Nonconvex Lipschitz function in plane which is locally convex outside a discontinuum

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Abstract. We construct a Lipschitz function on $\mathbb{R}^2$ which is locally convex on the complement of some totally disconnected compact set but not convex. Existence of such function disproves a theorem that appeared in a paper by L. Pasqualini and was also cited by other authors.

Keywords: convex function; convex set; exceptional set

Classification: 26B25, 52A20

1. Introduction

In his work from 1938 L. Pasqualini presents a theorem (see [4, Theorem 51, p. 43]) of which the following statement is a reformulation:

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function and $M \subset \mathbb{R}^d$ a set not containing any continuum of topological dimension $(d-1)$. Suppose that f is locally convex on the complement of M. Then f is convex on $\mathbb{R}^d$.

The proof however contains a gap. This result also appeared in the survey paper [1], where the (incorrect) proof was shortly repeated. Also V.G. Dmitriev mentions this result in [2], although he provides a wrong reference.

As a counterexample to the theorem of Pasqualini we present the following theorem:

Theorem 1.1. *There is a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ and $M \subset \mathbb{R}^2$ such that*

- *f is locally convex on $\mathbb{R}^2 \setminus M$,*
- *f is not convex on $\mathbb{R}^2$,*
- *M is compact and totally disconnected,*
- *f has compact support.*

Note that it is a simple observation that the set M from Theorem 1.1 cannot be of one dimensional Hausdorff measure 0.

The author was supported by a cooperation grant of the Czech and the German science foundation, GAČR project no. P201/10/J039.

DOI 10.14712/1213-7243.014.402

2. Preliminaries

In the paper we will use the following more or less standard notation and definitions. For $a, b \in \mathbb{R}^d$ and $r > 0$ we will denote by $B(a, r)$ the closed ball with center a and radius r and $[a, b]$ will denote the closed line segment with endpoints a and b. For $A \subset \mathbb{R}^d$ the symbol $\operatorname{co} A$ will mean the convex hull of A and A^c will mean the complement of A. If $l \subset \mathbb{R}^2$ is a line and $\varepsilon > 0$ then we define $l(\varepsilon) = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, l) < \varepsilon\}$.

A function f defined on a set $A \subset \mathbb{R}^2$ is called L-Lipschitz, if for every $x, y \in A$, $x \neq y$, we have $|f(x) - f(y)| \leq L|x - y|$.

We will call f locally convex on A if for every x, y such that $[x, y] \subset A$ and $\alpha \in [0, 1]$ we have $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$.

Finally, f will be called piecewise affine on A if there is a locally finite triangulation Δ of A such that f is affine on every triangle from Δ.

3. Construction of the function

Definition 3.1. Let $\mathcal{Q}$ be the system of all unions of finite systems of (closed) polytopes in $\mathbb{R}^2$. Let $L > 0$, $f : \mathbb{R}^2 \to \mathbb{R}$ and $P \in \mathcal{Q}$. We say that a pair (P, f) is L-good if

(1) f is L-Lipschitz,
(2) f is piecewise affine on P^c,
(3) f is locally convex on P^c.

The key technical result is the following:

Lemma 3.2. *Let $\delta, \varepsilon, L > 0$ and let l be a line in $\mathbb{R}^2$. Let (P, g) be an L-good pair. Then there is an $(L + \varepsilon)$-good pair (Q, h) such that*

(1) *$Q \subset P$,*
(2) *$h = g$ on P^c,*
(3) *if $x, y \in Q$ belong to different components of $\mathbb{R}^2 \setminus l(\delta)$ then they belong to different components of Q.*

We first prove Theorem 1.1 using Lemma 3.2

PROOF OF THEOREM 1.1: Choose a sequence $\{x_n\}_{n=1}^{\infty}$ dense in the plane and consider any sequence of lines $\{l_n\}_{n=1}^{\infty}$ with the property that for any $i, j \in \mathbb{N}$ there is some $k \in \mathbb{N}$ such that $x_i, x_j \in l_k$. Choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Then the sequence $\{l_n(\varepsilon_n)\}_{n=1}^{\infty}$ has the property that for every $x, y \in \mathbb{R}^2$, $x \neq y$, there is some $k \in \mathbb{N}$ such that x and y belong to the different component of $\mathbb{R}^2 \setminus l_k(\varepsilon_k)$.

In the proof we will proceed by induction and construct a sequence of functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ and a sequence $\{P_i\} \subset \mathcal{Q}$, $i = 0, 1, \ldots$, such that for every i the following conditions hold:

(1) pair (P_i, f_i) is $(1 + \sum_{n=1}^{i} \varepsilon_n)$-good,
(2) if $i > 0$ then $P_i \subset P_{i-1}$,
(3) if $i > 0$ then $f_i = f_{i-1}$ on $(P_{i-1})^c$,

(4) if $i > 0$ and if $x, y \in P_i$ belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$ then they belong to the different component of P_i.

To do this let f_0 be an arbitrary 1-Lipschitz function on $\mathbb{R}^2$ which is equal to 0 on $((-3,3)^2)^c$ and equal to 1 on $[-1,1]^2$ and put $P_0 := [-3,3]^2 \setminus (-1,1)^2$. Validity of conditions (1)–(4) is obvious.

Now, if we have constructed f_{i-1} and P_{i-1} we obtain f_i and P_i simply by applying Lemma 3.2 with $\varepsilon = \delta = \varepsilon_i$, $L = (1 + \sum_{n=1}^{i-1} \varepsilon_n)$, $l = l_i$, $P = P_{i-1}$ and $g = f_{i-1}$. The function f_i will be then equal to h from the statement of Lemma 3.2 and P_i will be equal to the corresponding Q. Validity of conditions (1)–(4) follows directly from Lemma 3.2.

Put $M := \bigcap P_i$. Due to property (2) M is compact and nonempty. To prove that M is totally disconnected consider $x, y \in M$, $x \neq y$. By the choice of the sequences $\{l_n\}_{n=1}^{\infty}$ and $\{\varepsilon_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ there is some i such that x and y belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$. By property (3) we have that x and y belong to the different component of P_i. Using property (2) again we then obtain that x and y belong to the different component of M as well.

Define $\tilde{f} : M^c \to \mathbb{R}$ in such a way that $\tilde{f}(x) = f_i(x)$ whenever $x \in (P_i)^c$. It is easy to see that the definition of $\tilde{f}$ is correct due to properties (2) and (3) and the definition of M, and also that by property (1) the function $\tilde{f}$ is $(1 + \sum_{n=1}^{\infty} \varepsilon_n)$-Lipschitz and locally convex on M^c. By Kirszbraun's theorem (see [3]) there is a $(1 + \sum_{n=1}^{\infty} \varepsilon_n)$-Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f = \tilde{f}$ on M^c. Therefore f is locally convex on M^c as well. Also, f has compact support due to properties (2) and (3), the fact that P_0 is compact and that f_0 is supported in P_0.

It remains to show that f is not convex on $\mathbb{R}^2$, but this is easy since

$$\frac{f(-3,0) + f(3,0)}{2} = 0 < 1 = f(0,0). \qquad \square$$

The proof of Lemma 3.2 is divided into several lemmas.

Lemma 3.3. *Let $H \subset \mathbb{R}^2$ be a closed halfplane, $x \in \mathbb{R}^2 \setminus H$, $w \in \partial H$ and $L > 0$. If $f : H \cup \{x\} \to \mathbb{R}$ is L-Lipschitz and affine on H, then the function*

$$g_w(u) = \begin{cases} f(u), & \text{if} \quad u \in H, \\ \alpha f(x) + (1-\alpha) f(w), & \text{for} \quad u = \alpha x + (1-\alpha) w, \alpha \in [0,1], \end{cases}$$

is L-Lipschitz as well.

Proof: Without any loss of generality we can suppose that $f(w) = 0$ and $w = (0,0)$. This means that g_w is in fact linear on both H and $[x, w]$. Choose $a \in H$

and $b = \alpha x$ for some $\alpha \in [0,1]$. Now,

$$\begin{aligned}|g_w(a) - g_w(b)| &= \alpha \left| g_w\left(\frac{1}{\alpha}a\right) - g_w\left(\frac{1}{\alpha}b\right)\right| = \alpha \left| g_w\left(\frac{1}{\alpha}a\right) - g_w\left(\frac{1}{\alpha}\alpha x\right)\right| \\ &= \alpha \left| g_w\left(\frac{1}{\alpha}a\right) - g_w(x)\right| \le \alpha L \left|\frac{1}{\alpha}a - x\right| = \alpha L \left|\frac{1}{\alpha}a - \frac{1}{\alpha}\alpha x\right| \\ &= L|a - \alpha x| = L|a-b|.\end{aligned}$$

Similarly, if $a = \alpha x$ and $b = \beta x$ for some $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$ we have

$$|g_w(a) - g_w(b)| = |\alpha f(x) - \beta f(x)| = |f(x)| \cdot |\alpha - \beta| \le L|x| \cdot |\alpha - \beta| = L|a-b|.$$

□

Lemma 3.4. *Let $\varepsilon, L, K > 0$. Let f be an L-Lipschitz function on $[-K,K]^2$, which is equal to an affine function f_1 on $[-K,0] \times [-K,K]$, and $z \in (0,K) \times (-K,K)$. Then there is an $x \in [(0,0),z]$ and $\gamma > 0$ such that for every $y \in B(x,\gamma)$ and every $w \in B((0,0),\gamma) \cap (\{0\} \times (-K,K))$ the function*

$$g_{y,w}(u) = \begin{cases} f(u), & \text{if} \quad u \in [-K,0] \times [-K,K], \\ \alpha f(w) + (1-\alpha) f(x), & \text{for} \quad u = \alpha w + (1-\alpha) y, \alpha \in [0,1], \end{cases}$$

is $(L+\varepsilon)$-Lipschitz and $|g_{y,w} - f| < \varepsilon$ on $[-K,0] \times [-K,K] \cup [w,y]$.

Proof: Without any loss of generality we can suppose that $\varepsilon < 1$, $L = 1$ and that $f(0,0) = 0$. Indeed, if $f(0,0) \neq 0$ we can just consider the function $u \mapsto f(u) - f(0,0)$ in the place of f and then add $f(0,0)$ to the resulting function $g_{y,w}$. If $L \neq 1$ then we can just consider the function $u \mapsto \frac{f(u)}{L}$ in the place of f and $\frac{\varepsilon}{L}$ in the place of ε and multiply the resulting function $g_{y,w}$ by L.

Since f is 1-Lipschitz we can find a sequence $\{x_i\}_{i=1}^{\infty} \subset [(0,0),z]$ converging to $(0,0)$ such that for some $s \in [-1,1]$

$$s_i := \frac{f(x_i)}{|x_i|} \to s \quad \text{as} \quad i \to \infty. \tag{3.1}$$

Denote $\tilde{z} := \frac{z}{|z|}$. Consider now the sequence of functions $h_i : [-\frac{K}{|x_i|}, 0] \times [-\frac{K}{|x_i|}, \frac{K}{|x_i|}] \cup \{\tilde{z}\} \to \mathbb{R}$ defined as

$$h_i(u) := \frac{1}{|x_i|} f(|x_i| \cdot u).$$

Then h_i is 1-Lipschitz for every i. Since f is equal to an affine function f_1 on $[-K,0] \times [-K,K]$ and $f(0,0) = 0$ we have $h_i = f_1$ on $[-\frac{K}{|x_i|}, 0] \times [-\frac{K}{|x_i|}, \frac{K}{|x_i|}]$. Also $h_i(\tilde{z}) = s_i$, because $\tilde{z} = \frac{z}{|z|} = \frac{x_i}{|x_i|}$. Therefore by (3.1) the function $h := \lim h_i : H \cup \{\tilde{z}\} \to \mathbb{R}$ which is equal to f_1 on $H := (-\infty, 0] \times (-\infty, \infty)$ and such that $h(\tilde{z}) = s$, is also 1-Lipschitz.

Consider $\tilde{\gamma} > 0$ such that $\tilde{\gamma} < \frac{\varepsilon \tilde{z}_1}{4}$ (here by $\tilde{z}_1$ we mean the first coordinate of $\tilde{z}$). This choice then implies

$$\frac{|v-\tilde{z}|}{|v-\tilde{z}|-\tilde{\gamma}} = 1 + \frac{\tilde{\gamma}}{|v-\tilde{z}|-\tilde{\gamma}} < 1 + \frac{\frac{\varepsilon \tilde{z}_1}{4}}{\tilde{z}_1 - \frac{\varepsilon \tilde{z}_1}{4}} = 1 + \frac{\varepsilon}{4-\varepsilon}$$

for $v \in H$, which gives us inequality

$$\frac{|v-\tilde{z}|}{|v-\tilde{z}|-\tilde{\gamma}} < 1 + \frac{\varepsilon}{2},$$

as $\varepsilon < 1$. Now, for every $\tilde{s} \in [s-\tilde{\gamma}, s+\tilde{\gamma}]$, $v \in H$ and $t \in B(\tilde{z}, \tilde{\gamma})$

$$\begin{aligned} \frac{f_1(v)-\tilde{s}}{|v-t|} &\leq \frac{|f_1(v)-s|}{|v-t|} + \frac{|s-\tilde{s}|}{|v-t|} \leq \frac{|f_1(v)-s|}{|v-\tilde{z}|-\tilde{\gamma}} + \frac{\tilde{\gamma}}{|v-\tilde{z}|-\tilde{\gamma}} \\ &\leq \frac{|f_1(v)-s|}{|v-\tilde{z}|} \cdot \frac{|v-\tilde{z}|}{|v-\tilde{z}|-\tilde{\gamma}} + \frac{2\tilde{\gamma}}{\tilde{z}_1} \leq \left(1+\frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = 1+\varepsilon. \end{aligned}$$

Therefore, by Lemma 3.3 for every $\tilde{s} \in [s-\tilde{\gamma}, s+\tilde{\gamma}]$, $w \in \{0\} \times (-\infty, \infty)$ and $t \in B(\tilde{z}, \tilde{\gamma})$ the function

$$\tilde{h}_{w,t,\tilde{s}}(u) = \begin{cases} f_1(u), & \text{if} \quad u \in H, \\ (1-\alpha)\tilde{s} + \alpha f_1(w), & \text{for} \quad u = (1-\alpha)t + \alpha w, \alpha \in [0,1], \end{cases}$$

is $(1+\varepsilon)$-Lipschitz as well.

Choose i such that $s_i \in [s-\tilde{\gamma}, s+\tilde{\gamma}]$ and put $x = x_i$ and $\gamma = \frac{|x|\tilde{\gamma}}{2}$. Now, consider some $y \in B(x, \gamma)$ and some $w \in B((0,0), \gamma) \cap \{0\} \times (-K, K)$ and let $g_{y,w}$ be as in the statement of the lemma. First we will prove that $g_{y,w}$ is $(1+\varepsilon)$-Lipschitz. To do this we first observe that $\frac{1}{|x|} g_{y,w}(|x| \cdot \xi)$ is equal to $\tilde{h}_{\frac{w}{|x|}, \frac{y}{|x|}, \frac{f(x)}{|x|}}(\xi)$, whenever the first function (as a function of ξ) is defined. Now, we have $\frac{w}{|x|} \in \{0\} \times (-\infty, \infty)$,

$$\left| \frac{y}{|x|} - \tilde{z} \right| = \left| \frac{y}{|x|} - \frac{x}{|x|} \right| = \frac{|y-x|}{|x|} \leq \frac{|x|\tilde{\gamma}}{2|x|} \leq \tilde{\gamma},$$

which means $\frac{y}{|x|} \in B(\tilde{z}, \tilde{\gamma})$ and finally $\frac{f(x)}{|x|} = s_i \in [s-\tilde{\gamma}, s+\tilde{\gamma}]$ and we are done since $\frac{1}{|x|} g_{y,w}(|x| \cdot \xi)$ (as a function of ξ) and $g_{y,w}$ have the same Lipschitz constant.

To finish the proof it is now sufficient to observe that if we additionally choose x_i small enough we obtain also $|g_{y,w} - f| < \varepsilon$ on $[-K, 0] \times [-K, K] \cup [w, y]$. □

Lemma 3.5. *Let $L, \varepsilon, \delta > 0$, $a < b$ and $c < d$ be given. Let*

$$P = \mathrm{co}\{(-1,a), (-1,b), (1,c), (1,d)\}$$

and

$$P^{\varepsilon} = \mathrm{co}\{(-1, a-\varepsilon), (-1, b+\varepsilon), (1, c-\varepsilon), (1, d+\varepsilon)\}.$$

Suppose that f is an L-Lipschitz function defined on $\mathbb{R}^2$ which is locally affine on $P^\varepsilon \setminus P$. Then there are

$$\frac{a+c}{2} =: a_0 < a_1 < \cdots < a_{n-1} < a_n := \frac{b+d}{2}$$

and $\frac{1}{2} > \kappa > 0$ such that, using the notation introduced below, the function $g_\kappa : \overline{P^\varepsilon \setminus (P \setminus [-\kappa,\kappa] \times \mathbb{R})} \to \mathbb{R}$ defined as $g_\kappa(z_i^\pm) = f(z_i^\pm)$ for $i = 0, n$, $g_\kappa(z_i^\pm) = f(z_i)$ for $i = 1, \dots, n-1$ and

$$g_\kappa(u) = \begin{cases} f(u), & \textit{if} \quad u \in P^\varepsilon \setminus P, \\ \alpha g(z_i^+) + \beta g(z_i^-) + \gamma g(z_{i+1}^+), & \textit{for} \quad u = \alpha z_i^+ + \beta z_i^- + \gamma z_{i+1}^+, \\ & \alpha, \beta, \gamma \ge 0, \alpha + \beta + \gamma = 1, \\ \alpha g(z_i^-) + \beta g(z_{i+1}^-) + \gamma g(z_{i+1}^+), & \textit{for} \quad u = \alpha z_i^- + \beta z_{i+1}^- + \gamma z_{i+1}^+, \\ & \alpha, \beta, \gamma \ge 0, \alpha + \beta + \gamma = 1 \end{cases}$$

is $(L+\delta)$-Lipschitz and such that $|f - g_\kappa| < \delta$ on $\mathbb{R}^2$. Here we denoted $z_0^\pm := \big(\pm\kappa, \frac{a+c}{2} \pm \frac{\kappa(c-a)}{2}\big)$, $z_n^\pm := \big(\pm\kappa, \frac{b+d}{2} \pm \frac{\kappa(d-b)}{2}\big)$, $z_i^\pm := (\pm\kappa, a_i)$ for $i = 1, \dots, n-1$ and $z_i := (0, a_i)$ for $i = 0, \dots, n$.

Proof: Without any loss of generality we can suppose $L = 1$. Denote P_i^ε the connectivity component of $\overline{P^\varepsilon \setminus P}$ containing z_i, $i = 0, n$. When we have found a_i we denote $P_i = \operatorname{co}\{z_i^\pm, z_{i+1}^\pm\}$ for $i = 0, \dots, n-1$. Put $S = \operatorname{co}\{z_1^\pm, z_{n-1}^\pm\}$ and $\alpha = \operatorname{dist}(S, P^\varepsilon \setminus P)$. We always assume κ to be small enough that $1 > \alpha > 0$.

First, we will use Lemma 3.4 twice to find points $a_1 \in B(a_0, \frac{\min(|a_0 - a_n|, 1)}{2})$, $a_{n-1} \in B(a_n, \frac{\min(|a_0 - a_n|, 1)}{2})$ and $\kappa_1 > 0$ such that for every $\kappa_1 > \kappa > 0$ the functions $g_\kappa|_{P_0^\varepsilon \cup P_0}$ and $g_\kappa|_{P_n^\varepsilon \cup P_{n-1}}$ are both $(1+\delta)$-Lipschitz and such that $|f - g_\kappa| < \delta$ on $P_0^\varepsilon \cup P_n^\varepsilon \cup P_0 \cup P_{n-1}$. Here, in the notation of the points z_i, the point z_1 corresponds to the point x guaranteed by Lemma 3.4 (when we identify z_0 with the origin) and similarly the point z_{n-1} corresponds to x in the case when we apply Lemma 3.4 centred in z_n. Note that although Lemma 3.4 guarantees $(1+\delta)$-Lipschitzness on P_0 (or on P_{n-1}) only on line segments with one endpoint in P_0^ε (or in P_n^ε), this is enough for our purposes. Indeed, if for instance $a, b \in \operatorname{co}\{z_0^-, z_0^+, z_1^+\}$, we can always find $\tilde{a}, \tilde{b}$ with $\tilde{a} \in P_0^\varepsilon$ and such that the vector $a - b$ is parallel to the vector $\tilde{a} - \tilde{b}$. In such situation of course

$$\frac{|g_\kappa(a) - g_\kappa(b)|}{|a-b|} = \frac{|g_\kappa(\tilde{a}) - g_\kappa(\tilde{b})|}{|\tilde{a} - \tilde{b}|}.$$

Also, if $a, b \in \operatorname{co}\{z_0^-, z_1^-, z_1^+\}$ one can always consider $\tilde{a} = z_1^-$ or $\tilde{a} = z_1^+$ such that

$$\frac{|g_\kappa(a) - g_\kappa(b)|}{|a-b|} \le \frac{|g_\kappa(\tilde{a}) - g_\kappa(z_0^-)|}{|\tilde{a} - z_0^-|}.$$

Similarly for P_{n-1}.

Observe that for every $u_0 \in P_0^\varepsilon \cup P_0$ and every $u_n \in P_n^\varepsilon \cup P_{n-1}$ we have

$$\frac{|g_\kappa(u_0) - g_\kappa(u_n)|}{|u_0 - u_n|} \le \frac{|g_\kappa(u_0) - g_\kappa(z_0)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_0) - g_\kappa(z_n)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_n) - g_\kappa(u_n)|}{|u_0 - u_n|}$$
$$\le \frac{|u_0 - z_0|}{|u_0 - u_n|} + \frac{|z_0 - z_n|}{|u_0 - u_n|} + \frac{|z_n - u_n|}{|u_0 - u_n|}.$$

and since the last expression can be smaller than $1+\delta$ when we assume $|a_0 - a_1|$ and $|a_{n-1} - a_n|$ to be small enough, we can additionally assume that $g|_{P^\varepsilon \cup P_0 \cup P_{n-1}}$ is $(1+\delta)$-Lipschitz.

Next, note that the function $g_\kappa|_{[z_1, z_{n-1}]}$ is actually independent on κ and that it is 1-Lipschitz for any choice of $a_2, \ldots, a_{n-2}$ (this is true because in one dimension the affine extension never increases the Lipschitz constant). This also means that for $S = \mathrm{co}\{z_1^\pm, z_{n-1}^\pm\}$ we have $g_\kappa|_S$ is 1-Lipschitz for any choice of $a_2, \ldots, a_{n-2}$ as well. Put $\alpha = \mathrm{dist}\,(S, P^\varepsilon \backslash P)$, we can assume κ_2 to be small enough that $1 > \alpha > 0$ (here we used the fact that $|a_0 - a_1|, |a_{n-1} - a_n| \le \frac{1}{2}$). Consider n big enough such that $\frac{|a_1 - a_{n-1}|}{n-1} \le \frac{\alpha\delta}{4}$, put $a_i = a_1 + \frac{i|a_1 - a_{n-1}|}{n-1}$ and pick $\kappa_3 < \min(\kappa_2, \frac{\alpha\delta}{4})$. Then for $\kappa < \kappa_3$ and $a \in S$

$$\begin{aligned} |g_\kappa(a) - f(a)| &\le |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - f(z_i)| + |f(z_i) - f(a)| \\ &\le |a - z_i| + 0 + |a - z_i| \le \frac{\delta}{2} < \delta, \end{aligned} \tag{3.2}$$

where i is chosen such that $a \in P_i$.

To finish the proof we need to observe that for $\kappa < \kappa_3$ the function g_κ is $(1+\delta)$-Lipschitz. Since $S \cup P_0 \cup P_{n-1}$ is convex, the remaining case we have to consider is $a \in S$ and $b \in P^\varepsilon \setminus P$. Find i such that $a \in P_i$. With this choice we have $|a - z_i| \le \frac{\alpha\delta}{2}$ and therefore

$$|b - z_i| \le |a - b| + |a - z_i| \le |a - b| + \frac{\alpha\delta}{2} \le (1+\delta)\,|a - b|.$$

Now, we have

$$\begin{aligned} |g_\kappa(a) - g_\kappa(b)| &\le |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - g_\kappa(b)| \\ &\le \frac{\delta\alpha}{2} + |f(z_i) - f(b)| \le \frac{\delta}{2}|a - b| + |b - z_i| \\ &\le \frac{\delta}{2}|a - b| + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| \le (1+\delta)|a - b|. \end{aligned}$$

□

Lemma 3.6. *Let $1 > \varepsilon > 0$ and $\alpha, L > 0$. Let f be a L-Lipschitz function on $[-1,1]^2$ which is affine on both $[-1,1] \times [-1,0]$ and $[-1,1] \times [0,1]$ (and equal to affine functions f_1 and f_2, respectively). Put*

$$A_1 = [-1, -1/2] \times [-1, 0], A_2 = [1/2, 1] \times [0, 1],$$

$$B_1^\varepsilon = [-1,\varepsilon] \times [0,\varepsilon], B_2^\varepsilon = [-\varepsilon,1] \times [-\varepsilon,0]$$

and

$$A = A_1 \cup A_2 \cup B_1^\varepsilon \cup B_2^\varepsilon.$$

Then either f is convex on $[-1,1]^2$ or the function $g_\varepsilon : A \to \mathbb{R}$ defined as

$$g(u) = \begin{cases} f_1(u), & \text{if} \quad u \in A_1 \cup B_1^\varepsilon, \\ f_2(u), & \text{if} \quad u \in A_2 \cup B_2^\varepsilon. \end{cases}$$

is locally convex on A. Moreover, if ε is small enough, g_ε is $(L+\alpha)$-Lipschitz and $|g_\varepsilon - f| < \alpha$ on A.

Proof: It follows from a direct computation. □

Lemma 3.7. *Let $L, \alpha > 0$ and $1 > \gamma > \varepsilon > 0$. Let f be a L-Lipschitz function on $[-4,4]^2 \cup [1,2] \times [4,5]$ which is affine on both $[-4,4] \times [-4,0]$ and $[-4,4] \times [0,4] \cup [1,2] \times [4,5]$ (and equal to affine functions f_1 and f_2, respectively). Put*

$$A_1 = [-3,-2] \times [0,\gamma], A_2 = [-3,0] \times [\gamma,\gamma+\varepsilon], A_3 = [-1,2] \times [\gamma-\varepsilon,\gamma],$$
$$A_4 = [1,2] \times [\gamma,4], B_1 = [-4,4] \times [-4,0], B_2 = [1,2] \times [4,5],$$

and

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2.$$

Then either f is locally convex on $[-4,4]^2 \cup [1,2] \times [4,5]$ or the function

$$g(u) = \begin{cases} f_1(u), & \text{if} \quad u \in A_1 \cup A_2 \cup B_1, \\ f_2(u) + \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}(u \cdot (0,1) - 4), & \text{if} \quad u \in A_3 \cup A_4, \\ f_2(u), & \text{if} \quad u \in B_2, \end{cases}$$

is $(L+\alpha)$-Lipschitz, locally convex on A and $|f - g| < \alpha$ on A, if ε and γ are small enough.

Proof: Without any loss of generality we can suppose $L = 1$. First we prove that g is continuous on A. To do this we need to prove that

$$f_1(a,\gamma) = f_2(a,\gamma) + \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}((a,\gamma) \cdot (0,1) - 4) \tag{3.3}$$

whenever $(\gamma, a) \in A_2 \cap A_3$ and that

$$f_2(a,4) = f_2(a,4) + \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}((a,4) \cdot (0,1) - 4) \tag{3.4}$$

whenever $(a,4) \in A$. Define an affine function f_3 on $\mathbb{R}^2$ as

$$f_3(u,v) = \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}((u,v) \cdot (0,1) - 4).$$

To prove (3.3) we can write

$$\begin{aligned} g(a,\gamma) &= f_2(a,\gamma)+f_3(a,\gamma)\\ &= f_2(a,\gamma)+\frac{f_1(0,\gamma)-f_2(0,\gamma)}{\gamma-4}\cdot(\gamma-4)\\ &= f_2(a,\gamma)+f_1(0,\gamma)-f_1(0,0)-f_2(0,\gamma)+f_2(0,0)\\ &= f_2(a,\gamma)+f_1(a,\gamma)-f_1(a,0)-f_2(a,\gamma)+f_2(a,0)\\ &= f_2(a,\gamma)+f_1(a,\gamma)-f_1(a,0)-f_2(a,\gamma)+f_1(a,0)=f_1(a,\gamma). \end{aligned}$$

To prove (3.4) we can write

$$\begin{aligned} g(a,4) &= f_2(a,4)+f_3(a,4)\\ &= f_2(a,4)+\frac{f_1(0,\gamma)-f_1(0,0)-f_2(0,\gamma)+f_1(0,0)}{\gamma-4}(4-4)=f_2(a,4). \end{aligned}$$

Next note that since both f_1 and f_2 are 1-Lipschitz we have

$$g \text{ is 1-Lipschitz on } B_1\cup A_1\cup A_2, \tag{3.5}$$

and

$$g \text{ is 1-Lipschitz on } B_2. \tag{3.6}$$

Since additionally f_3 is constant on all lines parallel to x-axis and since

$$\frac{f_3(0,\gamma)-f_3(0,4)}{4-\gamma}\le\frac{f_1(0,\gamma)-f_1(0,0)-f_2(0,\gamma)+f_2(0,0)}{3}\le\frac{2\gamma}{3}\le\gamma.$$

we have

$$g \text{ is } (1+\gamma)\text{-Lipschitz on } A_4\cup A_3 \tag{3.7}$$

and

$$|g-f_2|\le 4\gamma \text{ on } A_4\cup A_3. \tag{3.8}$$

Now, if $x\in B_1$ and $y\in A_3$ then $g(x)=f_1(x)$, $|g(y)-f_1(y)|\le 3\varepsilon$ and $|x-y|\ge\gamma-\varepsilon$ and therefore

$$|g(x)-g(y)|\le|g(x)-f_1(y)|+|f_1(y)-g(y)|\le|x-y|+3\varepsilon\le\frac{\gamma+2\varepsilon}{\gamma-\varepsilon}.$$

So we have

$$g \text{ is } \frac{\gamma+2\varepsilon}{\gamma-\varepsilon}\text{-Lipschitz on } B_1\cup A_3. \tag{3.9}$$

If $x\in B_1$ and $y\in A_4$ then $g(x)=f_1(x)$, $f(y)\le g(y)\le f_1(y)$ and therefore

$$g \text{ is 1-Lipschitz on } B_1\cup A_4. \tag{3.10}$$

Using (3.6) and (3.7) and continuity of g we obtain that

(3.11) $$g \text{ is } (1+\gamma)\text{-Lipschitz on } A_2 \cup A_3 \text{ and on } B_2 \cup A_4.$$

Finally, if $x \in A_1 \cup A_2$ and $y \in A_4 \cup B_2$ or $x \in A_1$ and $y \in A_3 \cup A_4 \cup B_2$ we have

(3.12) $$|g(x) - f_2(x)| \le 2(\gamma + \varepsilon) \le 4\gamma, \ \ |g(y) - f_2(y)| \le 4\gamma$$

and $|x - y| \ge 1$. This implies

(3.13) $$\begin{aligned} |g(x) - g(y)| &\le |g(x) - f_2(x)| + |f_2(x) - f_2(y)| + |f_2(y) - g(y)| \\ &\le 4\gamma + |x - y| + 4\gamma \le (1 + 8\gamma)|x - y|. \end{aligned}$$

Now, according to (3.5)–(3.12) it is sufficient to choose $\frac{\alpha}{4} > \gamma > \varepsilon > 0$ small enough such that

$$\max\left(1 + 8\gamma, \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}\right) < 1 + \alpha$$

to obtain that g is $(1+\alpha)$-Lipschitz on A and $|f - g| < \alpha$ on A. □

Lemma 3.8. *Under the assumptions of Lemma* 3.5 *there is* $\frac{1}{2} > \kappa > 0$, $R \subset P \cap (-\kappa, \kappa) \times \mathbb{R}$ *and a function* $h : \overline{P^\varepsilon \setminus P} \cup R \to \mathbb{R}$ *such that:*

- (a) $R \in \mathcal{Q}$,
- (b) $h = f$ *on* $\overline{P^\varepsilon \setminus P}$,
- (c) h *is locally convex on* $\overline{P^\varepsilon \setminus P} \cup R$,
- (d) $\overline{P^\varepsilon \setminus P} \cup R$ *is connected,*
- (e) h *is piecewise affine on* $\overline{P^\varepsilon \setminus P} \cup R$,
- (f) h *is* $(L + \delta)$*-Lipschitz.*

Proof: Without any loss of generality we can suppose $L = 1$. Let κ, z_i and g_κ be as in Lemma 3.5, but with $\frac{\delta}{2}$ in the place of δ. Consider the sets

$$X = [-4, 4]^2 \cup [1, 2] \times [4, 5] \quad \text{and} \quad Y = [-1, 1]^2.$$

Find homotheties $\Psi_i : x \mapsto \rho_i x + v_i$, $\rho_i > 0$, $v_i \in \mathbb{R}^2$, $i = 1, \dots, n-1$ and orientation preserving similarities Ψ_0 and Ψ_n, with scaling ratios ρ_0 and ρ_n, such that if we put $M_i = \Psi_i(X)$, $i = 0, n$ and $M_i = \Psi_i(Y)$, $i = 1, \dots, n-1$ we have

- (A) $M_i \cap M_j = \emptyset$ if $i \ne j$,
- (B) $\Psi_0([-4, 4] \times [-4, 0]) \subset \overline{P^\varepsilon \setminus P}$,
- (C) $\Psi_n([-4, 4] \times [-4, 0]) \subset \overline{P^\varepsilon \setminus P}$,
- (D) $M_i \subset (-\kappa, \kappa) \times \mathbb{R}$,
- (E) $[z_i^-, z_i^+] \subset \Psi_i(\mathbb{R} \times \{0\})$,

Put $\Omega = \min_{i \ne j} \operatorname{dist}(M_i, M_j)$ and note that $\Omega > 0$ due to property (A). Define

$$T_i := \operatorname{co}\{\Psi_i(\frac{1}{2}, 1), \Psi_i(1, 1), \Psi_{i+1}(-\frac{1}{2}, -1), \Psi_{i+1}(-1, -1)\},$$

for $i = 1, \dots, n-2$,

$$T_0 := \mathrm{co}\{\Psi_0(1,5), \Psi_0(2,5), \Psi_1(-\frac{1}{2}, -1), \Psi_1(-1,-1)\}$$

and

$$T_{n-1} := \mathrm{co}\{\Psi_n(1,5), \Psi_n(2,5), \Psi_{n-1}(\frac{1}{2}, 1), \Psi_{n-1}(1,1)\}.$$

Put

$$R := \left(\bigcup_{i=0}^{n-1} T_i\right) \cup \left(\bigcup_{i=0}^{n} M_i\right). \tag{3.14}$$

Let g_i, $i = 1, \dots, n-1$ be the function g from Lemma 3.6 with $\alpha = \frac{\Omega\delta\rho_i}{4}$ (and corresponding ε) and with $f_1(x) = \rho_i g_\kappa \circ \Psi_i$ and $f_2(x) = \rho_i g_\kappa \circ \Psi_i$ (with the exception when g_κ is already convex on M_i, in which case we put $g_i = g_\kappa|_{M_i}$). Let g_0 be the function g from Lemma 3.7 with $\gamma = \frac{\Omega\delta\rho_i}{4}$ (and corresponding ε and γ) and with $f_1 = \rho_0 g_\kappa \circ \Psi_0$ and $f_2 = \rho_0 g_\kappa \circ \Psi_0$ and finally, let g_n be the function g from Lemma 3.7 with $\gamma = \frac{\Omega\delta\rho_i}{4}$ (and corresponding ε and γ) and with $f_1 = \rho_n g_\kappa \circ \Psi_n$ and $f_2 = \rho_n g_\kappa \circ \Psi_n$.

Consider now the function h defined by the formula

$$h = \begin{cases} \frac{1}{\rho_i} g_i \circ \Psi_i^{-1} & \text{on} \quad M_i \\ g_\kappa & \text{otherwise.} \end{cases}$$

Property (a) follows from (3.14) and the fact that every M_i and every T_i is a polygon. Properties (b), (c) and (e) follow directly from the construction and corresponding properties of the functions g_i and property (d) is obvious. We will now finish the proof by proving property (f).

So suppose that $a, b \in (P^\varepsilon \setminus P) \cup R$. We need to prove that $|h(a) - h(b)| \le (1+\delta)|a-b|$. We can additionally suppose that either a or b belongs to some M_i since otherwise there is nothing to prove. We will prove only the case $a \in M_i$, $b \in M_j$, $i \ne j$, the other cases can be proved following the same lines. By Lemma 3.6 (for $i = 1, \dots, n-1$) and Lemma 3.7 (for $i = 0, n$) we can now write

$$\begin{aligned} |h(a) - h(b)| &\le |h(a) - g_\kappa(a)| + |g_\kappa(a) - g_\kappa(b)| + |g_\kappa(b) - h(b)| \\ &< \frac{1}{\rho_i} \cdot \frac{\Omega\delta\rho_i}{4} + \left(1 + \frac{\delta}{2}\right) \cdot |a-b| + \frac{1}{\rho_j} \cdot \frac{\Omega\delta\rho_j}{4} \\ &\le \frac{\delta}{2}|a-b| + \left(1 + \frac{\delta}{2}\right) \cdot |a-b| = (1+\delta)|a-b|, \end{aligned}$$

which is what we need. □

Proof of Lemma 3.2: Without any loss of generality we can suppose $L = 1$. Let V be the set of all points $v \in \partial P$ with the property that there is some $\varepsilon_v > 0$ such that $P \cap B(v, \varepsilon_v)$ is similar to $\{(x,y) : x \ge 0\} \cap B(0,1)$ and that g is affine on $P \cap B(v, \varepsilon_v)$. Since $P \in \mathcal{Q}$, the set $\partial P \setminus V$ is finite and without any loss of generality we can assume that $l(\delta) \cap (\partial P \setminus V) = \emptyset$. We can also assume that $l = \{0\} \times \mathbb{R}$ and that $\delta = 1$.

This means that the closure of every component P_i of $P \cap l(\delta)$ is of the form

$$\mathrm{co}\{(-1, a_i), (-1, b_i), (1, c_i), (1, d_i)\}$$

for some $a_i < b_i$, $c_i < d_i$ and such that, for some $\varepsilon_i > 0$, g is locally affine on $P_i^{\varepsilon_i} \setminus P_i$, where

$$P_i^{\varepsilon_i} := \mathrm{co}\{(-1, a_i - \varepsilon_i), (-1, b_i + \varepsilon_i), (1, c_i - \varepsilon_i), (1, d_i + \varepsilon_i)\}.$$

Then we have

$$\alpha = \min_{i \neq j} \mathrm{dist}\,(P_i, P_j) > 0.$$

Let κ_i, R_i and h_i be equal to κ, R and h obtained from Lemma 3.8 for ε_i in the place of ε, P_i in the place of P, g in the place of f and $\frac{\min(\alpha, \varepsilon_i, 1)\varepsilon}{4}$ in the place of δ.

Put $Q = P \setminus (\bigcup R_i)$ and define $\tilde{h} : Q^c \to \mathbb{R}$ by

$$\tilde{h}(u) = \begin{cases} h_i(u) & \text{on } R_i \\ g(u) & \text{otherwise.} \end{cases}$$

Let K be the Lipschitz constant of $\tilde{h}$. Using the Kirszbraun theorem we can find a K-Lipschitz function h on $\mathbb{R}^2$ such that $h = \tilde{h}$ on P^c.

Now, property (1) follows directly from the definition of Q and (a) in Lemma 3.8, property (2) from the definition of h and (b) in Lemma 3.8 and property (3) from (d) in Lemma 3.8.

It remains to prove that the pair (Q, h) is $(1 + \varepsilon)$-good. The local convexity and piecewise affinity of h on Q^c follow from (c) and (e) in Lemma 3.8 and the corresponding properties of g, so the proof will be finished, if we verify that $K \leq (1 + \varepsilon)$.

To do this pick $a, b \in \mathbb{R}^2$, we need to prove that $|h(a) - h(b)| \leq (1 + \varepsilon)|a - b|$. We can additionally suppose that either a or b belongs to some R_i since otherwise there is nothing to prove. We will prove only the case $a \in R_i$, $b \in R_j$, $i \neq j$, the other cases can be proved following the same lines.

Using the definition of h, namely property (f) from Lemma 3.8 we can now write

$$\begin{aligned} |h(a) - h(b)| = |h_i(a) - h_j(b)| &\leq |h_i(a) - f(a)| + |f(a) - f(b)| + |f(b) - h_j(b)| \\ &\leq \frac{\min(\alpha, \varepsilon_i, 1)\varepsilon}{4} + \left(1 + \frac{\varepsilon}{4}\right) \cdot |a - b| + \frac{\min(\alpha, \varepsilon_j, 1)\varepsilon}{4} \\ &\leq \frac{2\varepsilon}{4}|a - b| + \left(1 + \frac{\varepsilon}{2}\right) \cdot |a - b| < (1 + \varepsilon)|a - b|. \end{aligned}$$

□

Acknowledgment. I would like to thank Professor Luděk Zajíček for finding all the historical information and to Professor Jiří Jelínek for translating the original argument by Pasqualini and also for many comments on the previous versions of the manuscript.

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(*Received* July 9, 2013, *revised* April 15, 2014)