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The sup = max problem for the extent of generalized metric spaces

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. It looks not useful to study the sup = max problem for extent, because there are simple examples refuting the condition. On the other hand, the sup = max problem for Lindelöf degree does not occur at a glance, because Lindelöf degree is usually defined by not supremum but minimum. Nevertheless, in this paper, we discuss the sup = max problem for the extent of generalized metric spaces by combining the sup = max problem for the Lindelöf degree of these spaces.

Keywords: extent, Lindelöf degree, Σ-space, strict p-space, semi-stratifiable

Classification: Primary 54A25, 54D20; Secondary 03E10, 54E18

1. Introduction

Let φ be a cardinal function and X a space. Some cardinal functions are defined in terms of

$$\varphi(X) = \sup\{|S| : S \subset X \text{ has a property } \mathcal{P}_\varphi\} + \omega.$$

The sup = max *problem for* φ is the one when $\varphi(X) = |S|$ holds for some $S \subset X$ having the property $\mathcal{P}_\varphi$. Whenever we deal with the sup = max problem for φ, note that $\varphi(X)$ should be a limit cardinal. Otherwise, this problem becomes trivial.

As a typical cardinal function for the sup = max problem, let us recall the *spread* $s(X)$ of a space X which is defined by

$$s(X) = \sup\{|D| : D \text{ is a discrete subset in } X\} + \omega.$$

First, Hajnal-Juhász [5] proved that for a Hausdorff space X with $|X| \geq \kappa$, if κ is a singular strong limit cardinal, then there is a discrete subset of size κ in X. Moreover, they also proved the following.

Theorem 1.1 (Hajnal-Juhász [6]). *Let κ be a singular cardinal with* $\mathrm{cf}(\kappa) = \omega$. *If X is a regular T_1-space with $s(X) = \kappa$, then there is a discrete subset of size κ in X.*

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The case of κ being a singular cardinal with $\mathrm{cf}(\kappa) = \omega$ seemed to be specially interesting. In fact, Roitman [14] proved that there is consistently a zero-dimensional regular T_1-space X with $s(X) = \omega_{\omega_1}$ and with no discrete subset of size ω_{ω_1} in X. And it had been naturally asked whether Theorem 1.1 holds for a Hausdorff space X. A complete answer to this problem was given by the following.

Theorem 1.2 (Kunen-Roitman [11])**.** *Let κ be a singular cardinal with $\mathrm{cf}(\kappa) = \omega$. Then there is a Hausdorff space X with $s(X) = \kappa$ and with no discrete subset of size κ if and only if there is a set $S \subset 2^\omega$ of size κ such that every subset of S of size κ is not meager.*

Thus, the sup = max problem for spread seemed to be settled before 1980. The reader might find the details of the sup = max problem in the books [9], [10]. In particular, the details of Theorems 1.1 and 1.2 are found in [10, Chapter 4].

Now, let us recall that the *extent* $e(X)$ of a space X is defined by

$$e(X) = \sup\{|D| : D \text{ is a closed discrete subset in } X\} + \omega.$$

Obviously, we have $e(X) \leq s(X)$. Since the definition of extent looks similar to that of spread, it is natural to consider the sup = max problem for extent. However, it looks vain as seen from Example 2.1 below. Due to this kind of examples, the sup = max problem for extent seems to have been never dealt with so far. Nevertheless, the situation is changed when we restrict the extent to a generalized metric space such as a Σ-space, a strict p-space or a semi-stratifiable space. Our results depend on the topological structure of a space rather than the cardinal condition of extent. In fact, for a cardinal κ, we only assume $\mathrm{cf}(\kappa) > \omega$ instead of $\mathrm{cf}(\kappa) = \omega$.

Next, let us recall that the *Lindelöf degree* $L(X)$ of a space X is defined by

$$L(X) = \min\{\kappa : \text{ every open cover of } X \text{ has a subcover of cardinality } \leq \kappa\} + \omega.$$

Then $e(X) \leq L(X)$ holds. Since Lindelöf degree is usually defined not by supremum but minimum as above, the sup = max problem for it does not seem to occur at a glance. In order to consider the sup = max problem for extent, we introduce a new type of the sup = max problem for Lindelöf degree. Indeed, the sup=max problem for Lindelöf degree is rather easier to deal with than that of extent in some cases.

Throughout this paper, all spaces are assumed to be *Hausdorff*, and κ and τ denote uncountable cardinals. For a cardinal κ, $\mathrm{cf}(\kappa)$ denotes the cofinality of κ, and the spaces κ and $\kappa + 1$ mean the spaces $[0, \kappa)$ and $[0, \kappa]$ with the usual order topology, respectively.

2. A simple example and a motivation

The following simple example seems to be a reason why the sup = max problem for extent has been never discussed so far.

Example 2.1. For every limit cardinal κ, there is a space X_κ with one non-isolated point such that $e(X_\kappa) = |X_\kappa| = \kappa$, but there is no closed discrete subset of size κ in X_κ. If $\mathrm{cf}(\kappa) = \omega$, the space X_κ is metrizable.

Proof: Let κ be a limit cardinal. Take the subspace $X_\kappa = \{\alpha+1 : \alpha \in \kappa\} \cup \{\kappa\}$ of $\kappa + 1$. Then X_κ has the only one non-isolated point κ with $|X_\kappa| = \kappa$.

Since $\{\alpha + 1 \in X_\kappa : \alpha < \beta\}$ is a closed discrete subset in X_κ for each $\beta \in \kappa$, we have $e(X_\kappa) = \kappa$. Let D be a closed discrete subset in X_κ. Take an open neighborhood U_0 of κ in X_κ with $|U_0 \cap D| \le 1$. Take a $\beta_0 \in \kappa$ with $X_\kappa \cap (\beta_0, \kappa] \subset U_0$. Since $|D \setminus U_0| \le |\beta_0| < \kappa$, we have $|D| < \kappa$.

When $\mathrm{cf}(\kappa) = \omega$, let $\{\tau_n\}$ be a sequence of cardinals with $\tau_n < \tau_{n+1}$ for each $n \in \omega$ and $\kappa = \sup_{n\in\omega} \tau_n$. Let $\mathcal{B}_n = \{\{\alpha+1\} : \alpha < \tau_n\}$ and $\mathcal{B}_{\kappa,n} = \{X_\kappa \cap (\tau_n, \kappa]\}$ for each $n \in \omega$. Then $\bigcup_{n\in\omega}(\mathcal{B}_n \cup \mathcal{B}_{\kappa,n})$ is a σ-discrete base of X_κ. Hence X_κ is metrizable. □

Every metrizable space M has a σ-discrete base $\mathcal{B}$ with $|\mathcal{B}| = w(M)$, where $w(M)$ denotes the weight of M. It is well known that for a metrizable space M, we have $e(M) = s(M) = w(M) = \kappa$. So adding the assumption of $\mathrm{cf}(\kappa) > \omega$, the following is easy to see.

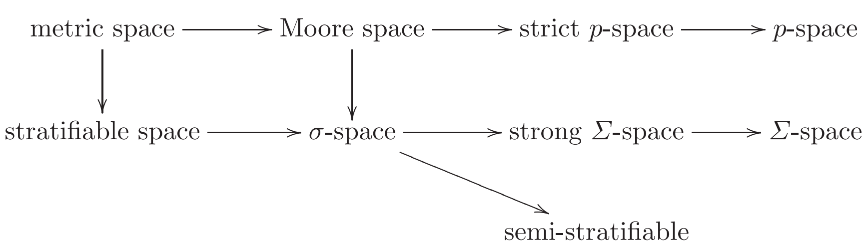
Proposition 2.2. *Let M be a metrizable space with $e(M) = \kappa$. Assume $\mathrm{cf}(\kappa) > \omega$. Then there is a closed discrete subset of size κ in M.*

In view of Example 2.1 and Proposition 2.2, it is natural to ask

Problem 0. Let X be a generalized metric space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. When is there a closed discrete subset of size κ in X?

This problem is a motivation of this paper, and we will give a couple of affirmative answers to this one. As Gruenhage gave a nice survey [4] for generalized metric spaces, we sometimes quote it.

For the reader's convenience, we state the following implications for generalized metric spaces which will be dealt with.



3. Σ-spaces

First, for the reader's convenience, we show

Fact 3.1 (folklore)**.** *Let $\mathcal{A}$ be an infinite collection of non-empty subsets in a space X. If $\mathcal{A}$ is locally finite in X, then there is a closed discrete subset D of size $|\mathcal{A}|$.*

Proof: Let $\kappa = |\mathcal{A}| \geq \omega$. We can inductively construct a subcollection $\{A_\alpha : \alpha \in \kappa\}$ of $\mathcal{A}$ and a sequence $\{x_\alpha : \alpha \in \kappa\}$ of points in X, satisfying that $x_\alpha \in A_\alpha$ and $\{x_\beta : \beta < \alpha\} \cap A_\alpha = \emptyset$ for each $\alpha \in \kappa$. Then $D := \{x_\alpha : \alpha \in \kappa\}$ is a closed discrete subset of X with $|D| = \kappa$. □

Let X be a space and $\mathcal{K}$ a closed cover of X. A closed cover $\mathcal{F}$ of X is a *(mod $\mathcal{K}$)-network* for X if, whenever $K \in \mathcal{K}$ and U is open in X with $K \subset U$, there is $F \in \mathcal{F}$ with $K \subset F \subset U$ (see [13]). A space X is a *(strong) Σ-space* [12] if there is a σ-locally finite (mod $\mathcal{K}$)-network for some closed cover $\mathcal{K}$ of X by countably compact (compact) sets (cf. [4, 4.13 Definition]).

Theorem 3.2. *If X is a Σ-space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, then there is a closed discrete subset of size κ in X.*

Proof: Let $\mathcal{K}$ be a closed cover of X by countably compact sets and $\mathcal{F}$ a σ-locally finite (mod $\mathcal{K}$)-network for X. First, we show $|\mathcal{F}| \geq \kappa$. Let D be any closed discrete subset in X. Let $\mathcal{F}_D = \{F \in \mathcal{F} : |F \cap D| < \omega\}$. Pick an $x \in D$. Take a $K_x \in \mathcal{K}$ with $x \in K_x$. Since K_x is countably compact, we have $|K_x \cap D| < \omega$. Let $U = X \setminus (D \setminus K_x)$. Then U is an open set in X with $K_x \subset U$. There is an $F_0 \in \mathcal{F}$ with $K_x \subset F_0 \subset U$. Then we have $K_x \cap D = F_0 \cap D$. We conclude that $F_0 \in \mathcal{F}_D$ and $x \in K_x \subset F_0$. Thus $\mathcal{F}_D$ covers D. This means that

$$|D| = |\bigcup\{F \cap D : F \in \mathcal{F}_D\}| \leq |\mathcal{F}_D| \cdot \omega \leq |\mathcal{F}| \cdot \omega.$$

Hence $\kappa = e(X) \leq |\mathcal{F}| \cdot \omega$ holds. By $\kappa > \omega$, we obtain $|\mathcal{F}| \geq \kappa = e(X)$.

Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, where each $\mathcal{F}_n$ is locally finite in X. By $\mathrm{cf}(\kappa) > \omega$, there is $m \in \omega$ with $|\mathcal{F}_m| \geq \kappa$. It follows from Fact 3.1 that there is a closed discrete subset D^* in X with $|D^*| = |\mathcal{F}_m| \geq \kappa$. By $e(X) = \kappa$, $|D^*|$ must be equal to κ. □

4. The sup = max problem for Lindelöf degree

Since Lindelöf degree is usually defined by minimum, the sup = max problem does not seem to occur. However, using another expression, Lindelöf degree can be defined by supremum.

For a collection $\mathcal{U}$ of open sets in a space X, let

$$L(\mathcal{U}) = \min\{|\mathcal{V}| : \mathcal{V} \subset \mathcal{U} \text{ with } \bigcup\mathcal{V} = \bigcup\mathcal{U}\} + \omega.$$

First, we have to check the following basic fact.

Fact 4.1. *For a space X, $L(X) = \sup\{L(\mathcal{U}) : \mathcal{U}$ is an open cover of $X\}$ holds.*

Proof: Let $\kappa = \sup\{L(\mathcal{U}) : \mathcal{U}$ is an open cover of $X\}$. Take any open cover $\mathcal{U}$ of X. Since there is a subcover $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \leq L(X)$, we have $L(\mathcal{U}) \leq L(X)$. Hence $\kappa \leq L(X)$ holds. Take any open cover $\mathcal{U}$ of X again. By $L(\mathcal{U}) \leq \kappa$, there is a subcover $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \leq \kappa$. Hence $L(X) \leq \kappa$ holds. □

Thus, we can consider the sup = max problem for Lindelöf degree as the problem when there is an open cover $\mathcal{U}$ of a space X with $L(X) = L(\mathcal{U})$.

As a trivial case, for a Lindelöf and non-compact space X, the sup = max problems for $L(X)$ are affirmative. On the other hand, taking the space X_κ as in Example 2.1, it is easily seen that $L(X_\kappa) = \kappa$ but $L(\mathcal{U}) < \kappa$ for any open cover $\mathcal{U}$ of X_κ.

A space X is *submetalindelöf* (or *$\delta\theta$-refinable*) if for every open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{V}_n\}$ of open refinements satisfying that for each $x \in X$ one can choose $n_x \in \omega$ such that $\mathcal{V}_{n_x}$ is point-countable at x. Related to this property, we have the following implications:

$$\begin{array}{ccccc}
\text{paracompact} & \longrightarrow & \text{metacompact} & \longrightarrow & \text{metalindelöf} \\
\downarrow & & \downarrow & & \downarrow \\
\text{subparacompact} & \longrightarrow & \text{submetacompact} & \longrightarrow & \text{submetalindelöf}
\end{array}$$

Lemma 4.2 (Aull). *If X is a submetalindelöf space, then $e(X) = L(X)$ holds.*

This can be shown by an easy modification of the proof of [1, Theorem 1].

For a space X, $A \subset X$ and a collection $\mathcal{U}$ of subsets in X, we let

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

In particular, we use $\mathrm{St}(x,\mathcal{U})$ instead of $\mathrm{St}(\{x\},\mathcal{U})$, where $x \in X$.

In the proof of Lemma 4.2 (or [1, Theorem 1]), the following result is used.

Lemma 4.3 ([1, Lemmas 1 and 3]). *Let X be a space, $A \subset X$ and $\mathcal{U}$ be an open cover of X. Then there is a closed discrete subset D in X such that $D \subset A \subset \mathrm{St}(D,\mathcal{U})$.*

Fact 4.4 (folklore). *Let X be a space with $e(X) = L(X) = \kappa$. If there is a closed discrete subset D of size κ in X, then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*

Proof: Let $\mathcal{U} = \{X \setminus (D \setminus \{d\}) : d \in D\}$. Since $\mathcal{U}$ has no proper subcover of X, it is an open cover of X with $L(\mathcal{U}) = |\mathcal{U}| = |D| = \kappa$. □

Theorem 4.5. *Let X be a submetalindelöf space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Then there is a closed discrete subset D of size κ in X if and only if there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = L(X) = \kappa$.*

Proof: First, by Lemma 4.2, note that $L(X) = e(X) = \kappa$ holds. It suffices by Fact 4.4 to show the "if" part. Let $\mathcal{U}$ be an open cover of X with $L(\mathcal{U}) = \kappa$. Since X is submetalindelöf, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ satisfying that for each $x \in X$ one can choose $n_x \in \omega$ such that $\mathcal{V}_{n_x}$ is point-countable at x. For each $n \in \omega$, let

$$X_n = \{x \in X : \mathcal{V}_n \text{ is point-countable at } x\}.$$

Then we have $X = \bigcup_{n\in\omega} X_n$. Pick an $n \in \omega$. By Lemma 4.3, there is a closed discrete subset D_n in X with $D_n \subset X_n \subset \bigcup \mathrm{St}(D_n, \mathcal{V}_n)$. Let $\mathcal{W}_n = \{V \in \mathcal{V}_n : V \cap D_n \neq \emptyset\}$. By the choice of X_n, we have $|\mathcal{W}_n| \le |D_n| \cdot \omega \le \kappa$.

Now, assume that $|D_n| < \kappa$ for each $n \in \omega$. Let $\tau = \sup_{n\in\omega} |D_n| \cdot \omega$. By $\mathrm{cf}(\kappa) > \omega$, we have $\tau < \kappa$. Let $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{W}_n$. Since $X_n \subset \mathrm{St}(D_n, \mathcal{V}_n) = \bigcup \mathcal{W}_n$ for each $n \in \omega$ and $X = \bigcup_{n\in\omega} X_n$, $\mathcal{W}$ covers X. So $\mathcal{W}$ is an open refinement of $\mathcal{U}$. On the other hand, since

$$|\mathcal{W}| = \sup_{n\in\omega} |\mathcal{W}_n| \le \sup_{n\in\omega} |D_n| \cdot \omega = \tau,$$

we conclude that $L(\mathcal{U}) \le \tau < \kappa = L(X)$. This contradicts $L(\mathcal{U}) = \kappa$. Hence we obtain $|D_m| = \kappa = e(X)$ for some $m \in \omega$. □

Since a regular strong Σ-space is subparacompact, it is submetalindelöf. So the following is an immediate consequence of Lemma 4.2, Theorems 3.2 and 4.5.

Corollary 4.6. *If a regular space X is a strong Σ-space with $L(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*

Remark 4.7. Since every space with one non-isolated point is paracompact, it follows from Example 2.1 that the sup = max problems for $L(X)$ and $e(X)$ are both negative for a submetalindelöf space X without any additional condition.

However, we do not know the following.

Problem 1. Assume that a space X has a point-countable base with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Is there a closed discrete subset of size κ in X?

5. Strict p-spaces

A Tychonoff space X is called a *p-space* (respectively, *strict p-space*) if there is a sequence $\{\mathcal{O}_n\}$ of collections of open sets in βX such that each $\mathcal{O}_n$ covers X and $\bigcap_{n\in\omega} \mathrm{St}(x, \mathcal{O}_n) \subset X$ (respectively, $\bigcap_{n\in\omega} \mathrm{St}(x, \mathcal{O}_n) = \bigcap_{n\in\omega} \overline{\mathrm{St}(x, \mathcal{O}_n)}^{\beta X} \subset X$) for each $x \in X$ (cf. [4, 3.15 Definition]).

Here we make use of the following characterization of p-spaces by Burke [2] (cf. [4, 3.21 Theorem]) instead of the definition.

Lemma 5.1. *A Tychonoff space X is a p-space if and only if there is a sequence $\{\mathcal{G}_n\}$ of open covers of X satisfying the following condition: If $G_n \in \mathcal{G}_n$ for each $n \in \omega$ with $\bigcap_{n\in\omega} G_n \neq \emptyset$, then*

(i) *$\bigcap_{n\in\omega} \overline{G_n}$ is compact, and*
(ii) *every open set U in X containing $\bigcap_{n\in\omega} \overline{G_n}$ contains some $\bigcap_{i\le m} \overline{G_i}$.*

Lemma 5.2. *Let X be a space and $\mathcal{K}$ a closed cover of X by compact sets. If $\mathcal{F}$ is a (mod $\mathcal{K}$)-network for X, then $L(X) \le |\mathcal{F}| \cdot \omega$ holds.*

Proof: Take any open cover $\mathcal{U}$ of X. Let

$$\mathcal{F}^* = \{F \in \mathcal{F} : \text{ there is a finite } \mathcal{W} \subset \mathcal{U} \text{ with } F \subset \bigcup \mathcal{W}\}.$$

For each $F \in \mathcal{F}^*$, one can assign a finite subcollection $\mathcal{V}(F)$ of $\mathcal{U}$ which covers F. Let $\mathcal{V} = \bigcup\{\mathcal{V}(F) : F \in \mathcal{F}^*\}$. Then we have $|\mathcal{V}| = |\mathcal{F}^*| \cdot \omega \leq |\mathcal{F}| \cdot \omega$. To show $L(X) \leq |\mathcal{F}| \cdot \omega$, it suffices to show that $\mathcal{V}$ covers X. Pick an $x \in X$. Take $K_x \in \mathcal{K}$ with $x \in K_x$. Since K_x is compact, there is a finite $\mathcal{W} \subset \mathcal{U}$ which covers K_x. Then there is $F_0 \in \mathcal{F}$ with $K_x \subset F_0 \subset \bigcup \mathcal{W}$. So we have $F_0 \in \mathcal{F}^*$. It follows that $x \in K_x \subset F_0 \subset \bigcup \mathcal{V}(F_0) \subset \bigcup \mathcal{V}$. Hence $\mathcal{V}$ is a subcover of $\mathcal{U}$. □

Lemma 5.3. *Let X be a p-space with $L(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*

Proof: Assume that $L(\mathcal{U}) < \kappa$ for any open cover $\mathcal{U}$ of X. There is a sequence $\{\mathcal{G}_n\}$ of open covers of X, described in Lemma 5.1. For each $n \in \omega$, letting $\tau_n = L(\mathcal{G}_n)$, there is a subcover $\mathcal{H}_n$ of $\mathcal{G}_n$ with $|\mathcal{H}_n| = \tau_n$. Let $\tau = \sup_{n\in\omega} \tau_n$. Since $\tau_n < \kappa$ for each $n \in \omega$ and $\mathrm{cf}(\kappa) > \omega$, we have $\tau < \kappa$. Let

$$\mathcal{F} = \{\bigcap_{i\leq n} \overline{H_i} : H_i \in \mathcal{H}_i, i \leq n \text{ and } n \in \omega\}.$$

Since $|\bigcup_{n\in\omega} \mathcal{H}_n| \leq \tau$, note that $|\mathcal{F}| \leq \tau$. Let

$$\mathcal{K} = \{\bigcap_{n\in\omega} \overline{H_n} : H_n \in \mathcal{H}_n \text{ for each } n \in \omega \text{ with } \bigcap_{n\in\omega} H_n \neq \emptyset\}.$$

Since each $\mathcal{H}_n$ covers X, it follows from Lemma 5.1(i) that $\mathcal{K}$ is a closed cover of X by compact sets. Take any $K \in \mathcal{K}$ and any open set U in X with $K \subset U$. Then there is a sequence $\{H_n\}$ of open sets in X such that $K = \bigcap_{n\in\omega} \overline{H_n}$, where $H_n \in \mathcal{H}_n$ with $\bigcap_{n\in\omega} H_n \neq \emptyset$. By Lemma 5.1(ii), there is $m \in \omega$ with $\bigcap_{i\leq m} \overline{H_i} \subset U$. Then we have $\bigcap_{i\leq m} \overline{H_i} \in \mathcal{F}$ such that $K \subset \bigcap_{i\leq m} \overline{H_i} \subset U$. Thus $\mathcal{F}$ is a (mod $\mathcal{K}$)-network for X. It follows from Lemma 5.2 and $\kappa > \omega$ that $\kappa = L(X) \leq |\mathcal{F}| \leq \tau < \kappa$ holds. This is a contradiction. □

Theorem 5.4. *If X is a strict p-space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, then there is a closed discrete subset of size κ in X.*

Proof: It follows from Jiang's result [7] that every strict p-space is submetacompact. Since X is submetalindelöf, it follows from Lemma 4.2 that $e(X) = L(X) = \kappa$ holds. Since X is p-space and $\mathrm{cf}(\kappa) > \omega$, it follows from Lemma 5.3 that there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$. It follows from Theorem 4.5 that there is a closed discrete subset D in X with $|D| = \kappa$. □

In view of Lemma 5.3 and Theorem 5.4, it is natural to ask

Problem 2. Let X be a p-space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Is there a closed discrete subset of size κ in X?

Since locally compact spaces are p-spaces, Lemma 5.3 is true for a locally compact space X. However, this is somewhat generalized in what follows. A space X is *locally Lindelöf* if each point of X has an open neighborhood whose closure is Lindelöf.

Proposition 5.5. *If X is a locally Lindelöf non-compact space with $L(X) = \kappa$, then there is an open cover $\mathcal{G}$ of X with $L(\mathcal{G}) = \kappa$.*

Proof: Since the case of X being Lindelöf is obvious, we may let $\kappa > \omega$. Let $\mathcal{U}$ be any open cover of X. Take an open cover $\mathcal{G}$ of X such that $\overline{G}$ is Lindelöf for each $G \in \mathcal{G}$. Assume that $L(\mathcal{G}) < \kappa$. There is a subcover $\mathcal{H}$ of $\mathcal{G}$ with $|\mathcal{H}| = L(\mathcal{G})$. For each $G \in \mathcal{H}$, there is a countable subcollection $\mathcal{V}(G)$ of $\mathcal{U}$ covering $\overline{G}$. Let $\mathcal{V} = \bigcup\{\mathcal{V}(G) : G \in \mathcal{H}\}$. Then $\mathcal{V}$ is a subcover of $\mathcal{U}$ with $|\mathcal{V}| \leq |\mathcal{H}| = L(\mathcal{G}) < \kappa$. This implies that $L(X) \leq L(\mathcal{G}) < \kappa = L(X)$, which is a contradiction. Hence we obtain $L(\mathcal{G}) = \kappa$. □

6. Semi-stratifiable spaces

A space X is *semi-stratifiable* [3] if there is a function $g : \omega \times X \to \mathrm{Top}(X)$, where $\mathrm{Top}(X)$ denotes the topology of X, satisfying

(i) $\bigcap_{n\in\omega} g(n,x) = \{x\}$ for each $x \in X$,

(ii) $y \in \bigcap_{n\in\omega} g(n,x_n)$ implies that $\{x_n\}$ converges to y

(see also [4, 5.6 Definition]).

For a space X, $d(X)$ denotes the *density* of X, that is,

$$d(X) = \min\{|S| : S \text{ is a dense subset in } X\}.$$

Lemma 6.1 (Creed)**.** *If X is a semi-stratifiable space, then $d(X) \leq L(X)$ holds.*

This was actually showed in the proof of (1)⇒(2) of [3, Theorem 2.8]. Moreover, the following is obtained by a modification of the proof.

Lemma 6.2. *Let X be a semi-stratifiable space with $L(X) = d(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*

Proof: Assume that $L(\mathcal{U}) < \kappa$ for any open cover $\mathcal{U}$ of X. Let $g : \omega \times X \to \mathrm{Top}(X)$ be a function described as above. Let $\mathcal{G}_n = \{g(n,x) : x \in X\}$ for each $n \in \omega$. Pick an $n \in \omega$. Let $\tau_n = L(\mathcal{G}_n)$. Since $\mathcal{G}_n$ is an open cover of X, we have $\tau_n < \kappa$. There is a subcover $\mathcal{H}_n$ of $\mathcal{G}_n$ with $|\mathcal{H}_n| = \tau_n$. Let $\mathcal{H}_n = \{g(n,x) : x \in T_n\}$, where $T_n \subset X$ with $|T_n| = \tau_n$. Let $\tau = \sup_{n\in\omega} \tau_n$. By $\mathrm{cf}(\kappa) > \omega$, we have $\tau < \kappa$. Let $T = \bigcup_{n\in\omega} T_n$. Pick an $x \in X$. For each $n \in \omega$, take $x_n \in T_n$ with $x \in g(n,x_n)$. By the choice of g, $\{x_n\}$ converges to x. Hence T is a dense subset in X with $|T| = \tau$. We conclude that $d(X) \leq |T| = \tau < \kappa = L(X)$. This contradicts the assumption. □

A space X is *metalindelöf* if every open cover of X has a point-countable open refinement. The following is easily seen.

Fact 6.3. *If X is a metalindelöf space, then $L(X) \leq d(X)$ holds.*

A space X is *collectionwise Hausdorff* if for every closed discrete subset D in X, there is a mutually disjoint collection $\{U_x : x \in D\}$ of open sets such that $x \in U_x$ for each $x \in D$.

Fact 6.4. *If X is a collectionwise Hausdorff space, then $e(X) \leq d(X)$ holds.*

Now, we obtain a main result in this section.

Theorem 6.5. *Let X be a semi-stratifiable space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. If X is either metalindelöf or collectionwise Hausdorff, then there is a closed discrete subset of size κ in X.*

PROOF: Since every semi-stratifiable space is subparacompact (cf. [4, 5.11 Theorem]), Lemma 4.2 assures that $e(X) = L(X) = \kappa$ holds. Moreover, it follows from Lemmas 6.1, Facts 6.3 and 6.4 that $e(X) = L(X) = d(X) = \kappa$ holds. Hence our conclusion follows from Theorem 4.5 and Lemma 6.2. □

This immediately yields

Corollary 6.6. *If X is a paracompact, semi-stratifiable space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, then there is a closed discrete subset of size κ in X.*

The following is well known as Jones' Lemma.

Lemma 6.7 ([8])**.** *If X is a normal space, then $2^{|D|} \leq 2^{d(X)}$ holds for every closed discrete subset D in X.*

Lemma 6.8. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$ such that $\{2^\tau : \tau$ is a cardinal $< \kappa\}$ has no maximum. If X is a normal space with $e(X) = \kappa$, then $e(X) \leq d(X)$ holds.*

PROOF: Assume that $d(X) < \kappa$ holds. Then we have $2^{d(X)} < 2^\kappa$. By the assumption of κ, there is a cardinal $\rho < \kappa$ with $2^{d(X)} < 2^\rho < 2^\kappa$. Take a closed discrete subset D in X with $\rho < |D| < \kappa$. Then we have $2^{d(X)} < 2^\rho \leq 2^{|D|}$, which contradicts Jones' Lemma above. □

Using Lemma 6.8 instead of Facts 6.3 and 6.4, the following is obtained analogously as Theorem 6.5.

Proposition 6.9. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$ such that $\{2^\tau : \tau$ is a cardinal $< \kappa\}$ has no maximum. If X is a normal, semi-stratifiable space with $e(X) = \kappa$, then there is a closed discrete subset of size κ in X.*

For a strong limit cardinal κ (i.e., $2^\tau < \kappa$ whenever $\tau < \kappa$), note that $\{2^\tau : \tau$ is a cardinal $< \kappa\}$ has no maximum.

Problem 3. Let X be a normal, semi-stratifiable space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Is there a closed discrete subset of size κ in X without such an assumption of κ as above?

Remark 6.10. As stated in [4, Theorem 7.8(i)], Σ-spaces, strict p-spaces and semi-stratifiable spaces are all β-spaces. However, the sup = max equality does not hold for the extent of β-spaces, because each of the space X_κ in Example 2.1 is a paracompact β-space.

7. Subspaces of a cardinal

In general, $e(X)$ cannot bound $L(X)$. In fact, for every countably compact non-compact space X, $\omega = e(X) < \omega_1 \leq L(X)$ holds. In particular, if $X = \kappa$ for a cardinal κ with $\mathrm{cf}(\kappa) > \omega$, then X is a locally compact space with $e(X) = \omega$ and $L(X) = \mathrm{cf}(\kappa)$. Moreover, we have the following result.

Theorem 7.1. *Let κ and τ be any cardinals with $\kappa \geq \tau \geq \omega$. Then there is a subspace X of κ such that $L(X) = \kappa$ and $e(X) = \tau$.*

Proof: Let $X = \kappa \setminus (R \cup L)$, where R is the set of all regular cardinals with $< \kappa$ and L is the set of all limit ordinals with $\leq \tau$. Obviously, $L(X) \leq |X| \leq \kappa$ holds. Let D be a closed discrete subset in X. Let $D \setminus \tau = \{\alpha_\xi : \xi \in \mu\}$ and $\alpha_\zeta < \alpha_\xi$ for every $\zeta < \xi < \mu$. Then, $\mu \leq \omega$ holds. Actually, assume that $\mu \geq \omega$. Taking $\{\alpha_n : n \in \omega\} \subset D \setminus \tau$, let $\beta = \sup\{\alpha_n : n \in \omega\}$. Then we have $\beta \notin X$ since D is closed discrete, and $\beta \notin R \cup L$ since $\mathrm{cf}(\beta) = \omega \leq \tau \leq \alpha_0 < \alpha_1 \leq \beta$. Therefore, $\beta = \kappa$ holds. If $\mu > \omega$, we have $\alpha_\omega \geq \beta = \kappa$, which contradicts $\alpha_\omega \in D \setminus \tau \subset \kappa$. So we obtain $\mu \leq \omega$. It follows from $|D \cap \tau| \leq \tau$ and $|D \setminus \tau| \leq \mu \leq \omega$ that $|D| \leq \tau$ holds. Hence we have $e(X) \leq \tau$.

Let λ be a regular cardinal. If $\lambda \leq \tau$, then $D_\lambda = \{\alpha + 1 : \alpha \in \lambda\}$ is a closed discrete subset of X, so $\lambda = |D_\lambda| \leq e(X)$. Therefore $\tau \leq e(X)$ holds. If $\lambda \leq \kappa$, then $\mathcal{U}_\lambda = \{X \cap [0, \alpha] : \alpha < \lambda\} \cup \{(\lambda, \kappa)\}$ is an open cover of X, so we have $\lambda = L(\mathcal{U}_\lambda) \leq L(X)$. Therefore $\kappa \leq L(X)$ holds. Thus, we conclude that $e(X) = \tau$ and $L(X) = \kappa$. □

Next, we construct a space X with $L(X) = e(X)$ such that Theorem 4.5 does not hold. Of course, such a space X must not be submetalindelöf. A typical example of non-submetalindelöf spaces is a stationary subspace of κ with $\mathrm{cf}(\kappa) > \omega$ (it is easily checked by the Pressing Down Lemma). So we try to find such a space in the class of subspaces of a cardinal κ.

For a subset S of κ, we denote by $\mathrm{Lim}(S)$ the set of all limit points of S in κ, that is, $\mathrm{Lim}(S) = \{\alpha \in \kappa : \alpha = \sup(S \cap \alpha)\}$.

Theorem 7.2. *Let κ be a regular limit cardinal $> \omega$. Then there is a subspace X of κ, satisfying the following;*

(i) $L(X) = e(X) = \kappa$,
(ii) *there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$ and*
(iii) *there is no closed discrete subset of size κ in X.*

Proof: Define a subspace X of κ by putting

$$X = \kappa \setminus \bigcup\{(\lambda, \lambda + \lambda] \cap \mathrm{Lim}(\kappa) : \lambda \text{ is a cardinal with } \lambda < \kappa\}.$$

Obviously, $e(X) \leq L(X) \leq |X| \leq \kappa$ holds. For each infinite cardinal $\lambda < \kappa$,

$$D_\lambda := \{\lambda + \alpha + 1 : \alpha \in \lambda\} \subset X \cap (\lambda, \lambda + \lambda)$$

is a closed discrete subset in X, and so we have $\lambda = |D_\lambda| \le e(X)$. Therefore $\kappa \le e(X)$ holds, thus $L(X) = e(X) = \kappa$. Since κ is regular, $\mathcal{U} := \{X \cap [0, \alpha] : \alpha \in \kappa\}$ is an open cover of X with $L(\mathcal{U}) = \kappa$. Let Z be a subset in X with $|Z| = \kappa$. Then Z is unbounded in κ. Take a sequence $\{\lambda_n : n \in \omega\}$ of infinite cardinals with $< \kappa$ and a sequence $\{\zeta_n : n \in \omega\}$ of members of Z inductively such that $\lambda_n \le \zeta_n < \lambda_{n+1}$ for each $n \in \omega$. Let $\lambda = \sup\{\lambda_n : n \in \omega\} (= \sup\{\zeta_n : n \in \omega\})$. Since λ is a cardinal and X contains all cardinals less than κ, we have $\lambda \in X \cap \mathrm{Lim}(Z)$. So Z is not a closed discrete subset in X. Hence, there is no closed discrete subset of size κ in X. □

Theorem 7.3. *Let κ be a singular limit cardinal. Then there is a subspace X of $\kappa + \kappa$ satisfying the following;*

(i) $L(X) = e(X) = \kappa$,
(ii) *there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$ and*
(iii) *there is no closed discrete subset of size κ in X.*

However, there is no subspace of κ satisfying these three conditions.

Proof: Take a strictly increasing sequence $\{\kappa_\xi : \xi \in \mathrm{cf}(\kappa) + 1\}$ in $\kappa + 1$ with $\kappa_{\mathrm{cf}(\kappa)} = \kappa$ such that for each $\xi \le \mathrm{cf}(\kappa)$, we have

- if ξ is not a limit ordinal, then κ_ξ is a regular uncountable cardinal,
- if ξ is a limit ordinal, then $\kappa_\xi = \sup\{\kappa_\eta : \eta \in \xi\}$.

We define a subspace X of $\kappa + \kappa$ by putting

$$X = (\kappa + \kappa) \setminus \big((\kappa \cap \mathrm{Lim}(\kappa)) \cup \{\kappa + \kappa_\xi : \xi \in \mathrm{cf}(\kappa)\}\big).$$

Obviously, $e(X) \le L(X) \le |X| \le |\kappa + \kappa| = \kappa$ holds. For each infinite cardinal $\lambda < \kappa$, $D_\lambda := \{\alpha + 1 : \alpha \in \lambda\} \subset X \cap \lambda$ is a closed discrete subset in X, and so we have $\lambda = |D_\lambda| \le e(X)$. Therefore $\kappa \le e(X)$ holds, thus $L(X) = e(X) = \kappa$.

Put $X_\xi = (\kappa + \kappa_\xi, \kappa + \kappa_{\xi+1})$ for each $\xi \in \mathrm{cf}(\kappa)$, $X_{-1} = (\kappa, \kappa + \kappa_0)$, and $X_{-2} = (\kappa \setminus \mathrm{Lim}(\kappa)) \cup \{\kappa\}$. Then we have $X = \bigoplus_{-2 \le \xi < \mathrm{cf}(\kappa)} X_\xi$. For each $\xi \in \mathrm{cf}(\kappa)$, let

$$\mathcal{U}_\xi := \{(\kappa + \kappa_\xi, \kappa + \kappa_\xi + \alpha] : 0 < \alpha < \kappa_{\xi+1}\}.$$

Then each $\mathcal{U}_\xi$ is an open cover of X_ξ with $L(\mathcal{U}_\xi) = \kappa_{\xi+1}$, since $\kappa_{\xi+1}$ is a regular cardinal. Therefore $\mathcal{U} := \{X_{-2}, X_{-1}\} \cup \bigcup_{\xi \in \mathrm{cf}(\kappa)} \mathcal{U}_\xi$ is an open cover of X with $L(\mathcal{U}) = \kappa$.

Let D be a closed discrete subset in X. For $\kappa \in X$, $D \cap \kappa$ is bounded in κ, and so $|D \cap X_{-2}| < \kappa$. Obviously, $|D \cap X_{-1}| \le \kappa_0 < \kappa$ holds. Pick a $\xi \in \mathrm{cf}(\kappa)$. Note that X_ξ is homeomorphic to $\kappa_{\xi+1} \setminus \kappa_\xi$. Since $\kappa_{\xi+1}$ is countably compact, so is X_ξ. Since X_ξ is clopen in X, $D \cap X_\xi$ must be finite. Hence we have

$$\begin{aligned} |D| &= |(D \cap X_{-2}) \cup (D \cap X_{-1}) \cup \bigcup_{\xi \in \mathrm{cf}(\kappa)} (D \cap X_\xi)| \\ &\le \max\{|D \cap X_{-2}|, \kappa_0, \mathrm{cf}(\kappa)\} < \kappa. \end{aligned}$$

Next, let us assume that there is a subspace X of κ with $e(X)=\kappa$. If $\mathrm{cf}(\kappa)>\omega$ and X is stationary in κ, then (ii) fails. Actually, if $\mathcal{U}$ is an open cover of X, then by the Pressing Down Lemma, there is $\gamma<\mathrm{cf}(\kappa)$ such that $\{X\cap(\kappa_\gamma,\kappa_\xi]:\xi\in(\gamma,\mathrm{cf}(\kappa))\}$ partially refines $\mathcal{U}$, so $L(\mathcal{U})\le\max\{\kappa_\gamma,\mathrm{cf}(\kappa)\}<\kappa$. If $\mathrm{cf}(\kappa)=\omega$ or X is non-stationary in κ with $\mathrm{cf}(\kappa)>\omega$, then (iii) fails. To see this, take an unbounded subset C of κ such that $X\cap\mathrm{Lim}(C)=\emptyset$. By induction, we can take a strictly increasing sequence $\{c(\xi):\xi\in\mathrm{cf}(\kappa)\}$ by members of C and a sequence $\{D_\xi:\xi\in\mathrm{cf}(\kappa)\}$ of closed discrete subsets in X such that $D_\xi\subset(c(\xi),c(\xi+1))$ and $|D_\xi|\ge\kappa_\xi$ for each $\xi\in\mathrm{cf}(\kappa)$. Actually, if $c(\xi)\in C$ is taken for $\xi\in\kappa$, then by $e(X)=\kappa$, we can take a closed discrete subset D'_ξ in X such that $|D'_\xi|=\max\{|c(\xi)|,\kappa_\xi,\mathrm{cf}(\kappa)\}^+<\kappa$. By $D'_\xi=\bigcup_{\zeta\in\mathrm{cf}(\kappa)}(D'_\xi\cap\kappa_\zeta)$ and $\mathrm{cf}(\kappa)<|D'_\xi|=\mathrm{cf}(|D'_\xi|)$, there is $D''_\xi\subset D'_\xi$ which is bounded in κ and $|D''_\xi|=|D'_\xi|$. Take $c(\xi+1)\in C$ with $D''_\xi\subset c(\xi+1)$ and let $D_\xi=D''_\xi\cap(c(\xi),c(\xi+1))$. By $D''_\xi=(D''_\xi\cap[0,c(\xi)])\cup D_\xi$ and $|D''_\xi\cap[0,c(\xi)]|\le\max\{|c(\xi)|,\omega\}<|D'_\xi|=|D''_\xi|$, we have $|D_\xi|=|D'_\xi|\ge\kappa_\xi$. So we can take the required sequences $\{c(\xi):\xi\in\mathrm{cf}(\kappa)\}$ and $\{D_\xi:\xi\in\mathrm{cf}(\kappa)\}$. Let $D=\bigcup_{\xi\in\mathrm{cf}(\kappa)}D_\xi$. For $X\cap\mathrm{Lim}(C)=\emptyset$, $\{X\cap(c(\xi),c(\xi+1)):\xi\in\mathrm{cf}(\kappa)\}$ is discrete in X. So $\{D_\xi:\xi\in\mathrm{cf}(\kappa)\}$ is also discrete in X, hence D is a closed discrete subset in X. For each $\xi\in\mathrm{cf}(\kappa)$, we have $\kappa_\xi\le|D_\xi|\le|D|$. Hence $|D|=\kappa$ holds, and so (iii) fails. □

As stated in Corollary 4.6, the sup = max equality holds for the Lindelöf degree of strong Σ-spaces. However, the following result shows that the sup = max equality does not hold for the Lindelöf degree of Σ-spaces.

Proposition 7.4. *Let κ be a limit cardinal. Then there is a countably compact subspace X of $\kappa+1$ with $L(X)=\kappa$ such that $L(\mathcal{U})<\kappa$ for any open cover $\mathcal{U}$ of X.*

Proof: Let $X=(\kappa+1)\setminus\{\xi\in\kappa:\mathrm{cf}(\xi)>\omega\}$. Since X contains $\{\xi\in\mathrm{Lim}(X):\mathrm{cf}(\xi)=\omega\}$, it is countably compact. Pick a regular uncountable cardinal $\lambda<\kappa$. Letting $\mathcal{U}_\lambda=\{X\setminus(\alpha,\lambda):\alpha\in\lambda\}$, it is an open cover of X. Moreover, we have $L(\mathcal{U}_\lambda)=\lambda\le L(X)$. By $L(X)\le|X|=\kappa$, we obtain $L(X)=\kappa$.

Let $\mathcal{U}$ be an open cover of X. For each $\alpha\in X$, take $U_\alpha\in\mathcal{U}$ with $\alpha\in U_\alpha$. By $\kappa\in U_\kappa$, there is $\gamma\in\kappa$ with $X\cap(\gamma,\kappa]\subset U_\kappa$. Then $\mathcal{V}:=\{U_\alpha:\alpha\in(X\cap[0,\gamma])\cup\{\kappa\}\}$ is a subcover of $\mathcal{U}$ with $|\mathcal{V}|\le|[0,\gamma]\cup\{\kappa\}|<\kappa$. Hence we have $L(\mathcal{U})<\kappa$. □

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