lemma 3.1. *Let G be an ω-narrow almost metrizable group, $\mathcal{N}$ the collection of all normal (= invariant under inner automorphisms) compact subgroups K of G such that the quotient group G/K is metrizable* (*equivalently, $\chi(K, G) = \omega$*).

Then for every neighborhood O of the neutral element there exists $K \in \mathcal{N}$ such that $K \subset O$.

Proof: Let $L \subset O$ be a compact subgroup of G such that the quotient space $G/L = \{xL : x \in G\}$ is metrizable. It suffices to prove that $K = \bigcap\{gLg^{-1} : g \in G\}$, the largest normal subgroup of G contained in L, belongs to $\mathcal{N}$.

There exists a compatible metric on G/L which is invariant under the action of G by left translations. To construct such a metric, consider a countable base $U_1, U_2, \dots$ of neighborhoods of L in G. We may assume that for each n we have $U_n = U_n^{-1} = U_n L$ and $U_{n+1}^2 \subset U_n$. Let $\gamma_n = \{gU_n : g \in G\}$. The open cover γ_n of G is invariant under left G-translations and under right L-translations, and γ_{n+1} is a barycentric refinement of γ_n. The pseudometric on G that can be constructed in a canonical way from the sequence (γ_n) of open covers (see [8, Theorem 8.1.10]) gives rise to a compatible G-invariant metric on G/L. A similar construction was used in [1, Lemma 4.3.19].

If an ω-narrow group transitively acts on a metric space X by isometries, then X is separable [1, 10.3.2]. Thus $X = G/L$ is separable. Let Y be a dense countable subset of X. Then $K = \{g \in G : gx = x \text{ for every } x \in X\} = \{g \in G : gx = x \text{ for every } x \in Y\}$ is a G_δ-subset of L, hence $\chi(K, L) = \omega$. It follows that $\chi(K, G) \le \chi(K, L)\chi(L, G) = \omega$ ([8, Exercise 3.1.E]). □

4. Main theorem

Theorem 4.1. *If G is a precompact almost metrizable group, then $\widehat{G}$ is discrete.*

Proof: Let ρ be an irreducible unitary representation of G. We must prove that $[\rho]$ is isolated in $\widehat{G}$. It suffices to find a discrete open subset $D \subset \widehat{G}$ such that $[\rho] \in D$.

Precompact groups are ω-narrow, so Lemma 3.1 applies to G. Let $\mathcal{N}$, as above, be the collection of all normal compact subgroups $K \subset G$ such that $\chi(K, G) = \omega$. Then $\mathcal{N}$ is closed under countable intersections, and it follows from Lemma 3.1 that for every G_δ-subset A of G containing the neutral element there exists $K \in \mathcal{N}$ such that $K \subset A$. In particular, there exists $K \in \mathcal{N}$ such that K lies in the kernel of ρ. Let $D \subset \widehat{G}$ be the set of all classes $[\sigma] \in \widehat{G}$ such that K is contained in the kernel of σ. Then $[\rho] \in D$. It suffices to verify that D is open and discrete.

Step 1. We verify that D is open. Let $\mathcal{R}$ be the set of equivalence classes of all finite-dimensional unitary representations (which may be reducible) of K without non-zero invariant vectors. Let τ_n be the trivial n-dimensional representation $1_K \oplus \cdots \oplus 1_K$ (n summands) of K, $n = 1, 2 \dots$. In the notation of Section 2, P_{τ_n} does not depend on n and is the set of non-negative constant functions on K. It follows that in the space $\mathcal{S} = \mathcal{R} \cup \{[\tau_n] : n = 1, 2, \dots\}$, equipped with the Fell topology, the points $[\tau_n]$ are indistinguishable: any open set containing one of these points contains all the others. Since K has property (T), $[\tau_1] = [1_K]$ is not in the closure of $\mathcal{R}$. Therefore $\mathcal{R}$ is closed in $\mathcal{S}$ and $\mathcal{S} \setminus \mathcal{R}$ is open in $\mathcal{S}$.

We claim that for every irreducible unitary representation σ of G the class of the restriction $\sigma|_K$ belongs to $\mathcal{S}$. In other words, the claim is that $\sigma|_K$ is trivial if it admits a non-zero invariant vector. Let V be the (finite-dimensional) space of the representation σ. For $g \in G$ and $x \in V$ we write gx instead of $\sigma(g)x$. The space $V' = \{x \in V : gx = x \text{ for all } g \in K\}$ of all K-invariant vectors is G-invariant. Indeed, if $x \in V'$, $g \in G$ and $h \in K$, then $g^{-1}hgx = x$ because $g^{-1}hg \in K$ and x is K-invariant. It follows that $hgx = gx$ which proves that $gx \in V'$. Since σ is irreducible, either $V' = \{0\}$ or $V' = V$. Accordingly, either $\sigma|_K$ admits no non-zero invariant vectors or else is trivial.

We have just proved that the restriction map $r : \widehat{G} \to \mathcal{S}$ is well-defined. Clearly r is continuous, and therefore $D = r^{-1}(\mathcal{S} \setminus \mathcal{R})$ is open in $\widehat{G}$.

Step 2. We verify that D is discrete. Let $p : G \to G/K$ be the quotient map. Then D is the image of the dual map $\hat{p} : \widehat{G/K} \to \widehat{G}$. According to [12, Theorem 4.1], the dual space of a metrizable precompact group is discrete. Thus $\widehat{G/K}$ is discrete. Since p is a perfect map, it is compact-covering, and Lemma 2.1 implies that $\hat{p} : \widehat{G/K} \to \widehat{G}$ is a homeomorphic embedding. Therefore, $D = \hat{p}(\widehat{G/K})$ is discrete. $\square$

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