On α-embedded sets and extension of mappings

Olena Karlova

Abstract. We introduce and study α-embedded sets and apply them to generalize the Kuratowski Extension Theorem.

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1. Introduction

A subset A of a topological space X is called *functionally open* (*functionally closed*) if there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}((0,1])$ ($A = f^{-1}(0)$).

Let $\mathcal{G}_0^*(X)$ and $\mathcal{F}_0^*(X)$ be the collections of all functionally open and functionally closed subsets of a topological space X, respectively. Assume that the classes $\mathcal{G}_\xi^*(X)$ and $\mathcal{F}_\xi^*(X)$ are defined for all $\xi < \alpha$, where $0 < \alpha < \omega_1$. Then, if α is odd, the class $\mathcal{G}_\alpha^*(X)$ ($\mathcal{F}_\alpha^*(X)$) consists of all countable intersections (unions) of sets of lower classes, and, if α is even, the class $\mathcal{G}_\alpha^*(X)$ ($\mathcal{F}_\alpha^*(X)$) consists of all countable unions (intersections) of sets of lower classes. The classes $\mathcal{F}_\alpha^*(X)$ for odd α and $\mathcal{G}_\alpha^*(X)$ for even α are said to be *functionally additive*, and the classes $\mathcal{F}_\alpha^*(X)$ for even α and $\mathcal{G}_\alpha^*(X)$ for odd α are called *functionally multiplicative*. If a set belongs to the α-th functionally additive and to the α-th functionally multiplicative class simultaneously, then it is called *functionally ambiguous of the α-th class*. For every $0 \le \alpha < \omega_1$ let

$$\mathcal{B}_\alpha^*(X) = \mathcal{F}_\alpha^*(X) \cup \mathcal{G}_\alpha^*(X)$$

and let

$$\mathcal{B}^*(X) = \bigcup_{0\le\alpha<\omega_1} \mathcal{B}_\alpha^*(X).$$

If $A \in \mathcal{B}^*(X)$, then A is said to be *a functionally measurable* set.

If P is a property of mappings, then by $P(X,Y)$ we denote the collection of all mappings $f : X \to Y$ with the property P. Let $P(X)$ ($P^*(X)$) be the collection of all real-valued (bounded) mappings on X with a property P.

By the letter C we denote, as usual, the property of continuity.

Let $K_0(X,Y) = C(X,Y)$. For an ordinal $0 < \alpha < \omega_1$ we say that a mapping $f : X \to Y$ *belongs to the α-th functional Lebesgue class*, $f \in K_\alpha(X,Y)$, if the preimage $f^{-1}(V)$ of an arbitrary open set $V \subseteq Y$ is of the α-th functionally additive class in X.

A subspace E of X is *P-embedded* (*P^*-embedded*) in X if every (bounded) function $f \in P(E)$ can be extended to a (bounded) function $g \in P(X)$.

A subset E of X is said to be *z-embedded* in X if every functionally closed set in E is the restriction of a functionally closed set in X to E. It is well-known that

$$E \text{ is } C\text{-embedded} \Rightarrow E \text{ is } C^*\text{-embedded} \Rightarrow E \text{ is } z\text{-embedded.}$$

Recall that sets A and B are *completely separated in* X if there exists a continuous function $f : X \to [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

The following theorem was proved in [2, Corollary 3.6].

Theorem 1.1 (Blair-Hager)**.** *A subset E of a topological space X is C-embedded in X if and only if E is z-embedded in X and E is completely separated from every functionally closed set in X disjoint from E.*

It is natural to consider P- and P^*-embedded sets if $P = K_\alpha$ for $\alpha > 0$. In connection with this we introduce and study a class of α-embedded sets which coincides with the class of z-embedded sets when $\alpha = 0$. In Section 3 we generalize the notion of completely separated sets to α-separated sets. Section 4 deals with ambiguously α-embedded sets which play the important role in the extension of bounded K_α-functions. In the fifth section we prove an analog of the Tietze-Uryhson Extension Theorem for K_α-functions. Section 6 concerns the question when K_1-embedded sets coincide with K_1^*-embedded sets. The seventh section presents a generalization of the Kuratowski Theorem [11, p. 445] on extension of K_α-mappings with values in Polish spaces.

2. α-embedded sets

Let $0 \le \alpha < \omega_1$. A subset E of a topological space X is *α-embedded* in X if for any set A of the α-th functionally additive (multiplicative) class in E there is a set B of the α-th functionally additive (multiplicative) class in X such that $A = B \cap E$.

Proposition 2.1. *Let X be a topological space, $0 \le \alpha < \omega_1$ and let $E \subseteq X$ be an α-embedded set of the α-th functionally additive* (*multiplicative*) *class in X. Then every set of the α-th functionally additive* (*multiplicative*) *class in E belongs to the α-th functionally additive* (*multiplicative*) *class in X.*

Proof: For a set C of the α-th functionally additive (multiplicative) class in E we choose a set B of the α-th functionally additive (multiplicative) class in X such that $C = B \cap E$. Then C belongs to the α-th functionally additive (multiplicative) class in X as the intersection of two sets of the same class. □

Proposition 2.2. *Let X be a topological space, $E \subseteq X$ and*

(i) *X is perfectly normal, or*
(ii) *X is completely regular and E is its Lindelöf subset, or*

(iii) *E is a functionally open subset of X, or*

(iv) *X is a normal space and E is its F_σ-subset,*

then E is 0*-embedded in X.*

Proof: Let G be a functionally open set in E.

(i) Choose an open set U in X such that $G = E \cap U$. Then U is functionally open in X by Vedenissoff's theorem [5, p. 45].

(ii) Let U be an open set in X such that $G = E \cap U$. Since X is completely regular, $U = \bigcup_{s\in S} U_s$, where U_s is a functionally open set in X for each $s \in S$. Notice that G is Lindelöf, provided G is F_σ in the Lindelöf space E [5, p. 192]. Then there exists a countable set $S_0 \subseteq S$ such that $G \subseteq \bigcup_{s\in S_0} U_s$. Let $V = \bigcup_{s\in S_0} U_s$. Then V is functionally open in X and $V \cap E = G$.

(iii) Consider continuous functions $\varphi : E \to [0,1]$ and $\psi : X \to [0,1]$ such that $G = \varphi^{-1}((0,1])$ and $E = \psi^{-1}((0,1])$. For each $x \in X$ we set

$$f(x) = \begin{cases} \varphi(x) \cdot \psi(x), & x \in E, \\ 0, & x \in X \setminus E. \end{cases}$$

Since $\varphi(x) \cdot \psi(x) = 0$ on $\overline{E} \setminus E$, $f : X \to [0,1]$ is continuous. Moreover, $G = f^{-1}((0,1])$. Hence, the set G is functionally open in X.

(iv) Let $\tilde{G}$ be an open set in X such that $G = \tilde{G} \cap E$. Since G is functionally open in E, G is F_σ in E. Consequently, G is F_σ in X, provided E is F_σ in X. Therefore, there exists a sequence $(F_n)_{n=1}^\infty$ of closed sets $F_n \subseteq X$ such that $G = \bigcup_{n=1}^\infty F_n$. Since X is normal, for every $n \in \mathbb{N}$ there exists a continuous function $f_n : X \to [0,1]$ such that $f_n(x) = 1$ if $x \in F_n$ and $f_n(x) = 0$ if $x \in X \setminus \tilde{G}$. Then the set $V = \bigcup_{n=1}^\infty f_n^{-1}((0,1])$ is functionally open in X and $V \cap E = G$. $\square$

Examples 2.3 and 2.4 show that none of the conditions (i)–(iv) on X and E in Proposition 2.2 can be weakened.

Recall that a topological space X is said to be *perfect* if every its closed subset is G_δ in X.

Example 2.3. There exist a perfect completely regular space X and its functionally closed subspace E which is not α-embedded in X for every $0 \le \alpha < \omega_1$.

Consequently, there is a bounded continuous function on E which cannot be extended to a $\mathcal{K}_\alpha$-function for every α.

Proof: Let X be the Niemytski plane [5, p. 22], i.e., $X = \mathbb{R} \times [0,+\infty)$ where a base of neighborhoods of $(x,y) \in X$ with $y > 0$ is formed by open balls with the center in (x,y), and a base of neighborhoods of $(x,0)$ is formed by the sets $U \cup \{(x,0)\}$ such that U is an open ball which tangent to $\mathbb{R} \times \{0\}$ in the point $(x,0)$. It is well-known that the space X is perfect and completely regular, but it is not normal.

Denote $E = \mathbb{R} \times \{0\}$. Since the function $f : X \to \mathbb{R}$, $f(x,y) = y$, is continuous and $E = f^{-1}(0)$, the set E is functionally closed in X.

Notice that every function $f : E \to \mathbb{R}$ is continuous. Therefore, $|\mathcal{B}^*_\alpha(E)| = 2^{2^{\omega_0}}$ for every $0 \le \alpha < \omega_1$. On the other hand, $|\mathcal{B}^*_\alpha(X)| = 2^{\omega_0}$ for every $0 \le \alpha < \omega_1$, provided the space X is separable. Hence, for every $0 \le \alpha < \omega_1$ there exists a set $A \in \mathcal{B}^*_\alpha(E)$ which cannot be extended to a set $B \in \mathcal{B}^*_\alpha(X)$.

Observe that a function $f : E \to [0,1]$ such that $f = 1$ on A and $f = 0$ on $E \setminus A$ is continuous on E. But there is no K_α-function $f : X \to [0,1]$ such that $g|_E = f$, since otherwise the set $B = g^{-1}(1)$ would be an extension of A. □

Example 2.4. There exist a compact Hausdorff space X and its open subspace E which is not α-embedded in X for every $0 \le \alpha < \omega_1$.

Proof: Let $X = D \cup \{\infty\}$ be the Alexandroff compactification of an uncountable discrete space D [5, p. 169] i $E = D$. Fix $0 \le \alpha < \omega_1$ and choose an arbitrary uncountable set $A \subseteq E$ with uncountable complement $X \setminus A$. Evidently, A is functionally closed in E. Assume that there is a set B of the α-th functionally multiplicative class in X such that $A = B \cap E$. Clearly, $B = A \cup \{\infty\}$. Moreover, there exists a function $f : X \to \mathbb{R}$ of the α-th Baire class such that $B = f^{-1}(0)$ [9, Lemma 2.1]. But every continuous function on X, and consequently every Baire function of the class α on X satisfies the equality $f(x) = f(\infty)$ for all but countably many points $x \in X$, which implies a contradiction. □

Proposition 2.5. *Let $0 \le \alpha \le \beta < \omega_1$ and let X be a topological space. Then every α-embedded subset of X is β-embedded.*

Proof: Let E be an α-embedded subset of X. If $\beta = \alpha$, the assertion of the proposition if obvious. Suppose the assertion is true for all $\alpha \le \beta < \xi$ and let A be a set of the ξ-th functionally additive class in E. Then there exists a sequence of sets A_n of functionally multiplicative classes $< \xi$ in E such that $A = \bigcup_{n=1}^\infty A_n$. According to the assumption, for every $n \in \mathbb{N}$ there is a set B_n of a functionally multiplicative class $< \xi$ in X such that $A_n = B_n \cap E$. Then the set $B = \bigcup_{n=1}^\infty B_n$ belongs to the ξ-th functionally additive class in X and $A = B \cap E$. □

The opposite proposition is not true, as the following result shows.

Theorem 2.6. *There exist a completely regular space X and its 1-embedded subspace $E \subseteq X$ which is not 0-embedded in X.*

Proof: Let $X_0 = [0,1]$, $X_s = \mathbb{N}$ for every $s \in (0,1]$, $Y = \prod_{s\in(0,1]} X_s$ and

$$X = [0,1] \times Y = \prod_{s\in[0,1]} X_s.$$

Then X is completely regular as a product of completely regular spaces X_s. Let

$$A_1 = (0,1] \quad \text{and} \quad A_2 = \{0\}.$$

For $i = 1,2$ we consider the set

$$F_i = \bigcap_{n\ne i} \{y = (y_s)_{s\in(0,1]} \in Y : |\{s \in (0,1] : y_s = n\}| \le 1\}.$$

Obviously, $F_1 \cap F_2 = \emptyset$ and the sets F_1 and F_2 are closed in Y.

Let

$$B_1 = A_1 \times F_1, \quad B_2 = A_2 \times F_2 \text{ and } E = B_1 \cup B_2.$$

It is easy to see that the sets B_1 and B_2 are closed in E, and consequently they are functionally clopen in E.

Claim 1. *The set B_i is 0-embedded in X for every $i = 1, 2$.*

PROOF: Let C be a functionally open set in B_1. Let us consider the set

$$H = \{x = (x_s)_{s\in[0,1]} \in X : |\{s \in [0,1] : x_s \neq 1\}| \leq \aleph_0\}.$$

Then the set $[0,1] \times F_i$ is closed in H for every $i = 1, 2$. Since H is the Σ-product of the family $(X_s)_{s\in[0,1]}$ (see [5, p. 118]), according to [10] the space H is normal. Consequently, $[0,1] \times F_i$ is normal as closed subspace of normal space for every $i = 1, 2$. Clearly, B_1 is functionally open in $[0,1] \times F_1$. Hence, B_1 is 0-embedded in $[0,1] \times F_1$ according to Proposition 2.2(iii). Then C is functionally open in $[0,1] \times F_1$ by Proposition 2.1. Notice that the set $[0,1] \times F_1$ is 0-embedded in H by Propositions 2.2(iv). Hence, there exists a functionally open set C' in H such that $C' \cap ([0,1] \times F_1) = C$. It follows from [3] that H is 0-embedded in X. Then there exists a functionally open set C'' in X such that $C'' \cap H = C'$. Evidently, $C'' \cap B_1 = C$. Therefore, the set B_1 in 0-embedded in X.

Analogously, it can be shown that the set B_2 is 0-embedded in X, using the fact that B_2 is 0-embedded in $[0,1] \times F_2$ according to Proposition 2.2(iv).

Claim 2. *The set E is not 0-embedded in X.*

PROOF: Assuming the contrary, we choose a functionally closed set D in X such that $D \cap E = B_1$. Then $D = f^{-1}(0)$ for some continuous function $f : X \to [0,1]$. It follows from [5, p. 117] that there exists a countable set $S = \{0\} \cup T$, where $T \subseteq (0,1]$, such that for any $x = (x_s)_{s\in[0,1]}$ and $y = (y_s)_{s\in[0,1]}$ of X the equality $x|_S = y|_S$ implies $f(x) = f(y)$. Let $y_0 \in Y$ be such that $y_0|_T$ is a sequence of different natural numbers which are not equal to 1 or 2. We choose $y_1 \in F_1$ and $y_2 \in F_2$ such that $y_0|_T = y_1|_T = y_2|_T$. Then

$$f(a, y_0) = f(a, y_1) = f(a, y_2)$$

for all $a \in [0,1]$. We notice that $f(0, y_1) = 0$. Therefore, $f(0, y_0) = 0$. But $f(a, y_2) > 0$ for all $a \in A_2$. Then $f(a, y_0) > 0$ for all $a \in A_2$. Hence, $A_1 = (f^{y_0})^{-1}(0)$, where $f^{y_0}(a) = f(a, y_0)$ for all $a \in [0,1]$, and f^{y_0} is continuous. Thus, the set $A_1 = (0,1]$ is closed in $[0,1]$, which implies a contradiction.

Claim 3. *The set E is 1-embedded in X.*

PROOF: Let C be a functionally G_δ-set in E. We put

$$E_1 = A_1 \times Y, \quad E_2 = A_2 \times Y.$$

Then the set E_1 is functionally open in X and the set E_2 is functionally closed in X. For $i = 1, 2$ let $C_i = C \cap B_i$. Since for every $i = 1, 2$ the set C_i is functionally G_δ in the set B_i 0-embedded in X, by Proposition 2.5 there exists a functionally G_δ-set $\tilde{C}_i$ in X such that $\tilde{C}_i \cap B_i = C_i$. Let

$$\tilde{C} = (\tilde{C}_1 \cap E_1) \cup (\tilde{C}_2 \cap E_2).$$

Then $\tilde{C}$ is functionally G_δ in X and $\tilde{C} \cap E = C$. □

3. α-separated sets and α-separated spaces

Let $0 \leq \alpha < \omega_1$. Subsets A and B of a topological space X are said to be *α-separated* if there exists a function $f \in K_\alpha(X)$ such that

$$A \subseteq f^{-1}(0) \quad \text{and} \quad B \subseteq f^{-1}(1).$$

Let us remark that 0-separated sets are also called *completely separated* [5, p. 42].

Lemma 3.1 ([8, Lemma 2.1])**.** *Let X be a topological space, $\alpha > 0$ and let $A \subseteq X$ be a subset of the α-th functionally additive class. Then there exists a sequence $(A_n)_{n=1}^\infty$ such that each A_n is functionally ambiguous of the class α in X, $A_n \cap A_m = \emptyset$ for $n \neq m$ and $A = \bigcup_{n=1}^\infty A_n$.*

Proof: Since A belongs to the α-th functionally additive class, $A = \bigcup_{n=1}^\infty B_n$, where each B_n belongs to the functionally multiplicative class $< \alpha$ in X. Therefore, each B_n is functionally ambiguous of the class α. Let $A_1 = B_1$ and $A_n = B_n \setminus \bigcup_{k<n} B_k$ for $n > 1$. Then $(A_n)_{n=1}^\infty$ is the required sequence. □

Lemma 3.2 ([8, Lemma 2.2])**.** *Let X be a topological space, $\alpha \geq 0$ and let A_n belongs to the α-th functionally additive class in X for every $n \in \mathbb{N}$ with $X = \bigcup_{n=1}^\infty A_n$. Then there exists a sequence $(B_n)_{n=1}^\infty$ of mutually disjoint functionally ambiguous sets of the class α in X such that $B_n \subseteq A_n$ and $X = \bigcup_{n=1}^\infty B_n$.*

Proof: If follows from Lemma 3.1 that for every $n \in \mathbb{N}$ there exists a sequence $(F_{n,m})_{m=1}^\infty$ such that each $F_{n,m}$ is functionally ambiguous of the class α in X, $F_{n,m} \cap F_{n,k} = \emptyset$ for $m \neq k$ and $A_n = \bigcup_{m=1}^\infty F_{n,m}$. Let $k : \mathbb{N}^2 \to \mathbb{N}$ be a bijection. Set

$$C_{n,m} = F_{n,m} \setminus \bigcup_{k(p,s)<k(n,m)} F_{p,s}.$$

Evidently, $\bigcup_{n,m=1}^\infty C_{n,m} = X$. Let $B_n = \bigcup_{m=1}^\infty C_{n,m}$. Then $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n = X$ and $B_n \subseteq \bigcup_{m=1}^\infty F_{n,m} = A_n$. Notice that each $C_{n,m}$ is functionally ambiguous of the class α. Therefore, B_n belongs to the functionally additive class α for every n. Moreover, $B_n \cap B_m = \emptyset$ for $n \neq m$. Since $X \setminus B_n = \bigcup_{k \neq n} B_k$, B_n is functionally ambiguous of the class α. □

Lemma 3.3. *Let $0 \leq \alpha < \omega_1$ and let A be a subset of the α-th functionally multiplicative class of a topological space X. Then there exists a function $f \in K_\alpha^*(X)$ such that $A = f^{-1}(0)$.*

PROOF: For $\alpha = 0$ the lemma follows from the definition of a functionally closed set. Let $\alpha > 0$. Since the set $B = X \setminus A$ is of the α-th functionally additive class, there exists a sequence of functionally ambiguous sets B_n of the α-th class in X such that $B = \bigcup_{n=1}^{\infty} B_n$ and $B_n \cap B_m = \emptyset$ for all $n \neq m$ by Lemma 3.1. Define a function $f : X \to [0, 1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in A, \\ \frac{1}{n}, & \text{if } x \in B_n. \end{cases}$$

Take an arbitrary open set $V \subseteq [0, 1]$. If $0 \notin V$ then $f^{-1}(V)$ is of the α-th functionally additive class as a union of at most countably many sets B_n. If $0 \in V$ then there exists such a number N that $\frac{1}{n} \in V$ for all $n > N$. Then the set $X \setminus f^{-1}(V) = \bigcup_{n=1}^{N} B_n$ belongs to the α-th functionally multiplicative class. Hence, $f^{-1}(V)$ is of the α-th functionally additive class in X. Therefore, $f \in K^*_\alpha(X)$. □

Proposition 3.4. *Let $0 \leq \alpha < \omega_1$ and let X be a topological space. Then any two disjoint sets A and B of the α-th functionally multiplicative class in X are α-separated.*

PROOF: By Lemma 3.3 we choose functions $f_1, f_2 \in K_\alpha(X)$ such that $A = f_1^{-1}(0)$ and $B = f_2^{-1}(0)$. For all $x \in X$ let

$$f(x) = \frac{f_1(x)}{f_1(x) + f_2(x)}\,.$$

It is easy to see that $f \in K_\alpha(X)$, $f(x) = 0$ on A and $f(x) = 1$ on B. □

Let $0 \leq \alpha < \omega_1$. A topological space X is *α-separated* if any two disjoint sets $A, B \subseteq X$ of the α-th multiplicative class in X are α-separated. It follows from Urysohn's Lemma [5, p. 41] that a topological space is 0-separated if and only if it is normal. Proposition 3.4 implies that every perfectly normal space is α-separated for each $\alpha \geq 0$. It is natural to ask whether there is an α-separated space for $\alpha \geq 1$ which is not perfectly normal.

Example 3.5. There exists a completely regular 1-separated space which is not perfectly normal.

PROOF: Let $D = D(\mathfrak{m})$ be a discrete space of the cardinality $\mathfrak{m}$, where $\mathfrak{m}$ is a measurable cardinal number [6, 12.1]. According to [6, 12.2], D is not a realcompact space. Let $X = \upsilon D$ be a Hewitt realcompactification of D [5, p. 218]. Then X is an extremally disconnected P-space, which is not discrete [6, 12H]. Thus, there exists a point $x \in X$ such that the set $\{x\}$ is not open. Then $\{x\}$, being a closed set, is not a G_δ-set, since X is a P-space (i.e. a space in which every G_δ-subset is open). Therefore, the space X is not perfect.

If A and B are disjoint G_δ-subsets of X, then A and B are open in X. Notice that in an extremally disconnected space any two disjoint open sets are completely separated [6, 1H]. Consequently, A and B are 1-separated, since every continuous function belongs to the first Lebesgue class. □

Clearly, every ambiguous set A of the class 0 in a topological space (i.e., every clopen set) is a functionally ambiguous set of the class 0. If A is an ambiguous set of the first class, i.e. A is an F_σ- and a G_δ-set, then A need not be a functionally F_σ- or a functionally G_δ-set. Indeed, let X be the Niemytski plane, E be a set which is not of the $G_{\delta\sigma}$-type in $\mathbb{R}$ and let $A = E \times \{0\}$ be a subspace of X. Then A is closed and consequently G_δ-subset of X, since the Niemytski plane is a perfect space. Assume that A is a functionally F_σ-set in X. Then $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is a functionally closed subset of X for every $n \in \mathbb{N}$. According to [13, Theorem 5.1], a closed subset F of X is a functionally closed set in X if and only if the set $\{x \in \mathbb{R} : (x, 0) \in F\}$ is a G_δ-set in $\mathbb{R}$. It follows that for every $n \in \mathbb{N}$ the set A_n is a G_δ-subset of $\mathbb{R}$, which implies a contradiction.

Theorem 3.6. *Let $0 \le \alpha < \omega_1$ and let X be an α-separated space.*

(1) *Every ambiguous set $A \subseteq X$ of the class α is functionally ambiguous of the class α.*
(2) *For any disjoint sets A and B of the $(\alpha+1)$-th additive class in X there exists a set C of the $(\alpha+1)$-th functionally multiplicative class such that*

$$A \subseteq C \subseteq X \setminus B.$$

(3) *Every ambiguous set A of the $(\alpha+1)$-th class in X is a functionally ambiguous set of the $(\alpha+1)$-th class.*
(4) *Any set of the α-th multiplicative class in X is α-embedded.*

Proof: (1) Since the set $B = X \setminus A$ belongs to the α-th multiplicative class in X, there exists a function $f \in K_\alpha(X)$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$. Then $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Hence, the sets A and B are of the α-th functionally multiplicative class. Consequently, A is a functionally ambiguous set of the class α.

(2) Choose two sequences $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$, where A_n and B_n belong to the α-th multiplicative class in X for every $n \in \mathbb{N}$, such that $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Since X is α-separated, for every $n, m \in \mathbb{N}$ there exists a function $f_{n,m} \in K_\alpha(X)$ such that $A_n \subseteq f_{n,m}^{-1}(1)$ and $B_m \subseteq f_{n,m}^{-1}(0)$. Set

$$C = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} f_{n,m}^{-1}((0, 1]).$$

Then the set C is of the $(\alpha + 1)$-th functionally multiplicative class in X and $A \subseteq C \subseteq X \setminus B$.

(3) Let $A \subseteq X$ be an ambiguous set of the $(\alpha+1)$-th class. Denote $B = X \setminus A$. Since A and B are disjoint sets of the $(\alpha + 1)$-th additive class in X, according to (3.6) there exists a set $C \subseteq X$ of the $(\alpha + 1)$-th functionally multiplicative class such that $A \subseteq C \subseteq X \setminus B$. It follows that $A = C$, consequently A is of the

$(\alpha + 1)$-th functionally multiplicative class. Analogously, it can be shown that B is also of the $(\alpha + 1)$-th functionally multiplicative class. Therefore, A is a functionally ambiguous set of the $(\alpha + 1)$-th class.

(4) If $\alpha = 0$ then X is a normal space. Therefore, any closed set F in X is 0-embedded by Proposition 2.2. Let $\alpha > 0$ and let $E \subseteq X$ be a set of the α-th multiplicative class in X. Choose any set A of the α-th functionally multiplicative class in E. Since the set $E \setminus A$ belongs to the α-th functionally additive class in E, there exists a sequence of sets B_n of the α-th functionally multiplicative class in E such that $E \setminus A = \bigcup_{n=1}^{\infty} B_n$. Then for every $n \in \mathbb{N}$ the sets A and B_n are disjoint and belong to the α-th multiplicative class in X. Since X is α-separated, we can choose a function $f_n \in K_\alpha(X)$ such that $A \subseteq f_n^{-1}(0)$ and $B_n \subseteq f_n^{-1}(1)$. Let $\tilde{A} = \bigcap_{n=1}^{\infty} f_n^{-1}(0)$. Then the set $\tilde{A}$ belongs to the α-th functionally multiplicative class in X and $\tilde{A} \cap E = A$. □

Proposition 3.7. *A topological space X is normal if and only if every its closed subset is 0-embedded.*

Proof: We only need to prove the sufficiency. Let A and B be disjoint closed subsets of X. Then A is a functionally closed subset of $E = A \cup B$. Since E is closed in X, E is a 0-embedded set. Therefore, there is a functionally closed set $\tilde{A}$ in X such that $A = E \cap \tilde{A}$. Then B is a functionally closed subset of the closed set $D = \tilde{A} \cup B$. Since D is 0-embedded in X, there exists a functionally closed set $\tilde{B}$ in X such that $B = D \cap \tilde{B}$. It is easy to check that $\tilde{A} \cap \tilde{B} = \emptyset$. If $f : X \to [0, 1]$ be a continuous function such that $\tilde{A} = f^{-1}(0)$ and $\tilde{B} = f^{-1}(1)$, then the sets $U = f^{-1}([0, 1/2))$ and $V = f^{-1}((1/2, 1])$ are disjoint and open in X, $A \subseteq U$ and $B \subseteq V$. Hence, X is a normal space. □

An analog of the previous proposition is valid for hereditarily α-separated spaces. We say that a topological space X is *hereditarily α-separated* if every its subspace is α-separated.

Proposition 3.8. *Let $0 \le \alpha < \omega_1$ and let X be a hereditarily α-separated space. If every subset of the $(\alpha + 1)$-th multiplicative class in X is $(\alpha + 1)$-embedded, then X is $(\alpha + 1)$-separated.*

Proof: Let $A, B \subseteq X$ be disjoint sets of the $(\alpha+1)$-th multiplicative class. Then A is ambiguous of the class $(\alpha+1)$ in $E = A\cup B$. Since E belongs to the $(\alpha+1)$-th multiplicative class in X, E is $(\alpha + 1)$-embedded. Moreover, E is α-separated as a subspace of the hereditarily α-separated space X. According to Theorem 3.6(3) A is functionally ambiguous of the $(\alpha + 1)$-th class in E. Therefore, there is a set $\tilde{A}$ of the $(\alpha + 1)$-th functionally multiplicative class in X such that $A = E \cap \tilde{A}$. Then B is a functionally ambiguous subset of the class $(\alpha + 1)$ in $D = \tilde{A} \cup B$. Since D belongs to the $(\alpha+1)$-th multiplicative class in X, D is $(\alpha+1)$-embedded. Therefore, there exists a set $\tilde{B}$ of the $(\alpha+1)$-th functionally multiplicative class in X such that $B = D\cap\tilde{B}$. It is easy to check that $\tilde{A}\cap\tilde{B} = \emptyset$. Hence, the sets $\tilde{A}$ and

$\tilde{B}$ are $(\alpha+1)$-separated by Proposition 3.4. Then A and B are $(\alpha+1)$-separated too. □

Remark that the Alexandroff compactification of the real line $\mathbb{R}$ endowed with the discrete topology is a hereditarily normal space which is not 1-separated.

We give some examples below of α-separated subsets of a completely regular space.

Proposition 3.9. *Let X be a completely regular space and $A, B \subseteq X$ are disjoint sets. Then*

(a) *if A and B are Lindelöf G_δ-sets, then they are 1-separated;*
(b) *if A is a Lindelöf hereditarily Baire space and B is a functionally G_δ-set, then A and B are 1-separated;*
(c) *if A is Lindelöf and B is an F_σ-set, then A and B are 2-separated.*

Proof: (a) Let $A = \bigcap_{n=1}^{\infty} U_n$, where U_n is an open set in X for every $n \in \mathbb{N}$. Since X is completely regular, $U_n = \bigcup_{s\in S_n} U_{s,n}$ for every $n \in \mathbb{N}$ such that all the sets $U_{s,n}$ are functionally open in X. Then for every $n \in \mathbb{N}$ there is a countable set $S_{n,0} \subseteq S_n$ such that $A \subseteq \bigcup_{s\in S_{n_0}} U_{s,n}$, since A is Lindelöf. Let $V_n = \bigcup_{s\in S_{n_0}} U_{s,n}$, $n \in \mathbb{N}$. Obviously, every V_n is a functionally open set and $A = \bigcap_{n=1}^{\infty} V_n$. Hence, A is a functionally G_δ-subset of X. Analogously, B is also a functionally G_δ-set. Therefore, the sets A and B are 1-separated by Proposition 3.4.

(b) According to [7, Proposition 12] there is a functionally G_δ-set C in X such that $A \subseteq C \subseteq X \setminus B$. Taking a function $f \in K_1(X)$ such that $C = f^{-1}(0)$ and $B = f^{-1}(1)$, we obtain that A and B are 1-separated.

(c) Let $X \setminus B = \bigcap_{n=1}^{\infty} U_n$, where $(U_n)_{n=1}^{\infty}$ is a sequence of open subsets of X. Then $U_n = \bigcup_{s\in S_n} U_{s,n}$ for every $n \in \mathbb{N}$ such that all the sets $U_{s,n}$ are functionally open in X. Since A is Lindelöf, $A \subseteq V_n = \bigcup_{s\in S_{n_0}} U_{s,n}$, where the set S_{n_0} is countable for every $n \in \mathbb{N}$. Denote $C = \bigcap_{n=1}^{\infty} V_n$. Then C is a functionally G_δ-set in X and $A \subseteq C \subseteq X \setminus B$. Since C is a functionally ambiguous set of the second class, A and B are 2-separated. □

The following example shows that the class of separation of sets A and B in Proposition 3.9(c) cannot be made lower.

Example 3.10. There exist a metrizable space X and its disjoint Lindelöf F_σ-subsets A and B, which are not 1-separated.

Proof: Let $X = \mathbb{R}$, $A = \mathbb{Q}$ and B is a countable dense subsets of irrational numbers. Assume that A and B are 1-separated, i.e. there exist disjoint G_δ-sets C and D in $\mathbb{R}$ such that $A \subseteq C$ and $B \subseteq D$. Then $\overline{C} = \overline{D} = \mathbb{R}$, which implies a contradictions, since X is a Baire space. □

4. Ambiguously α-embedded sets

Let $0 < \alpha < \omega_1$. A subset E of a topological space X is *ambiguously α-embedded in X* if for any functionally ambiguous set A of the class α in E there exists a functionally ambiguous set B of the class α in X such that $A = B \cap E$.

Proposition 4.1. *Let $0 < \alpha < \omega_1$ and let X be a topological space. Then every ambiguously α-embedded set E in X is α-embedded in X.*

Proof: Take a set $A \subseteq E$ of the α-th functionally additive class in E. Then A can be written as $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is a functionally ambiguous set of the class α in E for every $n \in \mathbb{N}$ by Lemma 3.1. Then there exists a sequence of functionally ambiguous sets B_n of the class α in X such that $A_n = B_n \cap E$ for every $n \in \mathbb{N}$. Let $B = \bigcup_{n=1}^{\infty} B_n$. Then the set B belongs to the α-th functionally additive class in X and $B \cap E = A$. □

We will need the following auxiliary fact.

Lemma 4.2 (Lemma 2.3 [8]). *Let $0 < \alpha < \omega_1$ and let X be a topological space. Then for any disjoint sets $A, B \subseteq X$ of the α-th functionally multiplicative class in X there exists a functionally ambiguous set C of the class α in X such that $A \subseteq C \subseteq X \setminus B$.*

Proof: Lemma 3.2 implies that there are disjoint functionally ambiguous sets E_1 and E_2 of the class α such that $E_1 \subseteq X \setminus A$, $E_2 \subseteq X \setminus B$ and $X = E_1 \cup E_2$. It remains to put $C = E_2$. □

Proposition 4.3. *Let $0 < \alpha < \omega_1$ and let X be a topological space. Then every α-embedded set E of the α-th functionally multiplicative class in X is ambiguously α-embedded in X.*

Proof: Consider a functionally ambiguous set A of the class α in E. Then there exists a set B of the α-th functionally multiplicative class in X such that $A = B \cap E$. Since E is of the α-th functionally multiplicative class in X, the set A is also of the same class in X. Analogously, the set $E \setminus A$ belongs to the α-th functionally multiplicative class in X. It follows from Lemma 4.2 that there exists a functionally ambiguous set C of the class α in X such that $A \subseteq C$ and $C \cap (E \setminus A) = \emptyset$. Clearly, $C \cap E = A$. Hence, the set E is ambiguously α-embedded in X. □

Example 4.4. There exists a 0-embedded F_σ-set $E \subseteq \mathbb{R}$ which is not ambiguously 1-embedded.

Proof: Let $E = \mathbb{Q}$. Obviously, E is a 0-embedded set. Consider any two disjoint A and B which are dense in E. Then A and B are simultaneously F_σ- and G_δ-sets in E. Assume that there exists an F_σ- and G_δ-set C in $\mathbb{R}$ such that $A = E \cap C$. Since $A \subseteq C$ and $B \subseteq \mathbb{R} \setminus C$, the sets C and $\mathbb{R} \setminus C$ are dense in $\mathbb{R}$. Moreover, the sets C and $\mathbb{R} \setminus C$ are G_δ in $\mathbb{R}$. It implies a contradiction, since $\mathbb{R}$ is a Baire space. □

Example 4.5. There exits a Borel non-measurable ambiguously 1-embedded subset of a perfectly normal compact space.

Proof: Let X be the "two arrows" space (see [5, p. 212]), i.e. $X = X_0 \cup X_1$ where $X_0 = \{(x,0) : x \in (0,1]\}$ and $X_1 = \{(x,1) : x \in [0,1)\}$. The topology base on X is generated by the sets

$$((x - \frac{1}{n}, x] \times \{0\}) \cup ((x - \frac{1}{n}, x) \times \{1\}) \text{ if } x \in (0,1] \text{ and } n \in \mathbb{N}$$

and

$$((x, x + \frac{1}{n}) \times \{0\}) \cup ([x, x + \frac{1}{n}) \times \{1\}) \text{ if } x \in [0,1) \text{ and } n \in \mathbb{N}.$$

For a set $A \subseteq X$ we denote

$$A^+ = \{x \in [0,1] : (x,1) \in A\} \text{ and } A^- = \{x \in [0,1] : (x,0) \in A\}.$$

It is not hard to verify that for every open or closed set $A \subseteq X$ we have $|A^+ \Delta A^-| \le \aleph_0$. It follows that $|B^+ \Delta B^-| \le \aleph_0$ for any Borel measurable set $B \subseteq X$.

Let $E = X_0$. Since $E^+ = \emptyset$ and $E^- = (0,1]$, the set E is non-measurable. We show that E is an ambiguously 1-embedded set. Indeed, let $A \subseteq E$ be an F_σ- and G_δ-subset of E. Then $B = E \setminus A$ is also an F_σ- and G_δ-subset of E. Let $\tilde{A}$ and $\tilde{B}$ be G_δ-sets in X such that $A = \tilde{A} \cap E$ and $B = \tilde{B} \cap E$. The inequalities $|\tilde{A}^+ \Delta \tilde{A}^-| \le \aleph_0$ and $|\tilde{B}^+ \Delta \tilde{B}^-| \le \aleph_0$ imply that $|C| \le \aleph_0$, where $C = \tilde{A} \cap \tilde{B}$. Hence, C is an F_σ-set in X. Moreover, C is a G_δ-set in X. Therefore, $\tilde{A} \setminus C$ and $\tilde{B} \setminus C$ are G_δ-sets in X. According to Lemma 4.2, there is an F_σ- and G_δ-set D in X such that $\tilde{A} \setminus C \subseteq D$ and $D \cap (\tilde{B} \setminus C) = \emptyset$. Then $D \cap E = A$. □

5. Extension of real-valued K_α-functions

Analogs of Proposition 5.1 and Theorem 5.3 for $\alpha = 1$ were proved in [7].

Proposition 5.1. *Let X be a topological space, $E \subseteq X$ and $0 < \alpha < \omega_1$. Then the following conditions are equivalent:*

(i) *E is K^*_α-embedded in X;*
(ii) *E is ambiguously α-embedded in X;*
(iii) *$(X, E, [c,d])$ has the K_α-extension property for any segment $[c,d] \subseteq \mathbb{R}$.*

Proof: (i)$\Longrightarrow$(ii) Take an arbitrary functionally ambiguous set A of the class α in E and consider its characteristic function χ_A. Then $\chi_A \in K^*_\alpha(E)$, as is easy to check. Let $f \in K_\alpha(X)$ be an extension of χ_A. Then the sets $f^{-1}(1)$ and $f^{-1}(0)$ are disjoint and belong to the α-th functionally multiplicative class in X. According to Lemma 4.2 there exists a functionally ambiguous set B of the class α in X such that $f^{-1}(1) \subseteq B$ and $B \cap f^{-1}(0) = \emptyset$. It remains to notice that $B \cap E = f^{-1}(1) \cap E = \chi_A^{-1}(1) = A$. Hence, E is an ambiguously α-embedded set in X.

(ii)$\Longrightarrow$(iii) Let $f \in K_\alpha(E,[c,d])$. Define

$$h_1(x) = \begin{cases} f(x), & \text{if } x \in E, \\ \inf f(E), & \text{if } x \in X \setminus E, \end{cases}$$

$$h_2(x) = \begin{cases} f(x), & \text{if } x \in E, \\ \sup f(E), & \text{if } x \in X \setminus E, \end{cases}$$

Then $c \le h_1(x) \le h_2(x) \le d$ for all $x \in X$.

We prove that for any reals $a < b$ there exists a function $h \in K_\alpha(X)$ such that

$$h_2^{-1}([c,a]) \subseteq h^{-1}(0) \quad \text{and} \quad h_1^{-1}([b,d]) \subseteq h^{-1}(1).$$

Fix $a < b$. Without loss of generality we may assume that

$$\inf f(E) \le a < b \le \sup f(E).$$

Denote

$$A_1 = f^{-1}([c,a]), \quad A_2 = f^{-1}([b,d]).$$

Then A_1 and A_2 are disjoint sets of the α-th functionally multiplicative class in E. Using Lemma 4.2, we choose a functionally ambiguous set C of the class α in E such that $A_1 \subseteq C$ and $C \cap A_2 = \emptyset$. Since E is an ambiguously α-embedded set in X, there exists such a functionally ambiguous set D of the class α in X that $D \cap E = C$. Moreover, by Proposition 4.1 there exist sets B_1 and B_2 of the α-th functionally multiplicative class in X such that $A_i = E \cap B_i$ when $i = 1, 2$. Let

$$\tilde{A}_1 = D \cap B_1, \quad \tilde{A}_2 = (X \setminus D) \cap B_2.$$

Then the sets $\tilde{A}_1$ and $\tilde{A}_2$ are disjoint and belong to the α-th functionally multiplicative class in X. Moreover, $A_1 = E \cap \tilde{A}_1$ and $A_2 = E \cap \tilde{A}_2$. According to Proposition 3.4 there is a function $h \in K_\alpha^*(X)$ such that

$$h^{-1}(0) = \tilde{A}_1 \quad \text{and} \quad h^{-1}(1) = \tilde{A}_2.$$

According to [12, Theorem 3.2] there exists a function $g \in K_\alpha(X)$ such that

$$h_1(x) \le g(x) \le h_2(x)$$

for all $x \in X$. Clearly, g is an extension of f and $g \in K_\alpha(X,[c,d])$.

(iii)$\Longrightarrow$(i) Let $f \in K_\alpha^*(E)$ and let $|f(x)| \le C$ for all $x \in E$. Consider a function $g \in K_\alpha(X)$ which is an extension of f. Define a function $r : \mathbb{R} \to [-C, C]$, $r(x) = \min\{C, \max\{x, -C\}\}$. Obviously, r is continuous. Let $h = r \circ g$. Then $h \in K_\alpha^*(X)$ and $h|_E = f$. Hence, E is K_α^*-embedded in X. □

Lemma 5.2. *Let $0 < \alpha < \omega_1$, X be a topological space and let $E \subseteq X$ be such an α-embedded set in X that for any set A of the α-th functionally multiplicative class in X with $E \cap A = \emptyset$ the sets E and A are α-separated. Then E is an ambiguously α-embedded set.*

PROOF: Consider a functionally ambiguous set C of the class α in E and denote $C_1 = C$, $C_2 = E \setminus C$. Then there exist sets $\tilde{C}_1$ and $\tilde{C}_2$ of the α-th functionally multiplicative class in X such that $\tilde{C}_i \cap E = C_i$ when $i = 1, 2$. Then the set $A = \tilde{C}_1 \cap \tilde{C}_2$ is of the α-th functionally multiplicative class in X and $A \cap E = \emptyset$. Let $h \in K_\alpha(X)$ be a function such that $E \subseteq h^{-1}(0)$ and $A \subseteq h^{-1}(1)$. Denote $H = h^{-1}(0)$ and $H_i = H \cap \tilde{C}_i$ when $i = 1, 2$. Since H_1 and H_2 are disjoint sets of the α-th functionally multiplicative class in X, by Lemma 4.2 there is a functionally ambiguous set D of the class α in X such that $H_1 \subseteq D \subseteq X \setminus H_2$. Obviously, $D \cap E = C$. □

Theorem 5.3. *Let $0 < \alpha < \omega_1$ and let E be a subset of a topological space X. Then the following conditions are equivalent:*

(i) *E is K_α-embedded in X;*
(ii) *E is α-embedded in X and for any set A of the α-th functionally multiplicative class in X such that $E \cap A = \emptyset$ the sets E and A are α-separated.*

PROOF: (i)$\Longrightarrow$(ii) Let $C \subseteq E$ be a set of the α-th functionally multiplicative class in E. Then by Lemma 3.3 we choose a function $f \in K^*_\alpha(E)$ such that $C = f^{-1}(0)$. If $g \in K_\alpha(X)$ is an extension of f, then the set $B = g^{-1}(0)$ belongs to the α-th functionally multiplicative class in X and $B \cap E = C$. Hence, E is an α-embedded set in X.

Now consider a set A of the α-th functionally multiplicative class in X such that $E \cap A = \emptyset$. According to Lemma 3.3 there is a function $h \in K^*_\alpha(X)$ such that $A = h^{-1}(0)$. For all $x \in E$ let $f(x) = \frac{1}{h(x)}$. Then $f \in K_\alpha(E)$. Let $g \in K_\alpha(X)$ be an extension of f. For all $x \in X$ let $\varphi(x) = g(x) \cdot h(x)$. Clearly, $\varphi \in K_\alpha(X)$. It is not hard to verify that $E \subseteq \varphi^{-1}(1)$ and $A \subseteq \varphi^{-1}(0)$.

(ii)$\Longrightarrow$(i) Let us remark that according to Lemma 5.2 the set E is ambiguously α-embedded in X.

Let $f \in K_\alpha(E)$ and let $\varphi : \mathbb{R} \to (-1, 1)$ be a homeomorphism. Using Proposition 5.1 to the function $\varphi \circ f : E \to [-1, 1]$ we have that there exists a function $h \in K_\alpha(X, [-1, 1])$ such that $h|_E = \varphi \circ f$. Let

$$A = h^{-1}(-1) \cup h^{-1}(1).$$

Then A belongs to the α-th functionally multiplicative class in X and $A \cap E = \emptyset$. Therefore, there exists a function $\psi \in K_\alpha(X)$ such that $A \subseteq \psi^{-1}(0)$ and $E \subseteq \psi^{-1}(1)$. For all $x \in X$ define

$$g(x) = \varphi^{-1}(h(x) \cdot \psi(x)).$$

Remark that $g \in K_\alpha(X)$ and $g|_E = f$. □

Corollary 5.4. *Let $0 < \alpha < \omega_1$ and let E be a subset of the α-th functionally multiplicative class of a topological space X. Then the following conditions are equivalent:*

(i) *E is K_α-embedded in X;*
(ii) *E is α-embedded in X.*

6. K_1^*-embedding versus K_1-embedding

A family $\mathcal{U}$ of non-empty open sets of a space X is called *a π-base* [4] if for any non-empty open set V of X there is $U \in \mathcal{U}$ with $V \subseteq U$.

Proposition 6.1. *Let X be a perfect space of the first category with a countable π-base. Then there exist disjoint F_σ- and G_δ-subsets A and B of X which are dense in X and $X = A \cup B$.*

Proof: Let $(V_n : n \in \mathbb{N})$ be a π-base in X and $X = \bigcup_{n=1}^\infty X_n$, where X_n is a closed nowhere dense subset of X for every $n \geq 1$. Let $E_1 = X_1$ and $E_n = X_n \setminus \bigcup_{k<n} X_k$ for $n \geq 2$. Then E_n is a nowhere dense F_σ- and G_δ-subset of X for every $n \geq 1$, $E_n \cap E_m = \emptyset$ if $n \neq m$, and $X = \bigcup_{n=1}^\infty E_n$.

Let $m_0 = 0$. We choose a number $n_1 \geq 1$ such that $(\bigcup_{n=1}^{n_1} E_n) \cap V_1 \neq \emptyset$ and let $A_1 = \bigcup_{n=1}^{n_1} E_n$. Since $\overline{X \setminus A_1} = X$, there exists a number $m_1 > n_1$ such that $(\bigcup_{n=n_1+1}^{m_1} E_n) \cap V_1 \neq \emptyset$. Set $B_1 = \bigcup_{n=n_1+1}^{m_1} E_n$. It follows from the equality $\overline{X \setminus (A_1 \cup B_1)} = X$ that there exists $n_2 > m_1$ such that $(\bigcup_{n=m_1+1}^{n_2} E_n) \cap V_2 \neq \emptyset$. Further, there is such $m_2 > n_2$ that $(\bigcup_{n=n_2+1}^{m_2} E_n) \cap V_2 \neq \emptyset$. Let $A_2 = \bigcup_{n=m_1+1}^{n_2} E_n$ and $B_2 = \bigcup_{n=n_2+1}^{m_2} E_n$. Repeating this process, we obtain the sequence of numbers

$$m_0 < n_1 < m_1 < \dots < n_k < m_k < n_{k+1} < \dots$$

and the sequence of sets

$$A_k = \bigcup_{n=m_{k-1}+1}^{n_k} E_n, \quad B_k = \bigcup_{n=n_k+1}^{m_k} E_n, \quad k \geq 1,$$

such that $A_k \cap V_k \neq \emptyset$ and $B_k \cap V_k \neq \emptyset$ for every $k \geq 1$.

Let $A = \bigcup_{k=1}^\infty A_k$ and $B = \bigcup_{k=1}^\infty B_k$. Clearly, $X = A \cup B$, $A \cap B = \emptyset$ and $\overline{A} = \overline{B} = X$. Moreover, A and B are F_σ-sets in X. Therefore, A and B are F_σ- and G_δ-subsets of X. □

We say that a topological space X *hereditarily has a countable π-base* if every its closed subspace has a countable π-base.

Proposition 6.2. *Let X be a hereditarily Baire space, E be a perfectly normal ambiguously 1-embedded subspace of X which hereditarily has a countable π-base. Then E is a hereditarily Baire space.*

Proof: Assume that E is not a hereditarily Baire space. Then there exists a nonempty closed set $C \subseteq X$ of the first category. Notice that C is a perfectly normal space with a countable π-base. According to Proposition 6.1 there exist disjoint dense F_σ- and G_δ-subsets A and B of C such that $C = A \cup B$. Since C is F_σ- and G_δ-set in E, the sets A and B are also F_σ and G_δ in E. Therefore there exist disjoint functionally F_σ- and G_δ-subsets $\tilde{A}$ and $\tilde{B}$ of X such that $A = \tilde{A} \cap E$ and $B = \tilde{B} \cap E$. Notice that the sets $\tilde{A}$ and $\tilde{B}$ are dense in $\overline{C}$. Taking into account that X is hereditarily Baire, we have that $\overline{C}$ is a Baire space. It follows a contradiction, since $\tilde{A}$ and $\tilde{B}$ are disjoint dense G_δ-subsets of $\overline{C}$. □

Remark that there exist a metrizable separable Baire space X and its ambiguously 1-embedded subspace E which is not a Baire space. Indeed, let $X = (\mathbb{Q} \times \{0\}) \cup (\mathbb{R} \times (0, 1])$ and $E = \mathbb{Q} \times \{0\}$. Then E is closed in X. Therefore, any F_σ- and G_δ-subset C of E is also F_σ- and G_δ- in X. Hence, E is an ambiguously 1-embedded set in X.

Theorem 6.3. *Let X be a hereditarily Baire space and let $E \subseteq X$ be its perfect Lindelöf subspace which hereditarily has a countable π-base. Then E is K_1^*-embedded in X if and only if E is K_1-embedded in X.*

Proof: Since the sufficiency is obvious, we only need to prove the necessity.

According to Proposition 5.1 the set E is ambiguously 1-embedded in X. Using Proposition 6.2, we have E is a hereditarily Baire space. Since E is Lindelöf, Proposition 3.9(b) implies that E is 1-separated from any functionally G_δ-set A of X such that $A \cap E = \emptyset$. Therefore, by Theorem 5.3 the set E is K_1-embedded in X. □

7. A generalization of the Kuratowski theorem

K. Kuratowski [11, p. 445] proved that every mapping $f \in K_\alpha(E, Y)$ has an extension $g \in K_\alpha(X, Y)$ in the case when X is a metric space, Y is a Polish space and $E \subseteq X$ is a set of the multiplicative class $\alpha > 0$.

In this section we will prove that the Kuratowski Extension Theorem is still valid if X is a topological space and E is a K_α-embedded subset of X.

We say that a subset A of a space X is *discrete* if any point $a \in A$ has a neighborhood $U \subseteq X$ such that $U \cap A = \{a\}$.

Theorem 7.1 ([8, Theorem 2.11]). *Let X be a topological space, Y be a metrizable separable space, $0 \le \alpha < \omega_1$ and $f \in K_\alpha(X, Y)$. Then there exists a sequence $(f_n)_{n=1}^\infty$ such that*

(i) *$f_n \in K_\alpha(X, Y)$ for every n;*
(ii) *$(f_n)_{n=1}^\infty$ is uniformly convergent to f;*
(iii) *$f_n(X)$ is at most countable and discrete for every n.*

Proof: Consider a metric d on Y which generates its topological structure. Since (Y, d) is metric separable space, for every n there is a subset $Y_n = \{y_{i,n} : i \in I_n\}$

of Y such that Y_n is discrete, $|I_n| \le \aleph_0$ and for any $y \in Y$ there exists $i \in I_n$ such that $d(y, y_{i,n}) < 1/n$ (see [11, p. 226]).

For every $n \in \mathbb{N}$ and $i \in I_n$ put $A_{i,n} = \{x \in X : d(f(x), y_{i,n}) < 1/n\}$. Then each $A_{i,n}$ belongs to the α-th functionally additive class in X and $\bigcup_{i\in I_n} A_{i,n} = X$ for every n. According to Lemma 3.2 for every n we can choose a sequence $(F_{i,n})_{i\in I_n}$ of disjoint functionally ambiguous sets of the class α such that $F_{i,n} \subseteq A_{i,n}$ and $\bigcup_{i\in I_n} F_{i,n} = X$.

For all $x \in X$ and $n \in \mathbb{N}$ let $f_n(x) = y_{i,n}$ if $x \in F_{i,n}$ for some $i \in I_n$. Notice that $f_n \in K_\alpha(X, Y)$ for every $n \in \mathbb{N}$.

It remains to prove that the sequence $(f_n)_{n=1}^\infty$ is uniformly convergent to f. Indeed, fix $x \in X$ and $n \in \mathbb{N}$. Then there exists $i \in I_n$ such that $x \in F_{i,n}$. Since $F_{i,n} \subseteq A_{i,n}$, $d(f(x), f_n(x)) = d(f(x), y_{i,n}) < \frac{1}{n}$, which completes the proof. □

Recall that a family $(A_s : s \in S)$ of subsets of a topological space X is called *a partition of* X if $X = \bigcup_{s\in S} A_s$ and $A_s \cap A_t = \emptyset$ for all $s \neq t$.

Proposition 7.2. *Let $0 < \alpha < \omega_1$, X be a topological space, $E \subseteq X$ be an α-embedded set which is α-separated from any disjoint set of the α-th functionally multiplicative class in X and let $(A_n : n \in \mathbb{N})$ be a partition of E by functionally ambiguous sets of the class α in E. Then there is a partition $(B_n : n \in \mathbb{N})$ of X by functionally ambiguous sets of the class α in X such that $A_n = E \cap B_n$ for every $n \in \mathbb{N}$.*

Proof: According to Proposition 5.2 for every $n \in \mathbb{N}$ there exists a functionally ambiguous set D_n of the class α in X such that $A_n = D_n \cap E$. By the assumption there exists a function $f \in K_\alpha(X)$ such that $E \subseteq f^{-1}(0)$ and $X \setminus \bigcup_{n=1}^\infty D_n \subseteq f^{-1}(1)$. Let $D = f^{-1}(0)$. Then the set $X \setminus D$ is of the α-th functionally additive class in X. Then there exists a sequence $(E_n)_{n=1}^\infty$ of functionally ambiguous set of the class α in X such that $X \setminus D = \bigcup_{n=1}^\infty E_n$. For every $n \in \mathbb{N}$ denote $C_n = E_n \cup D_n$. Then all the sets C_n are functionally ambiguous of the class α in X and $\bigcup_{n=1}^\infty C_n = X$. Let $B_1 = C_1$ and $B_n = C_n \setminus (\bigcup_{k<n} C_k)$ for $n \ge 2$. Clearly, every B_n is a functionally ambiguous set of the class α in X, $B_n \cap B_m = \emptyset$ if $n \neq m$ and $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty C_n = X$. Moreover, $B_n \cap E = A_n$ for every $n \in \mathbb{N}$. □

Let $0 \le \alpha < \omega_1$, X and Y be topological spaces and $E \subseteq X$. We say that a collection (X, E, Y) has *the K_α-extension property* if every mapping $f \in K_\alpha(E, Y)$ can be extended to a mapping $g \in K_\alpha(X, Y)$.

Theorem 7.3. *Let $0 < \alpha < \omega_1$ and let E be a subset of a topological space X. Then the following conditions are equivalent:*

(i) *E is K_α-embedded in X;*
(ii) *(X, E, Y) has the K_α-extension property for any Polish space Y.*

Proof: Since the implication (ii)⇒(i) is obvious, we only need to prove the implication (i)⇒(ii). Let Y be a Polish space with a metric d which generates its topological structure and (Y, d) is complete and let $f \in K_\alpha(E, Y)$.

It follows from Theorem 7.1 that there exists a sequence of mappings $f_n \in K_\alpha(E,Y)$ which is uniformly convergent to f on E. Moreover, for every $n\in\mathbb{N}$ the set $f_n(E)=\{y_{i_n,n}: i_n\in I_n\}$ is at most countable and discrete. We may assume that each $f_n(E)$ consists of distinct points.

For every $n\in\mathbb{N}$ and for each $(i_1,\dots,i_n)\in I_1\times\dots\times I_n$ let

$$B_{i_1,\dots,i_n}=f_1^{-1}(y_{i_1,1})\cap\dots\cap f_n^{-1}(y_{i_n,n}).$$

Then for each $i_1\in I_1$, …, $i_n\in I_n$ the set $B_{i_1\dots i_n}$ is functionally ambiguous of the class α in E and the family $(B_{i_1,\dots,i_n}: i_1\in I_1,\dots,i_n\in I_n)$ is a partition of E for every $n\in\mathbb{N}$. By Proposition 7.2 we choose a sequence of systems of functionally ambiguous sets $D_{i_1\dots i_n}$ of the class α in X such that

(1) $D_{i_1,\dots,i_n}\cap E=B_{i_1,\dots,i_n}$ for every $n\in\mathbb{N}$ and $(i_1,\dots,i_n)\in I_1\times\dots\times I_n$;
(2) $(D_{i_1,\dots,i_n}: i_1\in I_1,\dots,i_n\in I_n)$ is a partition of X for every $n\in\mathbb{N}$.

For all $n\in\mathbb{N}$ and $(i_1,\dots,i_n)\in I_1\times\dots\times I_n$ let

(3) $D_{i_1,\dots,i_n}=\emptyset$, if $B_{i_1,\dots,i_n}=\emptyset$.

Notice that the system $(B_{i_1,\dots,i_n,i_{n+1}}: i_{n+1}\in I_{n+1})$ forms a partition of the set $B_{i_1,\dots,i_n}$ for every $n\in\mathbb{N}$.

For all $i_1\in I_1$ let

$$C_{i_1}=D_{i_1}.$$

Assume that for some $n\geq 1$ the system $(C_{i_1,\dots,i_n}: i_1\in I_1,\dots,i_n\in I_n)$ of functionally ambiguous sets of the class α in X is already defined and

(A) $B_{i_1,\dots,i_n}=E\cap C_{i_1,\dots,i_n}$;
(B) $(C_{i_1,\dots,i_n}: i_1\in I_1,\dots,i_n\in I_n)$ is a partition of X;
(C) $C_{i_1,\dots,i_n}=\emptyset$ if $B_{i_1,\dots,i_n}=\emptyset$;
(D) $(C_{i_1\dots i_{n-1}i_n}: i_n\in I_n)$ is a partition of the set $C_{i_1,\dots,i_{n-1}}$.

Fix $i_1,\dots,i_n$. Since the set $K=C_{i_1,\dots,i_n}\setminus\bigcup_{k\in I_{n+1}}D_{i_1,\dots,i_n,k}$ is of the α-th functionally multiplicative class in X and $K\cap E=\emptyset$, there exists a set H of the α-th functionally multiplicative class in X such that $E\subseteq H\subseteq X\setminus K$. Using [8, Lemma 2.1] we obtain that there exists a sequence $(A_k)_{k=1}^\infty$ of disjoint functionally ambiguous sets of the class α in X such that

$$C_{i_1,\dots,i_n}\setminus H=\bigcup_{k=1}^\infty A_k.$$

Let

$$M_{i_1,\dots,i_n,i_{n+1}}=\emptyset,\quad\text{if}\quad D_{i_1,\dots i_n,i_{n+1}}=\emptyset,$$

and

$$M_{i_1,\dots i_n,i_{n+1}}=(A_{i_{n+1}}\cup D_{i_1,\dots,i_n,i_{n+1}})\cap C_{i_1,\dots,i_n},\quad\text{if}\quad D_{i_1,\dots,i_n,i_{n+1}}\neq\emptyset.$$

Now let

$$C_{i_1,\dots,i_n,1}=M_{i_1,\dots,i_n,1},$$

and

$$C_{i_1,\dots,i_n,i_{n+1}} = M_{i_1,\dots,i_n,i_{n+1}} \setminus \bigcup_{k<i_{n+1}} M_{i_1,\dots,i_n,k} \text{ if } i_{n+1} > 1.$$

Then for every $n \in \mathbb{N}$ the system $(C_{i_1,\dots,i_n} : i_1 \in I_1, \dots, i_n \in I_n)$ of functionally ambiguous sets of the class α in X has the properties (A)–(D).

For each $n \in \mathbb{N}$ and $x \in X$ let

$$g_n(x) = y_{i_n,n},$$

if $x \in C_{i_1,\dots,i_n}$. It is not hard to prove that $g_n \in K_\alpha(X,Y)$.

We show that the sequence $(g_n)_{n=1}^\infty$ is uniformly convergent on X. Indeed, let $x_0 \in X$ and $n, m \in \mathbb{N}$. Without loss of generality, we may assume that $n \geq m$. By the property (B), $x_0 \in C_{i_1,\dots,i_n} \cap C_{j_1,\dots,j_m}$. It follows from (B) and (D) that $i_1 = j_1, \dots, i_m = j_m$. Take an arbitrary point x from the set $B_{i_1,\dots,i_n}$, the existence of which is guaranteed by the property (C). Then $f_m(x) = y_{i_m,m} = g_m(x_0)$ and $f_n(x) = y_{i_n,n} = g_n(x_0)$. Since the sequence $(f_n)_{n=1}^\infty$ is uniformly convergent on E, $\lim_{n,m\to\infty} d(y_{i_m,m}, y_{i_n,n}) = 0$. Hence, the sequence $(g_n)_{n=1}^\infty$ is uniformly convergent on X.

Since Y is a complete space, for all $x \in X$ define $g(x) = \lim_{n\to\infty} g_n(x)$. According to the property (A), $g(x) = f(x)$ for all $x \in E$. Moreover, $g \in K_\alpha(X,Y)$ as a uniform limit of functions from the class K_α. □

8. Open problems

Question 8.1. *Does there exist a completely regular not perfectly normal space in which any functionally G_δ-set is* 1*-embedded?*

Question 8.2. *Does there exist a completely regular not perfectly normal space in which any set is* 1*-embedded?*

Question 8.3. *Do there exist a normal space and its functionally G_δ-subset which is not* 1*-embedded?*

Question 8.4. *Do there exist a topological space X and its subspace E such that E is K_1^*-embedded and is not K_1-embedded in X?*

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Chernivtsi National University, Department of Mathematical Analysis, Kotsjubyns'koho 2, Chernivtsi 58012, Ukraine

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