Asymptotic behavior of positive solutions of a Dirichlet problem involving supercritical nonlinearities

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Abstract. Let $p > 1$, $q > p$, $\lambda > 0$ and $s \in]1, p[$. We study, for $s \to p^-$, the behavior of positive solutions of the problem $-\Delta_p u = \lambda u^{s-1} + u^{q-1}$ in Ω, $u_{|\partial\Omega} = 0$. In particular, we give a positive answer to an open question formulated in a recent paper of the first author.

Keywords: elliptic boundary value problems; positive solutions; variational methods; asymptotic behavior; combined nonlinearities

Classification: 35J20, 35J25

1. Introduction

Throughout this paper, $\Omega \subset \mathbb{R}^N$ is a nonempty connected open bounded set with sufficiently regular boundary $\partial\Omega$. Let $p > 1$, $s \in]1, p[$ and $q > p$. Moreover, denote by λ_p the first eigenvalue of the p-Laplacian operator $\Delta_p(\cdot) := \operatorname{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$ on Ω. It is known that, for any $\lambda \in]0, \lambda_p[$, the problem

$$(P) \qquad \begin{cases} -\Delta_p u = \lambda u^{s-1} + u^{q-1} & \text{in } \Omega, \\ u_{|\partial\Omega} = 0 \end{cases}$$

has, for s sufficiently close to p, at least one positive (weak) solution of least energy, which we denote by $v_{\lambda,s}$, whenever the exponent q is subcritical, that is $q \leq p_N := \frac{pN}{N-p}$ if $N > p$ (see [2] or [5] for instance). In particular, in [2] (Theorem 4) the existence of a constant $c > 0$, depending only on p, N, Ω, such that

$$(1) \qquad \lim_{s \to p^-} \left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} \int_\Omega v_{\lambda,s}^s \, dx = c,$$

is established. The constant c also satisfies

$$(2) \qquad \lim_{s \to p^-} \left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} \int_\Omega u_{\lambda,s}^s \, dx = c,$$

where $u_{\lambda,s}$ is the unique positive solution of the problem

$$(P_0) \qquad \begin{cases} -\Delta_p u = \lambda u^{s-1} & \text{in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

When the exponent q is supercritical ($> p_N$) and $s < p$, using a sub-supersolution technique, the existence of at least one positive solution for problem (P) is proved in [3] for all $\lambda < \tilde{\Lambda}_{spq}$, where

$$\tilde{\Lambda}_{spq} = (\max_{\overline{\Omega}} |v_1|)^{-\frac{(p-1)(q-s)}{q-p}} \cdot \frac{(p-s)^{\frac{p-s}{q-p}}(q-p)}{(q-s)^{\frac{q-s}{q-p}}}$$

and $v_1 \in C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is the unique positive solution of the problem

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

Note that, since $(\max_{\overline{\Omega}} |v_1|)^{1-p} \le \lambda_p$ (see Remark 3.2 of [3]), we have $\lim_{s\to p^-} \tilde{\Lambda}_{spq} = (\max_{\overline{\Omega}} |v_1|)^{1-p} \le \lambda_p$. Also, from Theorem 2 of [3], we infer that, if $\lambda > \lambda_p$, problem (P) with $s = p$ cannot have positive solutions. However, by the results of [3], we do not know whether $\lim_{s\to p^-} \tilde{\Lambda}_{spq} = \lambda_p$. So, it could be interesting to know if there exists a constant $\Lambda_{spq} > 0$ such that, for all $\lambda \in]0, \Lambda_{spq}[$, problem (P) has a positive solution and $\lim_{s\to p^-} \Lambda_{spq} = \lambda_p$. Observe that the previous fact is true in the case of q subcritical (see [2], [5]). Our result in extending Theorem 4 of [2] to the case $q \in]p, +\infty[$ (so giving a positive answer to the open problem formulated in [2]), also gives a positive answer to the above question.

2. Main result

Throughout this section, we always assume $p \in]1, N[$. For all $m \in [1, \infty]$, we denote by $\|\cdot\|_m$ the standard norm in the $L^m(\Omega)$ space. Also, we equip the space $W_0^{1,p}(\Omega)$ with the norm $\|\cdot\| := \|\nabla(\cdot)\|_p$ and denote by

$$c_m := \sup_{\|u\|=1} \|u\|_m.$$

the best Sobolev embedding constant of $W_0^{1,p}(\Omega)$ in $L^m(\Omega)$, for all $m \in [1, \frac{pN}{N-p}]$.

The following lemma follows by applying the well known Moser's iterative scheme ([4], [7]) and standard regularity results ([6])

Lemma 1. *Let $r > \frac{N}{p}$, $f \in L^r(\Omega)$ (resp. $f \in L^\infty(\Omega)$) and let $u_f \in W_0^{1,p}(\Omega)$ be the (unique) weak solution of the problem*

$$\begin{cases} -\Delta_p u = f(x) & \textit{in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

Then, $u_f \in C^1(\overline{\Omega})$ and

$$C_0^r \stackrel{\text{def}}{=} \sup_{f \in L^r(\Omega)\setminus\{0\}} \frac{\max_{\overline{\Omega}} |u_f|}{\|f\|_r^{\frac{1}{p-1}}} \qquad \left(\textit{resp.} \quad C_0 \stackrel{\text{def}}{=} \sup_{f \in L^\infty(\Omega)\setminus\{0\}} \frac{\max_{\overline{\Omega}} |u_f|}{\|f\|_\infty^{\frac{1}{p-1}}} \right)$$

is a positive finite constant.

As announced in the introduction, our main result (Theorem 1 below) extends Theorem 4 of [2] to the case of $q \in]p, +\infty[$. We observe that, by the proof of Theorem 1, one can see that the same result is still true if u^q is replaced with a more general nonlinearity $f(x,u)$, where $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is a Carathèodory function fulfilling, for some $C > 0$ and $\delta > 0$, the inequality $|f(x,t)| \leq Ct^q$ for a.a. $x \in \Omega$, and $t \in]0, \delta]$.

Theorem 1. *Let $\lambda_0 \in]0, \lambda_p[$ and $q > p$. Then, there exists $s_0 \in]1, p[$ such that, for all $s \in]s_0, p[$ and all $\lambda \in]0, \lambda_0]$, problem (P) admits a positive solution $v_{\lambda,s} \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$. Moreover, the solution $v_{\lambda,s}$ satisfies* (1).

Proof: Fix $\varepsilon \in]0,1[$ such that $\varepsilon^{q-p} < \lambda_p - \lambda_0$ and $\bar{s} \in]1, p[$. Let $s \in]\bar{s}, p[$ and put

$$g(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \lambda t^{s-1} + t^{q-1} & \text{if } t \in [0, \varepsilon], \\ \lambda t^{s-1} + \varepsilon^{q-s} t^{s-1} & \text{if } t \geq \varepsilon. \end{cases}$$

For each $\lambda \in]0, \lambda_0[$, consider the functional

$$I_{\lambda,s}(u) = \frac{1}{p}\|u\|^p - \frac{1}{s}\int_\Omega \left(\int_0^{u(x)} g(t)\,dt \right) dx, \quad u \in W_0^{1,p}(\Omega).$$

By standard arguments, $I_{\lambda,s}$ is (strongly) continuous, sequentially weakly lower-semicontinuous and Gâtéuax differentiable in $W_0^{1,p}(\Omega)$. Moreover, from

$$0 \leq g(t) \leq (\lambda + \varepsilon^{q-p})|t|^{s-1}, \quad \text{for all } t \in \mathbb{R}, \tag{3}$$

we obtain

$$\lim_{\|u\| \to +\infty} I_{\lambda,s}(u) = +\infty.$$

This implies that $I_{\lambda,s}$ has a global minimum $v_{\lambda,s} \in W_0^{1,p}(\Omega)$. It follows that $v_{\lambda,s}$ is a critical point of $I_{\lambda,s}$. Consequently, $v_{\lambda,s}$ is a weak solution of the problem

$$(P_g) \qquad \begin{cases} -\Delta_p u = g(u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

From standard regularity results, one has $v_{\lambda,s} \in C^1(\overline{\Omega})$ (see [6]). Note that, since $g(t) \geq 0$ for all $t \in \mathbb{R}$, we have that $v_{\lambda,s}$ is nonnegative in Ω. Finally, again from (3), it is easy to infer that $\inf_{W_0^{1,p}(\Omega)} I_{\lambda,s} < 0$. Therefore, $v_{\lambda,s}$ must be nonzero. Applying the Strong Maximum Principle, it follows that $v_{\lambda,s}$ is, actually, positive in Ω. At this point, fix

$$r > \max\left\{\frac{N}{p}, \frac{p_N}{\bar{s}-1}\right\}.$$

Using the constant C_0^r introduced in Lemma 1 and inequality (3), if we put $\delta = \frac{r(s-1)-p_N}{r(p-1)}$, we have

$$\begin{aligned}
\max_{\overline{\Omega}} v_{\lambda,s} &\leq C_0^r \|g(v_{\lambda,s})\|_r^{\frac{1}{p-1}} \leq C_0^r(\lambda+\varepsilon^{q-p})^{\frac{1}{p-1}}\left(\int_\Omega v_{\lambda,s}^{r(s-1)}\,dx\right)^{\frac{1}{r(p-1)}} \\
&\leq (\max_{\overline{\Omega}} v_{\lambda,s})^{\delta}\cdot C_0^r(\lambda+\varepsilon^{q-p})^{\frac{1}{p-1}}\cdot\left(\int_\Omega v_{\lambda,s}^{p_N}\,dx\right)^{\frac{1}{r(p-1)}} \\
&= (\max_{\overline{\Omega}} v_{\lambda,s})^{\delta}\cdot C_0^r(\lambda+\varepsilon^{q-p})^{\frac{1}{p-1}}\cdot\|v_{\lambda,s}\|_{p_N}^{\frac{p_N}{r(p-1)}}.
\end{aligned}$$

From the previous inequality, we obtain

$$(\max_{\overline{\Omega}} v_{\lambda,s})^{1-\delta} \leq C_0^r\cdot c_{p_N}\cdot(\lambda_0+\varepsilon^{q-p})^{\frac{1}{p-1}}\|v_{\lambda,s}\|^{\frac{p_N}{r(p-1)}}.$$

Therefore,

$$\max_{\overline{\Omega}} v_{\lambda,s} \leq C^{\frac{1}{1-\delta}}\|v_{\lambda,s}\|^{\frac{p_N}{r(p-1)(1-\delta)}}, \tag{4}$$

where C is a constant depending only on p, N, r, Ω, λ_0.

Since $v_{\lambda,s}$ is a critical point of $I_{\lambda,s}$, one has $I'_{\lambda,s}(v_{\lambda,s})(v_{\lambda,s}) = 0$, that is

$$\|v_{\lambda,s}\|^p = \lambda\int_\Omega g(v_{\lambda,s})v_{\lambda,s}\,dx.$$

Thus, using (3), we obtain

$$\|v_{\lambda,s}\|^p \leq (\lambda+\varepsilon^{q-p})\|v_{\lambda,s}\|_s^s \leq (\lambda+\varepsilon^{q-p})c_s^s\|v_{\lambda,s}\|^s.$$

Consequently,

$$\|v_{\lambda,s}\| \leq \left(\frac{\lambda+\varepsilon^{q-p}}{\lambda_p}\right)^{\frac{1}{p-s}}(\lambda_p c_s^s)^{\frac{1}{p-s}} \leq \left(\frac{\lambda_0+\varepsilon^{q-p}}{\lambda_p}\right)^{\frac{1}{p-s}}(\lambda_p c_s^s)^{\frac{1}{p-s}}. \tag{5}$$

Moreover, since the limit $\lim_{s\to p^-}(\lambda_p c_s^s)^{\frac{1}{p-s}}$ is finite (see [1]) and $\frac{\lambda_0+\varepsilon^{q-p}}{\lambda_p} < 1$, it follows, from (5), that $\lim_{s\to p^-}\|v_{\lambda,s}\| = 0$. Therefore, taking in mind (4) and that $1-\delta \to \frac{p_N}{r(p-1)}$ as $s \to p^-$, we have $\lim_{s\to p^-}\max_{\overline{\Omega}}|v_{\lambda,s}| = 0$, *uniformly* with respect to λ. Consequently, there exists $s_0 \in [\bar{s}, p[$, *independent of* λ, such that, for all $s \in [s_0, p[$, one has

$$\max_{\overline{\Omega}} v_{\lambda,s} \leq \varepsilon. \tag{6}$$

This means that, for each $s \in [s_0, p[$, $v_{\lambda,s}$ is, actually, a weak solution of problem (P). At this point, it remains to show that the limit (1) holds. Fix $\lambda \in]0, \lambda_0]$, $\sigma \in]p, p_N[$ and $\tilde{\varepsilon} > 0$ such that

$$\tilde{\varepsilon}^{q-p} < \min\left\{\lambda_p - \lambda_0,\ \left(\frac{\lambda}{\lambda_p}\right)^{\frac{\sigma}{p_N}} \lambda_p - \lambda\right\}.$$

Repeating step by step the above proof, we can find $s_1 \in [s_0, p[$ such that, for all $s \in [s_1, p]$, inequality (6) holds with $\tilde{\varepsilon}$ in place of ε. After that, for each $s \in [s_1, p]$, define

$$\Psi_{\lambda,s}(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{s}\int_\Omega |u|^s\,dx$$

for all $u \in W_0^{1,p}(\Omega)$. It is known that the unique solution $u_{\lambda,s}$ of problem (P_0) is exactly the positive global minimum of $I_{\lambda,s}$ and

$$\frac{1}{\lambda}\left(\frac{1}{p} - \frac{1}{s}\right)^{-1} \Psi_{\lambda,s}(u_{\lambda,s}) = \|u_{\lambda,s}\|_s^s \tag{7}$$

(see [1] for instance). Consequently, we have

$$\Psi_{\lambda,s}(u_{\lambda,s}) - \frac{1}{q}\|v_{\lambda,s}\|_q^q \leq I_{\lambda,s}(v_{\lambda,s}) \leq I_{\lambda,s}(u_{\lambda,s}) \leq \Psi_{\lambda,s}(u_{\lambda,s}). \tag{8}$$

Now, taking into account (5) and (6), and in view of the fact that $(\lambda_p c_s^s)^{\frac{1}{p-s}}$ has a finite limit as $s \to p^-$, we obtain

$$\frac{1}{q}\|v_{\lambda,s}\|_q^q \leq \frac{\tilde{\varepsilon}^{q-p_N}}{q}\|v_{\lambda,s}\|_{p_N}^{p_N} \leq \frac{\tilde{\varepsilon}^{q-p_N}}{q} c_{p_N}^{p_N}\|v_{\lambda,s}\|^{p_N} \leq C_2\left(\frac{\lambda + \tilde{\varepsilon}^{q-p}}{\lambda_p}\right)^{\frac{p_N}{p-s}} \tag{9}$$

for some positive constant $C_2 > 0$ which does not depend on s. Note that, from the choice of $\tilde{\varepsilon}$, one has

$$\frac{(\lambda + \tilde{\varepsilon}^{q-p})^{p_N}}{\lambda_p^{p_N}}\frac{\lambda_p^s}{\lambda^s} \leq \left(\frac{\lambda}{\lambda_p}\right)^{\sigma - p} < 1.$$

Hence, by (7) and (9),

$$
\begin{aligned}
(10)\qquad & \left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} \frac{1}{q}\|v_{\lambda,s}\|^q \leq C_2 \left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} \left(\frac{\lambda+\tilde{\varepsilon}^{q-p}}{\lambda_p}\right)^{\frac{p_N}{p-s}} \\
& = C_2 \left(\frac{(\lambda+\tilde{\varepsilon}^{q-p})^{p_N}}{\lambda_p^{p_N}} \frac{\lambda_p^s}{\lambda^s}\right)^{\frac{1}{p-s}} \leq C_2 \left(\frac{\lambda}{\lambda_p}\right)^{\frac{\sigma-p}{p-s}}.
\end{aligned}
$$

Consequently, by (7), (8), and (10), we have

$$
\begin{aligned}
& \left|\frac{1}{\lambda}\left(\frac{1}{p}-\frac{1}{s}\right)^{-1}\left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} I_{\lambda,s}(v_{\lambda,s}) - \left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} \|u_{\lambda,s}\|_s^s\right| \\
(11)\qquad & = \frac{1}{\lambda}\left(\frac{1}{p}-\frac{1}{s}\right)^{-1}\left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} |I_{\lambda,s}(v_{\lambda,s}) - \Psi_{\lambda,s}(u_{\lambda,s})| \\
& \leq \frac{1}{\lambda}\left(\frac{1}{p}-\frac{1}{s}\right)^{-1}\left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} \frac{1}{q}\|v_{\lambda,s}\|_q^q \leq C_2 \frac{1}{\lambda}\left(\frac{1}{p}-\frac{1}{s}\right)^{-1}\left(\frac{\lambda}{\lambda_p}\right)^{\frac{\sigma-p}{p-s}} \to 0
\end{aligned}
$$

as $s \to p^-$. Then, by (2), we obtain

$$
(12)\qquad \lim_{s\to p^-} \frac{1}{\lambda}\left(\frac{1}{p}-\frac{1}{s}\right)^{-1}\left(\frac{\lambda_p}{\lambda}\right)^{\frac{s}{p-s}} I_{\lambda,s}(v_{\lambda,s}) = c.
$$

At this point, note that from (6) and the equality

$$
\|v_{\lambda,s}\|^p - \int_\Omega g(v_{\lambda,s})v_{\lambda,s}\,dx = I'_{\lambda,s}(v_{\lambda,s})(v_{\lambda,s}) = 0,
$$

we have $\|v_{\lambda,s}\|^p = \lambda\|v_{\lambda,s}\|_s^s + \|v_{\lambda,s}\|_q^q$. Therefore,

$$
(13)\qquad I_{\lambda,s}(v_{\lambda,s}) = \lambda\left(\frac{1}{p}-\frac{1}{s}\right)\|v_{\lambda,s}\|_s^p + \left(\frac{1}{p}-\frac{1}{q}\right)\|v_{\lambda,s}\|_q^q.
$$

The limit (1) now follows from (11), (12) and (13). □

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