M-weak and L-weak compactness
of b-weakly compact operators

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Abstract. We characterize Banach lattices under which each b-weakly compact (resp. b-AM-compact, strong type (B)) operator is L-weakly compact (resp. M-weakly compact).

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1. Introduction

The class of b-weakly compact operators was introduced by Alpay, Altin and Tonyali in [4] on vector lattices. After that, a series of papers, which gave different characterizations of this class of operators, were published [2], [3], [5], [6], [7].

Many relations between this class and other classes of operators was studied in [13], [14], [16]. In fact, in [14] the authors studied the b-weak compactness of semi-compact operators, and in [13] the authors studied the b-weak compactness of order weakly compact (resp. AM-compact) operators. Also, the compactness of b-weakly compact operator was studied in [16]. On the other hand, the M-weak compactness and the L-weak compactness of weakly compact operator was investigated in [17]. Also, Aqzzouz, Elbour and H’Michane [9] characterize Banach lattices on which each Dunford-Pettis operator is M-weakly compact (resp. L-weakly compact). After that, in [12] the authors characterize Banach lattices on which each semi compact operator is M-weakly compact (resp. L-weakly compact).

Our aim in this paper is to study the M-weak compactness and the L-weak compactness of b-weakly compact (resp. strong type (B), resp. b-AM-compact) operators. The article is organized as follows: we give in preliminaries all common notations and definitions of Banach lattice theory. In main results section, we study in the first subsection the L-weak compactness of b-weakly compact (resp. b-AM-compact, strong type (B)) operators and in the second subsection the M-weak compactness of b-weakly compact (resp. b-AM-compact, strong type (B)) operators.

2. Preliminaries

Let us recall from [4] that an operator $T$ from a Banach lattice $E$ into a Banach space $X$ is said to be b-weakly compact if it carries each b-order bounded subset
of \( E \) (i.e., order bounded in \( E'' \)) into a relatively weakly compact subset of \( X \). Recall from [10] that an operator defined from a Banach lattice \( E \) into a Banach space \( X \) is said to be b-AM-compact if it carries b-order bounded set of \( E \) into norm relatively compact set of \( X \).

Note that each b-AM-compact operator from a Banach lattice \( E \) into a Banach space \( X \) is b-weakly compact but the converse is not true in general. In fact, the identity operator of the Banach lattice \( L^1[0, 1] \) is b-weakly compact (because \( L^1[0, 1] \) is a KB-space, see [2, Proposition 2.1]) but it is not b-AM-compact (because \( L^1[0, 1] \) is not a discrete KB-space, see [10, Proposition 2.3]). Moreover, if \( E' \) is discrete then the class of b-weakly compact operators coincides with that of b-AM-compact operators (see [18, Theorem 3]).

An operator \( T \) defined from a Banach lattice \( E \) into a Banach space \( X \) is said to be strong type (B) if \( T''(B) \subset X \) where \( B \) is the band generated by \( E \) in \( E'' \).

Since \( E'' \) is Dedekind complete, every band in \( E'' \) is a projection band and in particular there is a projection of \( E'' \) onto \( B \). Thus, strong type (B) operators extend to \( E'' \). It is easy to see that each strong type (B) operator is a b-weakly compact operator but the converse is not true in general. Indeed, for \( p > 1 \) the operator \( T_p : X_p \rightarrow c_0 \) mentioned in [19] does not preserve any copy of \( c_0 \) and it follows from Proposition 2.10 of [15] that the operator \( T_p \) is b-weakly compact. On the other hand, \( T_p \) is not a strong type (B) operator. Otherwise, since the Banach lattice \( X_p \) does not contain a complemented copy of \( \ell^1 \) then, the norm of \( (X_p)' \) is order continuous and hence it follows from [8, Proposition 3.2] that the operator \( T_p \) is weakly compact, which is impossible. For more details on strong type (B) operators, we refer the reader to [8], [19], [20].

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space \( (E, \| \cdot \|) \) such that \( E \) is a vector lattice and its norm satisfies the following property: for each \( x, y \in E \) such that \( |x| \leq |y| \), we have \( \|x\| \leq \|y\| \). A norm \( \| \cdot \| \) of a Banach lattice \( E \) is order continuous if for each generalized sequence \( (x_\alpha) \) such that \( x_\alpha \downarrow 0 \) in \( E \), \( (x_\alpha) \) converges to 0 for the norm \( \| \cdot \| \) where the notation \( x_\alpha \downarrow 0 \) means that \( (x_\alpha) \) is decreasing, its infimum exists and \( \inf(x_\alpha) = 0 \). Note that if \( E \) is a Banach lattice, its topological dual \( E' \), endowed with the dual norm and the dual order, is also a Banach lattice.

A Banach lattice \( E \) is said to have the positive Schur property if every weakly convergent sequence to 0 in \( E^+ \) is norm convergent to zero. For example, the Banach space \( \ell^1 \) has the positive Schur property. A Banach lattice \( E \) is called a KB-space whenever every increasing norm bounded sequence of \( E^+ \) is norm convergent. As an example, each reflexive Banach lattice is a KB-space. A nonzero element \( x \) of a vector lattice \( E \) is discrete if the order ideal generated by \( x \) equals the lattice subspace generated by \( x \). The vector lattice \( E \) is discrete, if it admits a complete disjoint system of discrete elements. A subset \( A \) of a vector lattice \( E \) is called order bounded, if it is included in an order interval in \( E \). A linear mapping \( T \) from a vector lattice \( E \) into another \( F \) is order bounded if it carries an order bounded set of \( E \) into an order bounded set of \( F \). We will use the term operator \( T : E \rightarrow F \) between two Banach lattices to mean a bounded linear
mapping. It is positive if $T(x) \geq 0$ in $F$ whenever $x \geq 0$ in $E$. The operator $T$ is regular if $T = T_1 - T_2$ where $T_1$ and $T_2$ are positive operators from $E$ into $F$. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \rightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \rightarrow E'$ is likewise positive, where $T'$ is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminology concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

3. Main results

3.1 L-weak compactness of b-weakly compact operator. Recall that a non-empty bounded subset $A$ of a Banach lattice $E$ is said to be L-weakly compact if for every disjoint sequence $(x_n)$ in the solid hull of $A$, we have $\lim_{n \to \infty} \|x_n\| = 0$. An operator $T$ from a Banach space $X$ into $E$ is L-weakly compact if $T(B_X)$ is L-weakly compact in $E$, where $B_X$ denotes the closed unit ball of $X$.

Note that any L-weakly compact operator from a Banach space into a Banach lattice is weakly compact ([1, Theorem 5.61]) and any weakly compact operator is clearly b-weakly compact, but there exists a b-weakly compact (resp. b-AM-compact, resp. strong type (B)) operator which is not L-weakly compact. In fact, the identity operator of the Banach lattice $\ell^2$ is b-weakly compact (resp. b-AM-compact, resp. of strong type(B)), but it is not L-weakly compact. Also, the operator $T : C([0, 1]) \rightarrow c_0$ defined by:

$$T(f) = (\int_0^1 f r_n dt)_1^\infty$$

for each $f \in C([0, 1])$, is weakly compact ([17, Example 4.4]) and hence is b-weakly compact, where $r_n$ is the $n$-th Rademacher function on $[0, 1]$, but $T$ is not L-weakly compact ([17, Example 4.4]).

In the following result, we give the necessary conditions under which each b-weakly compact operator is L-weakly compact:

Theorem 3.1. Let $E$ and $F$ be two Banach lattices. If each b-weakly compact operator $T : E \rightarrow F$ is L-weakly compact, then one of the following assertions is valid:

1. $E = \{0\}$,
2. $F$ is finite dimensional,
3. the norms of $E'$ and $F$ are order continuous.

Proof: The proof follows along the lines of the proof of Theorem 3.3 of [9]. We prove separately the two following assertions.

(a) If the norm of $E'$ is not order continuous then $F$ is finite-dimensional.
(b) If the norm of $F$ is not order continuous, then $E = \{0\}$.

Assume that (a) is false. i.e., the norm of $E'$ is not order continuous and $F$ is infinite dimensional. It follows from Theorem 3.1 of [9] that there exists a disjoint
norm bounded sequence \((y_n)\) of \(F^+\) which does not converge in norm to zero. And since the norm of \(E'\) is not order continuous, then it follows from Theorem 2.4.14 and Proposition 2.3.11 of [22] that \(E\) contains a sub-lattice isomorphic to \(\ell^1\) and there exists a positive projection \(P : E \rightarrow \ell^1\).

To finish the proof, we have to construct a b-weakly compact operator which is not L-weakly compact.

Consider the operator \(S : \ell^1 \rightarrow F\) defined by
\[
S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n \quad \text{for each } (\lambda_n) \in \ell^1.
\]

The operator \(S\) is well defined and it is b-weakly compact because \(\ell^1\) is a KB-space (resp. \(\ell^1\) is a discrete KB-space, resp. \(S\) is b-weakly compact and \(\ell^1\) has an order continuous norm (see Proposition 2.11 of [4])). But \(S\) is not L-weakly compact. Otherwise, since \(S(e_n) = y_n\) for all \(n \geq 1\) where \((e_n)\) is the canonical basis of \(\ell^1\) and \((y_n)\) is a disjoint sequence, then \((y_n)\) is norm convergent to zero and this is false.

On the other hand, since the identity operator of the Banach lattice \(\ell^1\) is b-weakly compact then the composed operator \(T = S \circ P : E \rightarrow \ell^1 \rightarrow F\) is b-weakly compact because \(S \circ P = S \circ Id_{\ell^1} \circ P\). But \(T\) is not L-weakly compact. Otherwise, \(T \circ i = S\) is L-weakly compact where \(i : \ell^1 \rightarrow E\) is the canonical injection of \(\ell^1\) into \(E\), and this is a contradiction.

Now, assume that (b) is false, i.e., the norm of \(F\) is not order continuous and \(E \neq \{0\}\). Choose \(z \in E^+\) such that \(\|z\| = 1\). Hence, it follows from Theorem 39.3 of [21] that there exists \(\phi \in (E')^+\) such that \(\|\phi\| = 1\) and \(\phi(z) = \|\phi\| = 1\).

On the other hand, since the norm of \(F\) is not order continuous, there exists some \(y \in F^+\) and there exists a disjoint sequence \((y_n) \subset [0, y]\) which does not converge to zero in norm.

We consider the operator \(T : E \rightarrow F\) defined by
\[
T(x) = \phi(x) \cdot y \quad \text{for each } x \in E.
\]

It is clear that \(T\) is positive and compact (because its rank is one) and hence \(T\) is b-weakly compact. But \(T\) is not L-weakly compact. In fact, since \(\|z\| = 1\) and \(T(z) = \phi(z) \cdot y = y\) then \(y \in T(B_E)\). As \((y_n) \subset [0, y]\), we conclude that \((y_n)\) is a disjoint sequence in the solid hull of \(T(B_E)\). Hence, if \(T\) is L-weakly compact then \(\lim_{n \to \infty} \|y_n\| \to 0\), which is a contradiction.

**Remark 1.** The two necessary conditions (1) and (2) in Theorem 3.1 are sufficient, but the condition (3) is not. In fact, the identity operator of the Banach lattice \(\ell^2\) is b-weakly compact, but it is not L-weakly compact. However the norm of \((\ell^2)' = \ell^2\) is order continuous.

**Remark 2.** Since any strong type (B) operator is b-weakly compact and any b-AM-compact operator is b-weakly compact then the tree necessary conditions in Theorem 3.1 are also necessary if each strong type (B) operator \(T : E \rightarrow F\)
is $L$-weakly compact or each $b$-AM-compact operator $T : E \to F$ is $L$-weakly compact.

Now, we give sufficient conditions under which each strong type $(B)$ operator is $L$-weakly compact:

**Theorem 3.2.** Let $E$ and $F$ be two Banach lattices. Each strong type $(B)$ operator $T$ from $E$ into $F$ is $L$-weakly compact, if one of the following statements is valid:

1. $E = \{0\}$,
2. $F$ is finite dimensional,
3. $E'$ has an order continuous norm and $F$ has the positive Schur property.

**Proof:**

(1) Obvious.

(2) Since $F$ is finite dimensional, then it follows from Corollary 3.2 of [9] that $T$ is $L$-weakly compact.

(3) Let $T : E \to F$ be a strong type $(B)$ operator then $T'''(B) \subset F$ where $B$ is the band generated by $E$ in $E''$. As the norm of $E'$ is order continuous, then it follows from Theorem 2.4.14 of [22] that $B = E''$ and hence $T$ is weakly compact.

Now, since $F$ has the positive Schur property, then by Theorem 3.4 of [17] $T$ is $L$-weakly compact. □

Let us remark that if the norm of the Banach lattice $E$ is order continuous then it follows from [4, Proposition 2.11] that the strong type $(B)$ operators defined from $E$ into an arbitrary Banach space coincide with $b$-weakly compact operators. On the other hand, all $b$-AM-compact operators are $b$-weakly compact.

As a consequence of Theorem 3.2, we give the following result:

**Proposition 3.3.** Let $E$ and $F$ be two Banach lattices. Then each $b$-weakly compact (resp, $b$-AM-compact) operator $T : E \to F$ is $L$-weakly compact, if one of the following statements is valid:

1. $E = \{0\}$,
2. $F$ is finite dimensional,
3. the norms of $E'$ and $E$ are order continuous and $F$ has the positive Schur property.

As a consequence of Theorem 3.1 and Proposition 3.3, we obtain the following characterization:

**Corollary 3.4.** Let $E$ be a Banach lattice with order continuous norm and $F$ a Banach lattice with the positive Schur property. Then the following statements are equivalent.

1. Each $b$-weakly compact operator $T : E \to F$ is $L$-weakly compact.
2. Each positive $b$-weakly compact operator $T : E \to F$ is $L$-weakly compact.
3. One of the following conditions is valid:
   (a) $E = \{0\}$,
Corollary 3.5. Let $E$ and $F$ be two Banach lattices such that $F$ has the positive Schur property. Then the following statements are equivalent.

(1) Each strong type $(B)$ operator $T$ from $E$ into $F$ is $L$-weakly compact.

(2) One of the following conditions is valid:
   (a) $E = \{0\}$,
   (b) $E'$ has an order continuous norm,
   (c) $F$ is finite dimensional.

Remark 3. As a particular case of Corollary 3.4 and Corollary 3.5, we have the following characterizations.

(1) Let $E$ be a non-void Banach lattice with order continuous norm and $F$ an infinite-dimensional Banach lattice with the positive Schur property. Each $b$-weakly compact operator $T : E \rightarrow F$ is $L$-weakly compact, if and only if each positive $b$-weakly compact operator $T : E \rightarrow F$ is $L$-weakly compact, if and only if $E'$ has an order continuous norm.

(2) Let $E$ be a non-void Banach lattice and $F$ an infinite-dimensional Banach lattice with the positive Schur property. Then, each strong type $(B)$ operator $T$ from $E$ into $F$ is $L$-weakly compact, if and only if $E'$ has an order continuous norm.

3.2 M-weak compactness of $b$-weakly compact operator. An operator $T : E \rightarrow X$ from a Banach lattice $E$ into a Banach space $X$ is said to be $M$-weakly compact if for every disjoint sequence $(x_n)$ in $B_E$ we have $\lim_{n \rightarrow \infty} \|T(x_n)\| = 0$, where $B_E$ denotes the closed unit ball of $E$.

Note that every $M$-weakly compact operator from a Banach lattice into a Banach space is weakly compact ([1, Theorem 5.61]) and any weakly compact operator is clearly $b$-weakly compact. But there exists a $b$-weakly compact (resp. $b$-AM-compact, resp. strong type $(B)$) operator which is not $M$-weakly compact. In fact, $Id_{\ell^1}$ is $b$-weakly compact (resp. $b$-AM-compact, resp. strong type $(B)$) but it is not $M$-weakly compact.

Our following result gives necessary conditions under which each $b$-weakly compact (resp. $b$-AM-compact, resp. strong type $(B)$) operator is $M$-weakly compact:

Theorem 3.6. Let $E$ and $F$ be two Banach lattices. If each $b$-weakly compact (resp. $b$-AM-compact, resp. strong type $(B)$) operator $T : E \rightarrow F$ is $M$-weakly compact, then one of the following assertions is valid:

(1) $F = \{0\}$,

(2) $E'$ has an order continuous norm.

Proof: Assume by way of contradiction that the norm of $E'$ is not order continuous norm and $F \neq \{0\}$. To finish the proof, we have to construct a positive
b-weakly compact operator $T : E \rightarrow F$ (resp. b-AM-compact, resp. strong type (B)) operator which is not M-weakly compact. Since the norm of $E'$ is not order continuous norm, it follows from Theorem 2.4.14 and Proposition 2.3.11 of Meyer-Nieberg [22] that $E$ contains a closed sub-lattice which is isomorphic to $\ell^1$ and there exists a positive projection $P : E \rightarrow \ell^1$. On the other hand, as $F \neq \{0\}$, there exists a non-null element $y \in F^+$.

Now, we consider the operator $S : \ell^1 \rightarrow F$ defined by

$$S((\lambda_n)) = \left(\sum_{n=1}^{\infty} \lambda_n\right) y \quad \text{for each } (\lambda_n) \in \ell^1.$$

It is clear that $S$ is well defined and positive. Also, $S$ is compact (because its rank is one). Hence the positive operator

$$T = S \circ P : E \rightarrow \ell^1 \rightarrow F$$

is compact and $T$ is b-weakly compact (resp. b-AM-compact; resp. strong type (B)) but it is not M-weakly compact. In fact, if we denote by $(e_n)$ the canonical basis of $\ell^1 \subset E$, the sequence $(e_n)$ is disjoint and bounded in $E$, moreover we have $T((e_n)) = y$ for each $n \geq 1$. Then $\|T((e_n))\| \not\rightarrow 0$ (because $y \neq 0$). So, $T$ is not M-weakly compact and this proves the result. □

**Remark 4.** The necessary condition (1) in Theorem 3.6 is sufficient, but the condition (2) is not. In fact, the identity operator of the Banach lattice $\ell^2$ is b-weakly compact (resp. b-AM-compact, resp. strong type (B)) but is not M-weakly compact. However the norm of $(\ell^2)' = \ell^2$ is order continuous.

In the following result, we give sufficient conditions under which each b-weakly compact operator is M-weakly compact:

**Theorem 3.7.** Let $E$ and $F$ be two Banach lattices.

1. If $F = \{0\}$ or the norm of $E$ is order continuous and $E'$ has the positive Schur property then each b-weakly compact operator $T : E \rightarrow F$ is M-weakly compact.

2. If the norms of $E$ and $E'$ are order continuous and $F$ has the positive Schur property then each regular b-weakly compact operator $T : E \rightarrow F$ is M-weakly compact.

**Proof:** (1) If $F = \{0\}$, clearly each operator is M-weakly compact. In the latter case, let $T : E \rightarrow F$ be a b-weakly compact operator. Since the norm of $E$ is order continuous and the norm of $E'$ is order continuous (because $E'$ has the positive Schur property), then it follows from the proof of Proposition 3.3 that $T$ is weakly compact.

Now, since $E'$ has the positive Schur property, it follows from [17, Theorem 3.3] that $T$ is M-weakly compact.

(2) Let $T : E \rightarrow F$ be an order bounded b-weakly compact operator. Since the norms of $E$ and $E'$ are order continuous and $F$ has the positive Schur property,
then by Proposition 3.3 $T$ is $L$-weakly compact. Therefore, by [1, Theorem 5.67] $T$ is $M$-weakly compact. □

Now, we give sufficient conditions under which each operator of strong type (B) is $M$-weakly compact:

**Theorem 3.8.** Let $E$ and $F$ be two Banach lattices.

1. If $F = \{0\}$ or $E'$ has the positive Schur property then each strong type (B) operator $T : E \to F$ is $M$-weakly compact.
2. If the norm of $E'$ is order continuous and $F$ has the positive Schur property then each regular strong type (B) operator $T : E \to F$ is $M$-weakly compact.

**Proof:** (1) If $F = \{0\}$, clearly each operator is $M$-weakly compact. In the latter case, let $T : E \to F$ be a strong type (B) operator. Since the norm of $E'$ is order continuous, then it follows from [8, Proposition 3.2] that $T$ is weakly compact. Now, since $E'$ has the positive Schur property, then by [17, Theorem 3.3] $T$ is $M$-weakly compact.

(2) It follows from Theorem 3.6 of [17]. □

As a consequence of Theorem 3.6 and Theorem 3.8, we have the following characterization:

**Corollary 3.9.** Let $E$ and $F$ be two Banach lattices such that $F$ has the positive Schur property. Then the following statements are equivalent.

1. Each regular operator $T$ from $E$ into $F$ of strong type (B) is $M$-weakly compact.
2. One of the following conditions is valid:
   (a) $F = \{0\}$,
   (b) $E'$ has an order continuous norm.

**Remark 5.** As a particular case of Corollary 3.9, we have the following characterization: Let $E$ be a Banach lattice and $F$ a non-void Banach lattice with the positive Schur property. Then, each regular strong type (B) operator $T : E \to F$ is $M$-weakly compact if and only if $E'$ has an order continuous norm.

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