The sup = max problem for the extent and the Lindelöf degree of generalized metric spaces, II

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Abstract. In [*The sup = max problem for the extent of generalized metric spaces*, Comment. Math. Univ. Carolin. (The special issue devoted to Čech) **54** (2013), no. 2, 245–257], the author and Yajima discussed the sup = max problem for the extent and the Lindelöf degree of generalized metric spaces: (strict) p-spaces, (strong) Σ-spaces and semi-stratifiable spaces. In this paper, the sup = max problem for the Lindelöf degree of spaces having G_δ-diagonals and for the extent of spaces having point-countable bases is considered.

Keywords: extent; Lindelöf degree; G_δ-diagonal; point-countable base

Classification: Primary 54A25, 54D20; Secondary 03E10

1. Introduction

This is a continuation of the paper [6]. The *spread* $s(X)$ and the *extent* $e(X)$ of a space X are defined as below:

$$s(X) = \sup\{|D| : D \text{ is a discrete subset in } X\} + \omega,$$
$$e(X) = \sup\{|D| : D \text{ is a closed discrete subset in } X\} + \omega.$$

The *sup = max problem* for the spread and the extent of a space X are the following problems, respectively.

- For $\kappa = s(X)$, does X have a discrete subset of size κ?
- For $\kappa = e(X)$, does X have a closed discrete subset of size κ?

If the answer of each problem above is positive, we say that the sup = max condition holds. Obviously, the sup = max condition holds in case κ is a successor cardinal.

The sup = max problem of the spread was discussed in 60's-70's.

Theorem 1.1 (Hajnal-Juhaśz). *Let κ be a singular cardinal.*

(1) *If X is a Hausdorff space with $|X| \geq \kappa$ and κ is a strong limit cardinal, then X has a discrete subset of size κ* [4].

(2) *If X is a regular space with $s(X) = \kappa$ and $\mathrm{cf}(\kappa) = \omega$, then X has a discrete subset of size κ* [5].

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Theorem 1.2 (Roitman [10]). *Assume that $\aleph_{\omega_1} \leq 2^\omega$ and a first-countable Luzin space exists. Then there is a zero-dimensional Tychonoff space X with $s(X) = |X| = \aleph_{\omega_1}$ and with no discrete subset of size $\aleph_{\omega_1}$.*

The *Lindelöf degree* $L(X)$ of a space X is defined as below.

$$L(X) = \min\{\kappa : \text{every open cover of } X \text{ has a subcover of cardinality} \leq \kappa\} + \omega.$$

Then $L(X) = \sup\{L(\mathcal{U}) : \mathcal{U} \text{ is an open cover of } X\}$ holds, where $L(\mathcal{U})$ is defined by

$$L(\mathcal{U}) = \min\{|\mathcal{V}| : \mathcal{V} \subset \mathcal{U} \text{ with } \bigcup \mathcal{V} = \bigcup \mathcal{U}\} + \omega$$

for each collection $\mathcal{U}$ of subsets in X. The *sup = max problem* for the Lindelöf degree of a space X is the following problem.

- For $\kappa = L(X)$, does X have an open cover $\mathcal{U}$ with $L(\mathcal{U}) = \kappa$?

Recently in [6], the author and Yajima discussed the sup = max problem for the extent and the Lindelöf degree of some generalized metric spaces: (strict) p-spaces, (strong) Σ-spaces and semi-stratifiable spaces.

Theorem 1.3 ([6]). *Let κ be a cardinal with* $\mathrm{cf}(\kappa) > \omega$.

(1) *If X is a p-space with $L(X) = \kappa$, then X has an open cover $\mathcal{U}$ with $L(\mathcal{U}) = \kappa$.*
(2) *If X is a Σ-space with $e(X) = \kappa$, then X has a closed discrete subset of size κ.*
(3) *If X is a semi-stratifiable space with $e(X) = \kappa$ and one of the following conditions holds, then X has a closed discrete subset of size κ.*
 (3-1) *X is metalindelöf.*
 (3-2) *X is collectionwise Hausdorff.*
 (3-3) *X is normal and $\{2^\tau : \tau \text{ is a cardinal} < \kappa\}$ has no maximum.*

The assumption $\mathrm{cf}(\kappa) > \omega$ in the theorem above is essential since there is a simple example of metrizable space refuting the sup = max condition in case $\mathrm{cf}(\kappa) = \omega$.

Example 1.4 ([6, Example 2.1]). Let κ be a limit cardinal, and X_κ the subspace of $\kappa + 1$ defined by

$$X_\kappa = \{\alpha + 1 : \alpha \in \kappa\} \cup \{\kappa\}.$$

Then X_κ is a space having only one non-isolated point κ, and $e(X_\kappa) = L(X_\kappa) = |X_\kappa| = \kappa$ holds, but there is no closed discrete subset of size κ in X_κ. Moreover, if $\mathrm{cf}(\kappa) = \omega$, then the space X_κ is metrizable.

It is trivial that $e(X) \leq L(X) \leq |X|$ holds for every space X. Of course, $e(X) < L(X)$ easily happens in general, and the sup = max problem for the extent and for the Lindelöf degree are different in many cases even if $e(X) = L(X)$. On the other hand, there is no such difference for submetalindelöf spaces having the extent of uncountable cofinality. And it is well-known that strict p-spaces, strong

Σ-spaces, and semi-stratifiable spaces have some covering properties stronger than the submetalindelöf property.

Lemma 1.5. *Let X be a submetalindelöf space. Then*

(1) $e(X) = L(X)$ *holds* [1].

(2) *In case $e(X) = L(X) = \kappa$ and $\mathrm{cf}(\kappa) > \omega$, X has a closed discrete subset of size κ iff X has an open cover $\mathcal{U}$ with $L(\mathcal{U}) = \kappa$* [6, Theorem 4.5].

In this paper, we discuss the sup = max problem for the Lindelöf degree of spaces having G_δ-diagonals and for the extent of spaces having point-countable bases.

Preliminaries. All spaces are assumed to be T_1-topological spaces. The word 'countable' means countably infinite or finite. The cofinality of a cardinal κ is denoted by $\mathrm{cf}(\kappa)$. Regular cardinals are assumed to be infinite. Successor cardinals and limit cardinals are assumed to be uncountable.

We recall here definitions of some terms appearing in this paper. A space X is *metalindelöf* if every open cover of X has a point-countable open refinement. A space X is *submetalindelöf* if for every open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of open refinements, satisfying that for each $x \in X$ one can choose $n_x \in \omega$ such that $\mathcal{V}_{n_x}$ is point-countable at x. Obviously, metalindelöf spaces are submetalindelöf. A Hausdorff space X is *semi-stratifiable* [2] if there is a function $g : \omega \times X \to \mathrm{Top}(X)$, where $\mathrm{Top}(X)$ denotes the topology of X, satisfying:

(i) $\bigcap_{n\in\omega} g(n, x) = \{x\}$ for each $x \in X$,

(ii) $y \in \bigcap_{n\in\omega} g(n, x_n)$ implies that $\{x_n\}$ converges to y.

Let λ be an infinite cardinal. A tree T is called a *λ-Suslin tree* if $|T| = \lambda$ and T has neither a chain nor an antichain of size λ. A topological space is said to have the *λ-c.c.* if there is not a pairwise disjoint family of size λ by non-empty open sets. A *λ-Suslin line* is a LOTS (= linearly ordered topological space) having the λ-c.c. and with no dense subset of size less than λ. An ω_1-Suslin tree (line) is simply called a *Suslin tree* (*line*). It is well-known that a Suslin tree exists iff a Suslin line exists, (see [9]). In a similar way, it is seen that for each regular uncountable cardinal λ, a λ-Suslin tree exists iff a λ-Suslin line exists.

A subset F of X is said to be *nowhere dense* if $\mathrm{Int}_X(\mathrm{Cl}_X(F)) = \emptyset$, i.e. F is nowhere dense in X iff $F \subset \mathrm{Cl}_X(U) \setminus U$ for some open set U of X. A *Luzin space* is a regular space having uncountably many points but no isolated point, and every nowhere dense subset of which is countable. It is well-known that every Luzin space is a hereditarily Lindelöf and zero-dimensional Tychonoff space, and that every Suslin line has a first-countable Luzin subspace (see [8]).

2. Spaces having G_δ-diagonals

A space X has a *G_δ-diagonal* if there is a sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of open covers of X such that $\bigcap_{n\in\omega} \mathrm{St}(x, \mathcal{G}_n) = \{x\}$ for each $x \in X$. It is well-known that a space X has a G_δ-diagonal if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ of X is a G_δ-set in the square X^2 (cf. [3, 2.1 Definition]).

In this section, we prove the theorems below.

Theorem 2.1. *Let κ be a limit cardinal with $\mathrm{cf}(\kappa) > \omega$.*

(1) *Assume that $\tau^\omega < \kappa$ for each $\tau < \kappa$. If a space X has a G_δ-diagonal with $L(X) = \kappa$, then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*
(2) *Assume that $\kappa \leq \tau^\omega$ for some $\tau < \mathrm{cf}(\kappa)$. Then there is a Hausdorff space X having a G_δ-diagonal with $L(X) = \kappa$ such that $L(\mathcal{U}) < \kappa$ for every open cover $\mathcal{U}$ of X.*
(3) *Assume that $\kappa \leq \tau^\omega$ for some $\tau < \mathrm{cf}(\kappa)$, and a $\mathrm{cf}(\kappa)$-Suslin line exists. Then there is a zero-dimensional Tychonoff space X having a G_δ-diagonal with $L(X) = \kappa$ such that $L(\mathcal{U}) < \kappa$ for every open cover $\mathcal{U}$ of X.*

Theorem 2.2. *Assume that $\aleph_{\omega_1} \leq 2^\omega$ and a first-countable Luzin space exists. Then there is a zero-dimensional Tychonoff space X having a G_δ-diagonal with $L(X) = \aleph_{\omega_1}$ such that $L(\mathcal{U}) < \aleph_{\omega_1}$ for every open cover $\mathcal{U}$ of X.*

Theorem 2.1(1) for strong limit cardinals κ was pointed out by Yajima [11] before [6] was published and we started to write this article.

It is trivial and well-known that each semi-stratifiable space has a G_δ-diagonal. It is also well-known that semi-stratifiable spaces are subparacompact (cf. [3, 5.11 Theorem]), in particular, submetalindelöf. By Theorem 2.1(1) and Lemma 1.5, we obtain the corollary below.

Corollary 2.3. *Let X be a semi-stratifiable space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Assume that $\tau^\omega < \kappa$ for each $\tau < \kappa$. Then X has a closed discrete subset of size κ.*

Theorem 2.2 implies Theorem 2.1(3) for $\kappa = \aleph_{\omega_1}$ since each Suslin line has a first countable Luzin subspace. It is well-known that if ZFC is consistent, then ZFC+GCH is consistent, and it is also consistent with ZFC that $\aleph_{\omega_1} \leq 2^\omega$ and a Suslin line exists, (see [9]). So we obtain the corollary below.

Corollary 2.4. *The sup = max condition for the Lindelöf degree $L(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, of spaces having G_δ-diagonals is consistent with and independent from* ZFC.

First we prove Theorem 2.1(1). In fact, the assumption $L(X) = \kappa$ can be replaced by $|X| \geq \kappa$ as below.

Proposition 2.5. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$ such that $\tau^\omega < \kappa$ for each $\tau < \kappa$. And let X be a space with $|X| \geq \kappa$ which has a G_δ-diagonal. Then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) \geq \kappa$.*

Proof: Assume that $L(\mathcal{U}) < \kappa$ for any open cover $\mathcal{U}$ of X. Let $\{\mathcal{G}_n\}_{n\in\omega}$ be a sequence of open covers of X which witnesses X having a G_δ-diagonal. Take an $n \in \omega$. Since $\mathcal{G}_n$ is an open cover of X, letting $\tau_n = L(\mathcal{G}_n)$, we have $\tau_n < \kappa$. So there is a subcover $\mathcal{H}_n$ of $\mathcal{G}_n$ with $|\mathcal{H}_n| = \tau_n$. Let $\tau = \sup_{n\in\omega} \tau_n$. Then $\tau < \kappa$ by $\mathrm{cf}(\kappa) > \omega$. For each $x \in X$ and $n \in \omega$, since $\mathcal{H}_n$ covers X, we can take an $H_{x,n} \in \mathcal{H}_n$ with $x \in H_{x,n}$. Consider the correspondence $x \longmapsto \{H_{x,n}\}_{n\in\omega}$.

Since $\bigcap_{n\in\omega} H_{x,n} \subset \bigcap_{n\in\omega} \mathrm{St}(x,\mathcal{G}_n) = \{x\}$ for each $x \in X$, the correspondence is one-to-one. Since each $\{H_{x,n}\}_{n\in\omega}$ is a sequence of members of $\bigcup_{n\in\omega} \mathcal{H}_n$ and $|\bigcup_{n\in\omega} \mathcal{H}_n| = \tau$, the cardinality of all such sequences is not greater than τ^ω. Hence we have $|X| \le \tau^\omega < \kappa \le |X|$. This is a contradiction. □

To prove Theorem 2.1(2), (3) and Theorem 2.2, the lemma below is useful.

Lemma 2.6. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$ such that $\kappa \le \tau^\omega$ for some cardinal $\tau < \mathrm{cf}(\kappa)$. Then for each space X^* with $|X^*| \le \kappa$, there is a Hausdorff space X having a G_δ-diagonal and satisfying the following conditions:*

(1) *$X = X^*$ as a set,*
(2) *each open set in X^* is open in X,*
(3) *if $L(\mathcal{U}^*) < \kappa$ for each family $\mathcal{U}^*$ of open sets in X^*, then $L(\mathcal{U}) < \kappa$ for each family $\mathcal{U}$ of open sets in X,*
(4) *if X^* is a zero-dimensional Tychonoff space, then so is X.*

Proof: Put $X = X^*$ as a set. By $|X| = |X^*| \le \kappa \le \tau^\omega$, there is a one-to-one function f from X into ${}^\omega\tau$. For each $n \in \omega$ and $s \in {}^n\tau$, let $G(s) = \{x \in X : f(x) \restriction n = s\}$. Take a base $\mathcal{B}^*$ of X^*. Let us define a topology on X having a base

$$\mathcal{B} = \{B^* \cap G(s) : B^* \in \mathcal{B}^*,\ n \in \omega,\ s \in {}^n\tau\}.$$

Obviously, (1) and (2) hold. Let $\mathcal{G}_n = \{G(s) : s \in {}^n\tau\}$ for each $n \in \omega$. Then each $\mathcal{G}_n$ is a pairwise disjoint open cover of X, so it is also a clopen cover. Since f is one-to-one, it is easily seen that $\bigcap_{n\in\omega} \mathrm{St}(x,\mathcal{G}_n) = \{x\}$ for each $x \in X$. Hence X is a Hausdorff space having a G_δ-diagonal. If X^* is a zero-dimensional Tychonoff space, then we may chose $\mathcal{B}^*$ as a family of clopen sets of X^*, and it makes $\mathcal{B}$ a family of clopen sets of X, so X is also a zero-dimensional Tychonoff space, hence (4) holds.

Assume that $L(\mathcal{U}^*) < \kappa$ for any family $\mathcal{U}^*$ of open sets in X^*. Let $\mathcal{U}$ be any family of open sets in X. For each $n \in \omega$ and $s \in {}^n\tau$, let $\mathcal{U}^*(s)$ be the family of all open sets U^* in X^* such that $U^* \cap G(s)$ is contained by some member of $\mathcal{U}$. By $L(\mathcal{U}^*(s)) < \kappa$, there is a subfamily $\mathcal{V}^*(s)$ of $\mathcal{U}^*(s)$ with $|\mathcal{V}^*(s)| < \kappa$ such that $\bigcup \mathcal{V}^*(s) = \bigcup \mathcal{U}^*(s)$. For each $V^* \in \mathcal{V}^*(s)$, take a $U(V^*, s) \in \mathcal{U}$ with $V^* \cap G(s) \subset U(V^*, s)$. We let

$$\mathcal{V} = \{U(V^*, s) : n \in \omega,\ s \in {}^n\tau,\ V^* \in \mathcal{V}^*(s)\}.$$

By $\omega, \tau < \mathrm{cf}(\kappa)$, note that $|\mathcal{V}| < \kappa$.

Let $x \in \bigcup \mathcal{U}$. Take a $U \in \mathcal{U}$ with $x \in U$. There are a $B^* \in \mathcal{B}^*$, an $n \in \omega$, and an $s \in {}^n\tau$ with $x \in B^* \cap G(s) \subset U$. Then we have $B^* \in \mathcal{U}^*(s)$. Since $x \in B^* \in \mathcal{U}^*(s)$ and $\bigcup \mathcal{V}^*(s) = \bigcup \mathcal{U}^*(s)$, there is a $V^* \in \mathcal{V}^*(s)$ with $x \in V^*$. Then we have $x \in V^* \cap G(s) \subset U(V^*, s) \in \mathcal{V}$. Hence $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ is true. Since $|\mathcal{V}| < \kappa$, we conclude $L(\mathcal{U}) < \kappa$. (3) is satisfied. □

The following easy fact is used to see Theorem 2.1(2).

Lemma 2.7 (folklore)**.** *Let κ be a limit cardinal. Then there is a space X^* with $e(X^*) = |X^*| = \kappa$ (which is T_1 but not Hausdorff) such that $L(\mathcal{U}^*) < \kappa$ for each family $\mathcal{U}^*$ of open sets of X^*.*

Proof: Define a topology on $X^* = \kappa$ by letting that $U^* \subset X^*$ is open iff $U^* = \emptyset$ or $(\gamma, \kappa) \subset U^*$ for some $\gamma < \kappa$. Then X^* is a required one. $\square$

Now we are ready to prove Theorem 2.1(2). It suffices to show the proposition below.

Proposition 2.8. *Let κ be a limit cardinal with $\mathrm{cf}(\kappa) > \omega$. Assume that $\kappa \leq \tau^\omega$ for some $\tau < \mathrm{cf}(\kappa)$. Then there is a Hausdorff space X having a G_δ-diagonal with $e(X) = L(X) = |X| = \kappa$ such that $L(\mathcal{U}) < \kappa$ for every family $\mathcal{U}$ of open sets of X.*

Proof: Let X^* be the space obtained by Lemma 2.7. And let X be the space which is obtained by applying Lemma 2.6 for X^*. For each cardinal $\lambda < \kappa$, there is a closed discrete subset D in X^* with $|D| = \lambda$ since $e(X^*) = \kappa$. And such D is also closed discrete in X since each open set in X^* is also open in X. Therefore $\kappa \leq e(X) \leq L(X) \leq |X| \leq \kappa$ holds. $\square$

The space X in the proof of the proposition above is not regular. To find a regular example X, we need another space X^*. Fortunately, Roitman's example of Theorem 1.2 is a required one for $\kappa = \aleph_{\omega_1}$.

Corollary 2.9. *Assume that $\aleph_{\omega_1} \leq 2^\omega$ and a first-countable Luzin space exists. Then there is a zero-dimensional Tychonoff space X^* with $e(X^*) = |X^*| = \aleph_{\omega_1}$ such that $L(\mathcal{U}^*) < \aleph_{\omega_1}$ for each family $\mathcal{U}^*$ of open sets of X^*.*

Proof: Let X be the Roitman's example of Theorem 1.2 constructed in [10]. Reading the proof, we see that X satisfies the following conditions.

(1) For a Luzin space Y with $|Y| = \omega_1$, $X = \bigcup_{y \in Y} X_y$ is a pairwise disjoint union by closed discrete subsets.
(2) $\{|X_y| : y \in Y\}$ is an unbounded subset of $\aleph_{\omega_1}$.
(3) For each $x \in X$ and for each neighborhood U of x in X, there is an open set V in Y with $y(x) \in \mathrm{Cl}_Y(V)$, where $y(x) \in Y$ with $x \in X_{y(x)}$, such that $\bigcup_{y \in V} X_y \subset U$.

We show that $X^* = X$ witnesses the Corollary. By (1) and (2), we have $e(X) = |X| = \aleph_{\omega_1}$. Let $\mathcal{U}$ be a family of open sets in X. It suffices to show that $L(\mathcal{U}) < \aleph_{\omega_1}$. Let $\mathcal{V}$ be the family of all open sets V in Y such that $\bigcup_{y \in V} X_y \subset U(V)$ for some $U(V) \in \mathcal{U}$. Take and fix such $U(V)$ for each $V \in \mathcal{V}$. Put $\hat{V} = \bigcup \mathcal{V}$ and $F = \mathrm{Cl}_Y(\hat{V}) \setminus \hat{V}$. Since F is a nowhere dense subset of a Luzin space Y, we have $|F| \leq \omega < \omega_1 = \mathrm{cf}(\aleph_{\omega_1})$, hence $|\bigcup_{y \in F} X_y| < \aleph_{\omega_1}$ holds. It is known that every Luzin space is hereditarily Lindelöf [8]. Therefore, there is a countable subfamily $\mathcal{V}_0$ of $\mathcal{V}$ with $\bigcup \mathcal{V}_0 = \hat{V}$.

For each $x \in \bigcup \mathcal{U}$, take and fix a $U_x \in \mathcal{U}$ with $x \in U_x$. By (3), we can take an open set V_x in Y with $y(x) \in \mathrm{Cl}_Y(V_x)$, where $y(x) \in Y$ with $x \in X_{y(x)}$, such that $\bigcup_{y\in V_x} X_y \subset U_x$. Then U_x witnesses that $V_x \in \mathcal{V}$. By $V_x \subset \hat{V}$, we have $y(x) \in \mathrm{Cl}_Y(V_x) \subset \mathrm{Cl}_Y(\hat{V})$. In case $y(x) \notin \hat{V}$, by $y(x) \in F$, we have $x \in X_{y(x)} \subset \bigcup_{y\in F} X_y$. In case $y(x) \in \hat{V}$, there is a $V_0 \in \mathcal{V}_0$ with $y(x) \in V_0$, so we have $x \in X_{y(x)} \subset \bigcup_{y\in V_0} X_y \subset U(V_0)$. Hence, a subfamily $\mathcal{U}_0$ of $\mathcal{U}$ with $\bigcup \mathcal{U}_0 = \bigcup \mathcal{U}$ is obtained by putting $\mathcal{U}_0 = \{U(V_0) : V_0 \in \mathcal{V}_0\} \cup \{U_x : x \in (\bigcup_{y\in F} X_y) \cap \bigcup \mathcal{U}\}$. And we have $L(\mathcal{U}) \le |\mathcal{U}_0| \le |\mathcal{V}_0| + |\bigcup_{y\in F} X_y| < \aleph_{\omega_1}$. □

Now we are ready to prove Theorem 2.2. It suffices to show the proposition below.

Proposition 2.10. *Assume that $\aleph_{\omega_1} \le 2^\omega$ and a first countable Luzin space exists. Then there is a zero-dimensional Tychonoff space X having a G_δ-diagonal with $e(X) = L(X) = |X| = \aleph_{\omega_1}$ such that $L(\mathcal{U}) < \aleph_{\omega_1}$ for each family $\mathcal{U}$ of open sets of X.*

Proof: Let X^* be the space obtained by Corollary 2.9. And let X be the space which is obtained by applying Lemma 2.6 for X^*, $\kappa = \aleph_{\omega_1}$ and $\tau = 2$. For each cardinal $\lambda < \aleph_{\omega_1}$, there is a closed discrete subset D in X^* with $|D| = \lambda$ since $e(X^*) = \aleph_{\omega_1}$. And such D is also closed discrete in X since each open set in X^* is also open in X. Therefore $\aleph_{\omega_1} \le e(X) \le L(X) \le |X| \le \aleph_{\omega_1}$ holds. □

Modifying the proofs of Theorem 1.2 and Corollary 2.9, we obtain the theorem below. We give a sketch of the proof in Section 4 for readers convenience.

Theorem 2.11 (Modifying Roitman's Theorem [10])**.** *Let κ be a limit cardinal. Assume that $\kappa \le \sup\{2^\theta : \theta$ is a cardinal $< \mathrm{cf}(\kappa)\}$ and a $\mathrm{cf}(\kappa)$-Suslin line exists. Then there is a zero-dimensional Tychonoff space X^* with $e(X^*) = |X^*| = \kappa$ such that $L(\mathcal{U}^*) < \kappa$ for every family $\mathcal{U}^*$ of open sets of X^*.*

Now we are ready to prove Theorem 2.1(3). It suffices to show the proposition below.

Proposition 2.12. *Let κ be a limit cardinal with $\mathrm{cf}(\kappa) > \omega$. Assume that $\kappa \le \tau^\omega$ for some $\tau < \mathrm{cf}(\kappa)$, and a $\mathrm{cf}(\kappa)$-Suslin line exists. Then there is a zero-dimensional Tychonoff space X having a G_δ-diagonal with $e(X) = L(X) = |X| = \kappa$ such that $L(\mathcal{U}) < \kappa$ for every family $\mathcal{U}$ of open sets of X.*

Proof: We may assume that $\tau \ge \omega$. By $\kappa \le \tau^\omega \le 2^\tau \le \sup\{2^\theta : \theta$ is a cardinal $< \mathrm{cf}(\kappa)\}$, we can apply Theorem 2.11 and obtain a space X^*. And let X be the space which is obtained by applying Lemma 2.6 for X^*. For each cardinal $\lambda < \kappa$, there is a closed discrete subset D in X^* with $|D| = \lambda$ since $e(X^*) = \kappa$. And such D is also closed discrete in X since each open set in X^* is also open in X. Therefore $\kappa \le e(X) \le L(X) \le |X| \le \kappa$ holds. □

In our proof of Theorem 2.1(2) and (3), we use the assumption that $\tau < \mathrm{cf}(\kappa)$. It is natural to consider the case that $\mathrm{cf}(\kappa) \le \tau < \kappa$, but the author does not reach any result about it.

Problem 1. Let κ be a limit cardinal with $\mathrm{cf}(\kappa) > \omega$ such that $\tau^\omega < \kappa$ for every $\tau < \mathrm{cf}(\kappa)$, and there is some cardinal τ_0 with $\mathrm{cf}(\kappa) \le \tau_0 < \kappa \le {\tau_0}^\omega$. Is there a space X with $L(X) = \kappa$ such that $L(\mathcal{U}) < \kappa$ for every open cover $\mathcal{U}$ of X?

3. Spaces having point-countable bases

In this section, we discuss the sup = max problem for the extent of spaces having point-countable bases. There is no difference from the sup = max problem for the Lindelöf degree since such spaces are (sub)metalindelöf. It is well-known that each metrizable space has a σ-locally finite base, and it is trivial that such base is point-countable. So having a point-countable base is one of the generalized metric properties. If κ is a limit cardinal with $\mathrm{cf}(\kappa) = \omega$, then as seen in Example 1.4, the sup = max condition for the extent does not always hold even for metrizable spaces X with $e(X) = \kappa$. So we are interested in the case of $\mathrm{cf}(\kappa) > \omega$.

Problem 2 ([6, Problem 1])**.** Assume that a space X has a point-countable base with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Is there a closed discrete subset of size κ in X?

Answering the problem partially, we prove in this section the theorem below. (In fact, the condition $e(X) = \kappa$ can be replaced by $|X| \ge \kappa$.)

Theorem 3.1. *Let X be a space having a point-countable base with $e(X) = \kappa$. Assume that*

(i) $\tau^\omega < \kappa$ *for each cardinal* $\tau < \kappa$,

(ii) $\tau^\omega < \mathrm{cf}(\kappa)$ *for each cardinal* $\tau < \mathrm{cf}(\kappa)$.

Then X has a closed discrete subset of size κ.

In the theorem above, the condition $\mathrm{cf}(\kappa) > \omega$ automatically holds. Actually, it follows from $2 < \omega \le \mathrm{cf}(\kappa)$ that $\omega < 2^\omega < \mathrm{cf}(\kappa)$ holds by applying the assumption (ii) for $\tau = 2$. To prove the theorem, we use the well-known lemma below.

Lemma 3.2 (The Δ-system lemma. See [9, Chapter II, Theorem 1.6])**.** *Let κ be an infinite cardinal, and $\mathcal{A}$ a family of sets such that $|\mathcal{A}| = \theta > \kappa$ and $|A| < \kappa$ for each $A \in \mathcal{A}$. If θ is regular and $|\alpha^{<\kappa}| < \theta$ for each $\alpha < \theta$, then $\mathcal{A}$ has a subfamily $\mathcal{B}$ with $|\mathcal{B}| = \theta$ which forms a Δ-system. I.e., there is a set R such that $A \cap B = R$ holds for every distinct members A, B of $\mathcal{B}$.*

The set R in the lemma above is called the *root* of a Δ-system $\mathcal{B}$. The corollary below is easily obtained by applying the Δ-system lemma for $\kappa = \omega_1$.

Corollary 3.3. *Let θ be a regular cardinal such that $\tau^\omega < \theta$ for every cardinal $\tau < \theta$. Let $\mathcal{U}$ be a family of subsets of a space X, and $\{\mathcal{W}_j : j \in J\}$ a collection of countable subfamilies of $\mathcal{U}$ with $|J| = \theta$. Then there is a subset J' of J with $|J'| = \theta$ such that $\{\mathcal{W}_j : j \in J'\}$ forms a Δ-system, i.e., the root $\mathcal{R}$ exists and $\mathcal{W}_j \cap \mathcal{W}_k = \mathcal{R}$ holds for every distinct members j, k of J. In particular, the following hold.*

(1) $\omega_1 \le 2^\omega < \theta$.

(2) $\mathcal{R} = \bigcap_{j \in J'} \mathcal{W}_j$, *and so $\mathcal{R}$ is a countable subfamily of $\mathcal{U}$.*

(3) $\{\mathcal{W}_j \setminus \mathcal{R} : j \in J'\}$ *is pairwise disjoint.*

If $\mathcal{B}$ is a base of a space X, then $\mathcal{B}$ is an open cover of X, and since a space X is assumed to be a T_1-space, $\bigcap \mathcal{B}_x = \{x\}$ holds for each $x \in X$, where $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. So the proposition below suffices to derive Theorem 3.1.

Proposition 3.4. *Let κ be an infinite cardinal such that*

(i) $\tau^\omega < \kappa$ *for each cardinal* $\tau < \kappa$,

(ii) $\tau^\omega < \mathrm{cf}(\kappa)$ *for each cardinal* $\tau < \mathrm{cf}(\kappa)$.

And let X be a space with $|X| \geq \kappa$ which has a point-countable open over $\mathcal{U}$ such that $\sup\{|\bigcap \mathcal{U}_x| : x \in X\} < \kappa$, *where* $\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\}$. *Then X has a closed discrete subset of size κ.*

Proof: Let Θ be the set of all regular cardinals θ with $\sup\{|\bigcap \mathcal{U}_x| : x \in X\} < \theta \leq \kappa$ such that $\tau^\omega < \theta$ for every cardinal $\tau < \theta$. And let $\theta \in \Theta$. Take a subset A_θ of X with $|A_\theta| = \theta$. Then $\mathcal{U}_x$ is a countable subfamily of $\mathcal{U}$ for each $x \in A_\theta$ since $\mathcal{U}$ is point-countable. By the Δ-system lemma, we obtain a subset B_θ of A_θ with $|B_\theta| = \theta$ such that $\{\mathcal{U}_x : x \in B_\theta\}$ forms a Δ-system, and let $\mathcal{V}_\theta$ be the root of it. Then $\mathcal{V}_\theta = \bigcap_{x \in B_\theta} \mathcal{U}_x$ is a countable subfamily of $\mathcal{U}$. We show that $\mathcal{U} \setminus \mathcal{V}_\theta$ is an open cover of X. To see this, let $x \in X$. Since $|\bigcap \mathcal{U}_x| < \theta = |B_\theta|$, we can take a $y \in B_\theta \setminus \bigcap \mathcal{U}_x$ and a $U \in \mathcal{U}_x \subset \mathcal{U}$ with $y \notin U$. By $U \notin \mathcal{U}_y \supset \mathcal{V}_\theta$, we have $U \in \mathcal{U} \setminus \mathcal{V}_\theta$ and $x \in U$. Hence, $\mathcal{U} \setminus \mathcal{V}_\theta$ covers X.

In case κ is regular, by the assumptions, we have $\kappa \in \Theta$, so a subset B_κ of X with $|B_\kappa| = \kappa$ and a subfamily $\mathcal{V}_\kappa$ of $\mathcal{U}$ had been taken. It suffices to show that B_κ is closed discrete in X. Let $x \in X$. Since $\mathcal{U} \setminus \mathcal{V}_\kappa$ is an open cover of X, there is an open neighborhood U of x which belongs to $\mathcal{U} \setminus \mathcal{V}_\kappa$. Such U witnesses that B_κ is closed discrete, that is $|U \cap B_\kappa| \leq 1$ holds. Otherwise, there are distinct $y, z \in U \cap B_\kappa$. Then we have $U \in \mathcal{U}_y \cap \mathcal{U}_z = \mathcal{V}_\kappa$, and it is contradiction.

In case κ is singular, Θ is an unbounded subset of κ. Actually, $\kappa \notin \Theta$ since κ is singular, and $(\mu^\omega)^+ \in \Theta$ holds for every cardinal $\mu > 1$ with $\sup\{|\bigcap \mathcal{U}_x| : x \in X\} \leq \mu < \kappa$. Take a subset Θ_0 of $\Theta \setminus \mathrm{cf}(\kappa)$ which is unbounded in κ and of order type $\mathrm{cf}(\kappa)$. For each $\mu \in \Theta_0$, put

$$D_\mu = \{y \in B_\mu : (\mathcal{U}_y \setminus \mathcal{V}_\mu) \cap \bigcup \{\mathcal{U}_z : \nu \in \Theta_0 \cap \mu, z \in B_\nu\} = \emptyset\}.$$

We show that $|D_\mu| = \mu$. Let $\mathcal{U}[< \mu] = \bigcup \{\mathcal{U}_z : \nu \in \Theta_0 \cap \mu, z \in B_\nu\}$. Then $|\mathcal{U}[< \mu]| < \mu$ holds since μ is regular, $|\Theta_0 \cap \mu| < \mathrm{cf}(\kappa) \leq \mu$, $|B_\nu| = \nu < \mu$ for each $\nu \in \Theta_0 \cap \mu$, and $|\mathcal{U}_z| \leq \omega < 2^\omega < \mu$ for each $z \in B_\nu$. We have $|B_\mu \setminus D_\mu| \leq |\mathcal{U}[< \mu]| < \mu$ since $\{\mathcal{U}_y \setminus \mathcal{V}_\mu : y \in B_\mu\}$ is pairwise disjoint and $\mathcal{U}_y \setminus \mathcal{V}_\mu$ meets $\mathcal{U}[< \mu]$ for each $y \in B_\mu \setminus D_\mu$. Hence $|D_\mu| = \mu$ holds by $|B_\mu| = \mu$.

For each $\theta \in \Theta_0$, a countable subfamily $\mathcal{V}_\theta$ of $\mathcal{U}$ had been taken. And $|\Theta_0| = \mathrm{cf}(\kappa)$ holds. By the assumption (ii), we can apply the Δ-system lemma, and obtain a subset Θ_1 of Θ_0 with $|\Theta_1| = \mathrm{cf}(\kappa)$ such that $\{\mathcal{V}_\theta : \theta \in \Theta_1\}$ forms a Δ-system. Let $\mathcal{V}$ be the root. Then $\mathcal{V} = \bigcap_{\theta \in \Theta_1} \mathcal{V}_\theta$ is a countable subfamily of $\mathcal{U}$. Let $D = \bigcup_{\theta \in \Theta_1} D_\theta$. Then $|D| = \kappa$ since $|D_\theta| = \theta$ for each $\theta \in \Theta_1$, and Θ_1 is an unbounded subset in κ. So it suffices to show that D is closed discrete in X.

Let $x \in X$. Take an open neighborhood U_1 of x and a $\theta \in \Theta_1$ such that: if $\mathcal{U}_x \cap \bigcup_{\lambda\in\Theta_1} \mathcal{V}_\lambda \setminus \mathcal{V}$ is non-empty, then $U_1 \in \mathcal{V}_\theta \setminus \mathcal{V}$. Since $\mathcal{U} \setminus \mathcal{V}_\theta$ is an open cover of X, we can take an open neighborhood U of x such that $U \subset U_0 \cap U_1$ for some $U_0 \in \mathcal{U} \setminus \mathcal{V}_\theta$. It suffices to show that $|U \cap D| \leq 1$ holds. Otherwise, there are distinct $y, z \in U \cap D$. Take $\mu, \nu \in \Theta_1$ with $y \in D_\mu$ and $z \in D_\nu$. We may assume that $\nu \leq \mu$. By $x, y, z \in U \subset U_0$ and $U_0 \in \mathcal{U} \setminus \mathcal{V}_\theta$, we have $U_0 \in \mathcal{U}_x \cap \mathcal{U}_y \cap \mathcal{U}_z$ and $U_0 \notin \mathcal{V}$. If $\nu = \mu$, then by $y, z \in D_\mu \subset B_\mu$, we have $\mathcal{U}_y \cap \mathcal{U}_z = \mathcal{V}_\mu$. Otherwise, $\nu < \mu$, then by $\nu \in \Theta_1 \cap \mu \subset \Theta_0 \cap \mu$, $z \in D_\nu \subset B_\nu$, and $y \in D_\mu$, we have $(\mathcal{U}_y \setminus \mathcal{V}_\mu) \cap \mathcal{U}_z = \emptyset$, and so $\mathcal{U}_y \cap \mathcal{U}_z \subset \mathcal{V}_\mu$. Hence, $U_0 \in \mathcal{U}_y \cap \mathcal{U}_z \subset \mathcal{V}_\mu$ holds in any case. It follows that $\mathcal{U}_x \cap \mathcal{V}_\mu \setminus \mathcal{V} \subset \mathcal{U}_x \cap \bigcup_{\lambda\in\Theta_1} \mathcal{V}_\lambda \setminus \mathcal{V}$ is non-empty since U_0 is a member of it. And so $x \in U \subset U_1$ for some $U_1 \in \mathcal{V}_\theta \setminus \mathcal{V}$. By $U_1 \in \mathcal{V}_\theta \subset \mathcal{U}$ and $y, z \in U \subset U_1$, we have $U_1 \in \mathcal{U}_y \cap \mathcal{U}_z \subset \mathcal{V}_\mu$. By $U_0 \notin \mathcal{V}_\theta$ and $U_0 \in \mathcal{V}_\mu$, we have $\theta \neq \mu$. Therefore, $U_1 \in \mathcal{V}_\theta \cap \mathcal{V}_\mu = \mathcal{V}$. This is contradiction. □

The author still does not know any example of a space having a point-countable base which refutes the sup = max condition for the extent of uncountable cofinality.

Problem 3. Can we remove the assumptions (i) and (ii) from Theorem 3.1?

In particular, we have

Problem 4. Is it consistent with ZFC that there is a space X having a point-countable base with $e(X) = \aleph_{\omega_1}$ and with no closed discrete subset of size $\aleph_{\omega_1}$?

The theorem in this section does not give an answer for the problem above since the assumption of it requires that $\mathrm{cf}(\aleph_{\omega_1}) = \omega_1 \leq 2^\omega < \mathrm{cf}(\kappa)$.

If a space X has a point-countable base, then X is hereditarily meta-lindelöf and first-countable. But only assuming that a space X is hereditarily metalindelöf, it is not sufficient for deriving Theorem 3.1 by Example 1.4. And only assuming that a space X is first-countable, it is also not sufficient for deriving Theorem 3.1 as the example below shows.

Example 3.5. Let κ be a limit cardinal with $\mathrm{cf}(\kappa) > \omega$, and set

$$X = \{\alpha + 1 : \alpha \in \kappa\} \cup \{\theta \in \kappa : \theta \text{ is a cardinal}, \mathrm{cf}(\theta) = \omega\}.$$

Then X is first-countable, $e(X) = |X| = \kappa$, but there is no closed discrete subset in X of size κ.

Proof: Obviously, $e(X) \leq |X| \leq \kappa$ holds. For each infinite cardinal $\lambda < \kappa$, there is no cardinal θ with $\lambda < \theta \leq \lambda + \lambda$, so $\{\alpha + 1 : \lambda \leq \alpha < \lambda + \lambda\}$ is a closed discrete subset in X of size λ, hence $\lambda \leq e(X)$ holds. We have $e(X) = |X| = \kappa$ since κ is a limit cardinal.

Let $Z \subset X$ be unbounded in κ. Since κ is a limit cardinal again, we can inductively take strictly increasing sequences $\{\beta_n : n \in \omega\}$ by members of Z and $\{\lambda_n : n \in \omega\}$ by cardinals less than κ such that $\beta_n \leq \lambda_n < \beta_{n+1}$ for each $n \in \omega$. Put $\theta = \sup\{\beta_n : n \in \omega\} = \sup\{\lambda_n : n \in \omega\}$. Then $\theta \in \kappa$ by $\mathrm{cf}(\kappa) > \omega$, and θ is a limit cardinal with $\mathrm{cf}(\theta) = \omega$, so we have $\theta \in X$. On the other hand, θ is a limit point of $\{\beta_n : n \in \omega\} \subset Z$, so Z is not a closed discrete subset in X. Therefore, any closed discrete subset D in X is bounded in κ, so $|D| < \kappa$. Hence, X does not have a closed discrete subset of size κ. $\square$

The space X in the example above is not (sub)metalindelöf. On the other hand, the space X_κ in Example 1.4 is not first-countable in case $\mathrm{cf}(\kappa) > \omega$. For spaces having the both property, what happens?

Problem 5. Let κ be an infinite cardinal satisfying the conditions (i) and (ii) in the assumption of Theorem 3.1. And let X be a hereditarily metalindelöf and first-countable space with $e(X) = \kappa$. Does X have a closed discrete subset of size κ?

4. A sketch of a proof of Theorem 2.11

A proof of Theorem 2.11 is obtained by modifying proofs of Theorem 1.2 and Corollary 2.9. We give here a sketch of a proof for reader's convenience.

First, check that the lemma below holds. Proofs are routine.

Lemma 4.1 (folklore)**.** *Let λ be a regular uncountable cardinal, and K a LOTS having the λ-c.c. Then, the following hold.*

(1) *There is neither a strictly increasing sequence nor a strictly decreasing sequence, of length λ, by members of K.*
(2) *The character of K at each point is less than λ.*
(3) *There is neither a strictly ascending sequence nor a strictly descending sequence, of length λ, by convex subsets of K.*
(4) *If $\mathcal{U}$ is a family of open sets in K such that*
    for each subfamily $\mathcal{U}'$ of $\mathcal{U}$ with $|\mathcal{U}'| < \lambda$, there is a $U \in \mathcal{U}$ with $\bigcup \mathcal{U}' \subset U$,
    then there is a pairwise disjoint family $\mathcal{J}$ of non-empty open convex subsets of K partially refining $\mathcal{U}$ and satisfying that:
    for each non-empty open convex subset J' of K, if $J' \subset U$ for some $U \in \mathcal{U}$, then $J' \subset J$ for some $J \in \mathcal{J}$.
(5) *$L(\mathcal{U}) < \lambda$ for each family $\mathcal{U}$ of open sets in K.*
(6) *For each open set U in a subspace Z of K, there is a subset S of U with $|S| < \lambda$ such that $\mathrm{Cl}_Z(U) \setminus U \subset \mathrm{Cl}_Z(S)$.*

It is well-known that if a Suslin line exists, then a Suslin tree also exists, (see [9]). Modifying the proof of this fact, we obtain the lemma below.

Lemma 4.2 (folklore)**.** *Let λ be a regular uncountable cardinal and K a λ-Suslin line. Then for each subset E of K with $|E| < \lambda$, there is a λ-Suslin tree $T = (T, <_T)$ such that*

- *each member of T is an open convex set in K and disjoint from E,*
- *for each $J_0, J_1 \in T$, $J_0 <_T J_1$ holds iff $J_0 \supsetneq J_1$,*
- *each members J_0 and J_1 of T are incompatible in T iff $J_0 \cap J_1 = \emptyset$.*

In particular, for any ordinal $\alpha < \lambda$, there are an open convex subset J of K which is disjoint from E and a sequence, of length α, by members of J which is either strictly increasing or strictly decreasing.

Let α be an ordinal having the linear order topology by the usual order. If J is a convex subset of a compact LOTS K, and there is a sequence, of length α, by members of J which is either strictly increasing or strictly decreasing, then we can take such a sequence as a topological embedding. So we obtain the lemma below.

Lemma 4.3 (folklore)**.** *Let λ be a regular uncountable cardinal, and K a compact λ-Suslin line. Then there is a subspace $Z = \bigcup_{\alpha\in\lambda} Z_\alpha$ of K such that Z_α is homeomorphic to α and $Z_\alpha \bigcap \mathrm{Cl}_Z(\bigcup_{\beta<\alpha} Z_\beta) = \emptyset$ for each $\alpha < \lambda$.*

It is well-known that any LOTS K^* is embedded into some compact LOTS K as a dense subset. And it is easily seen that if K^* is a λ-Suslin line, where λ is a regular uncountable cardinal, then so is K.

Lemma 4.4 (folklore)**.** *Let λ be a regular uncountable cardinal. If a λ-Suslin line exists, then there is a compact λ-Suslin line.*

It is routine to check that the lemma below holds.

Lemma 4.5 (folklore)**.** *Let θ be an infinite cardinal, Z a GO-space (= subspace of a LOTS), and $\varphi : (\theta + 1) \to Z$ a topological embedding with $z = \varphi(\theta)$. Then there is a pairwise disjoint sequence $\{Q(\zeta) : \zeta < \theta\}$ of non-empty open subsets of Z such that for each neighborhood W of z in Z, $\{\zeta < \theta : Q(\zeta) \not\subset W\}$ is bounded in θ.*

The space Z in Lemma 4.3 witnesses the lemma below.

Lemma 4.6 (folklore)**.** *Let λ be a regular uncountable cardinal and assume that a λ-Suslin line exists. Then there is a regular space Z such that*

- (i') *$L(\mathcal{U}) < \lambda$ for each family $\mathcal{U}$ of open sets in Z,*
- (ii') *$|F| < \lambda$ for each nowhere dense subset F in Z,*
- (iii') *the character of Z at each point is less than λ,*
- (iv') *for each $E \subset Z$ with $|E| < \lambda$ and for each infinite cardinal $\theta < \lambda$, there are a pairwise disjoint sequence $\{Q(\zeta) : \zeta < \theta\}$ of non-empty open sets of Z, and a point $z \in Z \setminus \mathrm{Cl}_Z(E)$ such that for each neighborhood W of z, $\{\zeta < \theta : Q(\zeta) \not\subset W\}$ is bounded in θ.*

Let θ be an infinite cardinal. A collection $\{\Theta_\alpha : \alpha \in \Omega\}$ of subsets of θ is called an *independent family* if for each disjoint finite subsets I and O of Ω, there are unbounded many $\zeta \in \theta$ such that $\zeta \in \Theta_\alpha$ for each $\alpha \in I$, and $\zeta \notin \Theta_{\alpha'}$ for each $\alpha' \in O$.

Lemma 4.7 (Hausdorff, see [7])**.** *Let θ be an infinite cardinal. Then there is an independent family $\{\Theta_\alpha : \alpha < 2^\theta\}$ of subsets of θ.*

Lemma 4.8. *Let κ be an uncountable cardinal. And assume that $\kappa \le \sup\{2^\theta : \theta$ is a cardinal $< \mathrm{cf}(\kappa)\}$ and a $\mathrm{cf}(\kappa)$-Suslin line exists. Then there are a space Y with $|Y| = \mathrm{cf}(\kappa)$ and an unbounded subset $\{\kappa_y : y \in Y\}$ of κ such that*

(i) *$L(\mathcal{V}) < \kappa$ for each family $\mathcal{V}$ of open sets in Y,*
(ii) *$|F| < \mathrm{cf}(\kappa)$ for each nowhere dense subset F in Y,*
(iii) *for each $y \in Y$, there is a collection $\{\mathcal{W}_\alpha : \alpha \in \kappa_y\}$ of filters on Y such that for each $\alpha \in \kappa_y$:*
  (iii-1) *each neighborhood of y in Y belongs to $\mathcal{W}_\alpha$,*
  (iii-2) *for each $W \in \mathcal{W}_\alpha$, there is a $V \in \mathcal{W}_\alpha$ with $V \subset W$ which is open in Y such that $\mathrm{Cl}_Y(V) = V \cup \{y\}$,*
  (iii-3) *there is a $W_\alpha \in \mathcal{W}_\alpha$ such that $Y \setminus W_\alpha \in \mathcal{W}_\beta$ for every $\beta \in \kappa_y$ except α,*
  (iii-4) *$Y \setminus \{y\} \in \mathcal{W}_\alpha$ if $\kappa_y > 1$.*

Proof: Let $\lambda = \mathrm{cf}(\kappa)$ and Z be the space in Lemma 4.6. Take an unbounded subset $\{\lambda_\xi : \xi < \lambda\}$ of κ. By induction on $\xi < \lambda$, we take an ascending sequence $\{E_\xi : \xi < \lambda\}$ of subsets of Z with $|E_\xi| < \lambda$. Put $E_0 = \emptyset$ as the first step. Put $E_\xi = \bigcup_{\xi' < \xi} E_{\xi'}$ in case $\xi < \lambda$ is a limit ordinal. To take $E_{\xi+1}$ for $\xi < \lambda$, assume that a subset E_ξ of Z with $|E_\xi| < \lambda$ is determined. Take a cardinal θ_ξ with $\omega \le \theta_\xi < \lambda$ and $2^{\theta_\xi} \ge \lambda_\xi$, a pairwise disjoint sequence $\{Q_\xi(\zeta) : \zeta < \theta_\xi\}$ of non-empty open subsets of Z, and a point $z_\xi \in Z \setminus \mathrm{Cl}_Z(E_\xi)$ such that for each neighborhood W of z_ξ in Z, $\{\zeta < \theta_\xi : Q_\xi(\zeta) \not\subset W\}$ is bounded in θ_ξ. Let $Q_\xi = \bigcup_{\zeta < \theta_\xi} Q_\xi(\zeta)$. Then $z_\xi \notin Q_\xi$. Actually, $z_\xi \notin Q_\xi(\zeta')$ for any $\zeta' < \theta_\xi$ since $\{\zeta < \theta_\xi : Q_\xi(\zeta) \not\subset Q_\xi(\zeta')\} = \theta_\xi \setminus \{\zeta'\}$ is unbounded in θ_ξ. Moreover, we may assume that $\mathrm{Cl}_Z(Q_\xi) \cap E_\xi = \emptyset$ since Z is regular and $Z \setminus \mathrm{Cl}_Z(E_\xi)$ is a neighborhood of z_ξ. For each $\zeta < \theta_\xi$, take and fix a $q_\xi(\zeta) \in Q_\xi(\zeta)$. Since the character of Z at q_ξ is less than λ, we can take an open neighborhood base $\mathcal{B}_\xi(\zeta)$ at $q_\xi(\zeta)$ with $|\mathcal{B}_\xi(\zeta)| < \lambda$. We may assume that $\mathrm{Cl}_Z(V) \subset Q_\xi(\zeta)$ for every $V \in \mathcal{B}_\xi(\zeta)$. Put $Y_\xi = \{z_\xi\} \cup \{q_\xi(\zeta) : \zeta < \theta_\xi\}$ and $D_\xi = \bigcup\{\mathrm{Cl}_Z(V) \setminus V : \zeta < \theta_\xi, V \in \mathcal{B}_\xi(\zeta)\} \cup ((\mathrm{Cl}_Z(Q_\xi) \setminus Q_\xi) \setminus \{z_\xi\})$. Then E_ξ, Y_ξ and D_ξ are pairwise disjoint subsets of Z, and $|E_\xi|, |Y_\xi|, |D_\xi| < \lambda$ holds. Let $E_{\xi+1} = E_\xi \cup Y_\xi \cup D_\xi$, and continue the induction.

After finishing induction, we obtain pairwise disjoint families $\{Y_\xi : \xi < \lambda\}$ and $\{D_\xi : \xi < \lambda\}$ of subsets of Z. Put $Y = \bigcup_{\xi < \lambda} Y_\xi$ and $D = \bigcup_{\xi < \lambda} D_\xi$. Then we have $|Y| = \lambda = \mathrm{cf}(\kappa)$ and $Y \cap D = \emptyset$. We show that Y, as a topological subspace of Z, satisfies the required conditions.

(i) and (ii) hold for Y by (i') and (ii') for Z since Y is a subspace of Z and $\lambda = \mathrm{cf}(\kappa) \le \kappa$.

(iii) Let $y \in Y$ and take the $\xi < \lambda$ with $y \in Y_\xi$. For each $J \subset \theta_\xi$, put $Q_\xi[J] = \bigcup_{\zeta \in J} Q_\xi(\zeta)$, then $Q_\xi[J]$ is an open set of Z.

In the case of $y = z_\xi$, put $\kappa_y = \lambda_\xi$. Take an independent family $\{\Theta_\alpha : \alpha < 2^{\theta_\xi}\}$ of subsets of θ_ξ. Since $\kappa_y = \lambda_\xi \le 2^{\theta_\xi}$, a subset Θ_α of θ_ξ is defined for each $\alpha \in \kappa_y$. Let $\alpha \in \kappa_y$. For each $O \subset \kappa_y \setminus \{\alpha\}$ with $|O| < \omega$ and for each $\gamma < \theta_\xi$, put $\hat{\Theta}_\alpha(O, \gamma) = \{\zeta \in \Theta_\alpha \setminus (\bigcup_{\alpha' \in O} \Theta_{\alpha'}) : \zeta \ge \gamma\}$ and $V_\alpha(O, \gamma) = Y \cap Q_\xi[\hat{\Theta}_\alpha(O, \gamma)]$.

And let $\mathcal{V}_\alpha = \{V_\alpha(O,\gamma) : O \subset \kappa_y \setminus \{\alpha\}, |O| < \omega, \gamma < \theta_\xi\}$. Obviously, $\mathcal{V}_\alpha$ is a filter base on Y. Let $\mathcal{W}_\alpha$ be the filter on Y generated by $\mathcal{V}_\alpha$. It is routine to check that $\{\mathcal{W}_\alpha : \alpha \in \kappa_y\}$ satisfies the conditions (iii-1)–(iii-4).

In the case of $y \neq z_\xi$, put $\kappa_y = 1$. Take the $\zeta < \theta_\xi$ with $y = q_\xi(\zeta)$. Put $\mathcal{V}_0 = \{V \cap Y : V \in \mathcal{B}_\xi(\zeta)\}$. Obviously, $\mathcal{V}_0$ is a filter base on Y. Let $\mathcal{W}_0$ be the filter on Y generated by $\mathcal{V}_0$. It is routine to check that $\{\mathcal{W}_\alpha : \alpha \in \kappa_y\}$ satisfies the conditions (iii-1)–(iii-4).

It is trivial that $\{\kappa_y : y \in Y\}$ is an unbounded subset of κ. □

Now we are ready to prove Theorem 2.11.

Proof: Let κ be a limit cardinal. Assume that $\kappa \leq \sup\{2^\theta : \theta \text{ is a cardinal} < \mathrm{cf}(\kappa)\}$ and a $\mathrm{cf}(\kappa)$-Suslin line exists. We would like to find a zero-dimensional Tychonoff space X^* with $e(X^*) = |X^*| = \kappa$ such that $L(\mathcal{U}^*) < \kappa$ for every family $\mathcal{U}^*$ of open sets in X^*.

Take a space Y with $|Y| = \mathrm{cf}(\kappa)$ and an unbounded subset $\{\kappa_y : y \in Y\}$ of κ satisfying the conditions (i), (ii), (iii) in Lemma 4.8. Let $y \in Y$. Put $X_y = \{y\} \times \kappa_y$. Take a collection $\{\mathcal{W}_\alpha : \alpha \in \kappa_y\}$ of filters on Y satisfying (iii-1)–(iii-4) for y, and let $\mathcal{W}^*_x = \mathcal{W}_\alpha$ for each $x = \langle y, \alpha\rangle \in X_y$ with $\alpha \in \kappa_y$. Set $X = \bigcup_{y\in Y} X_y$. For each $U \subset X$, put $W[U] = \{y \in Y : X_y \subset U\}$. Define a topology on X such that $U \subset X$ is a neighborhood of $x \in X$ iff $x \in U$ and $W[U] \in \mathcal{W}^*_x$ hold. It is routine to check that we can define such topology on X. Then X is a zero-dimensional Tychonoff space and satisfies that:

1. (1) $X = \bigcup_{y\in Y} X_y$ is a pairwise disjoint union by closed discrete subsets.
2. (2) $\{|X_y| : y \in Y\}$ is an unbounded subset of κ.
3. (3) For each $x \in X$ and for each neighborhood U of x in X, there is an open set V in Y with $y(x) \in \mathrm{Cl}_Y(V)$, where $y(x) \in Y$ with $x \in X_{y(x)}$, such that $\bigcup_{y\in V} X_y \subset U$.

The rest part is similar to the proof of Corollary 2.9. We show that $X^* = X$ witness the theorem. By (1) and (2), we have $e(X) = |X| = \kappa$. Let $\mathcal{U}$ be a family of open sets in X. It suffices to show that $L(\mathcal{U}) < \kappa$. Let $\mathcal{V}$ be the family of all open sets V in Y such that $\bigcup_{y\in V} X_y \subset U(V)$ for some $U(V) \in \mathcal{U}$. Take and fix such $U(V)$ for each $V \in \mathcal{V}$. Put $\hat{V} = \bigcup \mathcal{V}$ and $F = \mathrm{Cl}_Y(\hat{V}) \setminus \hat{V}$. Since F is a nowhere dense subset in Y, we have $|F| < \mathrm{cf}(\kappa)$ by the condition (ii), hence $|\bigcup_{y\in F} X_y| < \kappa$ holds. By the condition (i), there is a subfamily $\mathcal{V}_0$ of $\mathcal{V}$ with $|\mathcal{V}_0| < \kappa$ and $\bigcup \mathcal{V}_0 = \hat{V}$.

For each $x \in \bigcup \mathcal{U}$, take and fix a $U_x \in \mathcal{U}$ with $x \in U_x$. By (3), we can take an open set V_x in Y with $y(x) \in \mathrm{Cl}_Y(V_x)$, where $y(x) \in Y$ with $x \in X_{y(x)}$, such that $\bigcup_{y\in V_x} X_y \subset U_x$. Then U_x witnesses that $V_x \in \mathcal{V}$. By $V_x \subset \hat{V}$, we have $y(x) \in \mathrm{Cl}_Y(V_x) \subset \mathrm{Cl}_Y(\hat{V})$. In case $y(x) \notin \hat{V}$, by $y(x) \in F$, we have $x \in X_{y(x)} \subset \bigcup_{y\in F} X_y$. In case $y(x) \in \hat{V}$, there is a $V_0 \in \mathcal{V}_0$ with $y(x) \in V_0$, so we have $x \in X_{y(x)} \subset \bigcup_{y\in V_0} X_y \subset U(V_0)$. Hence, a subfamily $\mathcal{U}_0$ of $\mathcal{U}$ with $\bigcup \mathcal{U}_0 = \bigcup \mathcal{U}$ is obtained by putting $\mathcal{U}_0 = \{U(V_0) : V_0 \in \mathcal{V}_0\} \cup \{U_x : x \in (\bigcup_{y\in F} X_y) \cap \bigcup \mathcal{U}\}$. And we have $L(\mathcal{U}) \leq |\mathcal{U}_0| \leq |\mathcal{V}_0| + |\bigcup_{y\in F} X_y| < \kappa$. $\square$

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