Compacta are maximally G_δ-resolvable

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. It is well-known that compacta (i.e. compact Hausdorff spaces) are maximally resolvable, that is every compactum X contains $\Delta(X)$ many pairwise disjoint dense subsets, where $\Delta(X)$ denotes the minimum size of a non-empty open set in X. The aim of this note is to prove the following analogous result: Every compactum X contains $\Delta_\delta(X)$ many pairwise disjoint G_δ-dense subsets, where $\Delta_\delta(X)$ denotes the minimum size of a non-empty G_δ set in X.

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It is well-known that compacta (i.e. compact Hausdorff spaces) are maximally resolvable, that is, every compactum X contains $\Delta(X)$ many pairwise disjoint dense subsets, where $\Delta(X)$ denotes the minimum size of a non-empty open set in X. The aim of this note is to prove the following analogous result: Every compactum X contains $\Delta_\delta(X)$ many pairwise disjoint G_δ-dense subsets, where $\Delta_\delta(X)$ denotes the minimum size of a non-empty G_δ set in X. Of course, a subset of X is called G_δ-dense iff it intersects every non-empty G_δ set in X. Clearly, this is equivalent with the statement that the G_δ-modification X_δ of X is maximally resolvable in the usual sense, where X_δ carries, on the underlying set of X, the topology generated by all G_δ subsets of X.

The proof of this result is based on the following lemma that may be of independent interest. In proving it we shall make use of the following two easy facts concerning the weight and character of the G_δ-modification of a space: For any topological space X we have

- $w(X_\delta) \le w(X)^\omega$,
- if $p \in X$ then $\chi(p, X_\delta) \le \chi(p, X)^\omega$.

Lemma 1. *Let X be a compactum with $|X| = \Delta(X) = \kappa > \omega$. Then $\pi(X_\delta) \le \kappa$. Consequently, X_δ is κ-resolvable.*

Proof: We distinguish two cases: (i) $\kappa = \kappa^\omega$ or (ii) $\kappa < \kappa^\omega$. In case (i), as $w(X) \le |X| = \kappa$ by the compactness of X, we even have

$$\pi(X_\delta) \le w(X_\delta) \le w(X)^\omega \le \kappa^\omega = \kappa\,.$$

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In case (ii) we first consider the smallest cardinal λ whose ωth power is greater than κ. Then $\lambda \le \kappa$, moreover for every cardinal $\mu < \lambda$ we have $\mu^\omega < \lambda$. We note that in this case we must have $2^\omega < \kappa$, hence $2^\omega < \lambda$ as well.

Next we show that the set $A = \{p \in X : \chi(p, X) < \lambda\}$ is G_δ-dense in X. Assume, on the contrary, that A is not G_δ-dense in X. Then, as every non-empty G_δ set in X includes a non-empty closed G_δ, there is a (non-empty) closed G_δ set H with $A \cap H = \emptyset$. But then, as H is also compact Hausdorff, for every point $p \in H$ we have

$$\psi(p, H) = \chi(p, H) = \psi(p, X) = \chi(p, X) \ge \lambda\,,$$

consequently the classical Čech-Pospíšil theorem from [1], see also [4], implies

$$|H| \ge 2^\lambda \ge \lambda^\omega > \kappa\,,$$

a contradiction. So A is indeed G_δ-dense. Now, for every point $p \in A$ we have $\chi(p, X_\delta) \le \chi(p, X)^\omega < \lambda \le \kappa$, which together with $|A| \le \kappa$ trivially implies $\pi(X_\delta) \le \kappa$.

The κ-resolvability of X_δ now follows from the classical Bernstein-Kuratowski disjoint refinement theorem, applied to any π-base of X_δ of cardinality at most κ. □

We are now ready to present our main result.

Theorem 2. *Every compactum X is maximally G_δ-resolvable.*

PROOF: This is obvious if X_δ has an isolated point, i.e. $\Delta(X_\delta) = 1$. So assume that $\Delta(X_\delta) = \kappa > 1$. Again by the Čech-Pospíšil theorem then $\kappa \ge 2^{\omega_1}$.

A standard argument shows that every non-empty G_δ set in X includes a non-empty closed G_δ set H with $|H| = \Delta_\delta(H) = \Delta(H_\delta) \ge \kappa$. But then our lemma implies that H_δ is $|H|$-resolvable, hence κ-resolvable as well. Consequently we have that every non-empty open set in X_δ includes a κ-resolvable subset, hence by a result of El'kin [3] (see also [2]), X_δ is κ-resolvable, which completes the proof. □

There are a number of other natural questions that we can raise concerning the G_δ-modifications of compacta. In fact, while working on the problem of this paper and before founding the simple and short solution presented above, we came up with the following problem.

Problem 3. *Let X be a compactum such that for every point $x \in X$ we have $\chi(x, X) \ge \omega_1$, or equivalently, no singleton set is a G_δ in X. Is there then a dense-in-itself subspace of X_δ of cardinality ω_1?*

Note that the affirmative answer to this question could be considered as a natural counterpart of the well-known (and non-trivial) fact that any compactum with no isolated points contains a countably infinite dense-in-itself subspace.

We should point out that, in ZFC, we cannot even prove the following weaker version of the affirmative answer to the above question: Under the same assumptions on X, its G_δ-modification X_δ has a non-discrete subset of cardinality ω_1. However, this weak version does follow from an old conjecture of the first author which was formulated in [5] and so far has not been refuted. This conjecture states that every countably tight compactum has a point of character $\le \omega_1$.

Now assume that all points of a compactum X have character $\ge \omega_1$. If X is countably tight then the conjecture implies the existence of a point $p \in X$ with $\chi(p, X) = \omega_1$, and then p is the limit of a non-trivial convergent sequence of length ω_1 in X. If, on the other hand, X is not countably tight then by [6] there is again a convergent free (hence non-trivial) sequence of length ω_1 in X. But such a sequence together with its limit clearly yields in X a non-discrete subset of cardinality ω_1.

References

[1] Čech E., Pospíšil B., *Sur les espaces compacts*, Publ. Fac. Sci. Univ. Masaryk **258** (1938), 1–14.

[2] Comfort W.W., Garcia-Ferreira S. *Resolvability: A selective survey and some new results*, Topology Appl. **74** (1996), 149–167.

[3] El'kin A.G., *Resolvable spaces which are not maximally resolvable*, Vestnik Moskov. Univ. Ser. I Mat. Meh. **24** (1969), no. 4, 66–70.

[4] Juhász I., *Cardinal functions in topology – 10 years later*, Mathematical Centre Tracts, 123, Mathematisch Centrum, Amsterdam, 1980.

[5] Juhász I., *On the minimum character of points in compact spaces*, in: Proc. Top. Conf. (Pécs, 1989), 365–371, Colloq. Math. Soc. János Bolyai, 55, North-Holland, Amsterdam, 1993.

[6] Juhász I., Szentmiklóssy Z., *Convergent free sequences in compact spaces*, Proc. Amer. Math. Soc. **116** (1992), 1153–1160.

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