Ideal independence, free sequences, and the ultrafilter number

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Abstract. We make use of a forcing technique for extending Boolean algebras. The same type of forcing was employed in Baumgartner J.E., Komjáth P., *Boolean algebras in which every chain and antichain is countable*, Fund. Math. **111** (1981), 125–133, Koszmider P., *Forcing minimal extensions of Boolean algebras*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3073–3117, and elsewhere. Using and modifying a lemma of Koszmider, and using CH, we obtain an atomless BA, A such that $\mathfrak{f}(A) = \mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{u}(A)$, answering questions raised by Monk J.D., *Maximal irredundance and maximal ideal independence in Boolean algebras*, J. Symbolic Logic **73** (2008), no. 1, 261–275, and Monk J.D., *Maximal free sequences in a Boolean algebra*, Comment. Math. Univ. Carolin. **52** (2011), no. 4, 593–610.

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This paper is concerned with some "small" cardinal functions defined on Boolean algebras. To describe the results we need the following definition. For notation concerning Boolean algebras, we follow [KMB89].

Definition 1.1.
1. A subset Y of a BA is ideal-independent if $\forall y \in Y$, $y \notin \langle Y \setminus \{y\}\rangle^{\mathrm{id}}$.
2. We define $\mathrm{s}_{\mathrm{mm}}(\mathrm{A})$ to be the minimal size of an ideal-independent family of A that is maximal with respect to inclusion.
3. A free sequence in a BA is a sequence $X = \{x_\alpha : \alpha < \gamma\}$ such that whenever F and G are finite subsets of γ such that $\forall i \in F\ \forall j \in G[i < j]$, then
$$\left(\prod_{\alpha\in F} x_\alpha\right) \cdot \left(\prod_{\beta\in G} -x_\beta\right) \neq 0.$$
Here empty products equal 1 by definition.

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4. We define $\mathfrak{f}(A)$ to be the minimal size of a free sequence in A that is maximal with respect to end-extension.
5. We define $\mathfrak{u}(A)$ to be the minimal size of a nonprincipal ultrafilter generating set of A.
6. If A is a Boolean algebra and u is a nonprincipal ultrafilter on A, let $P(A,u)$ be the partial order consisting of pairs (p_0,p_1) where $p_0,p_1 \in A\backslash u$, and $p_0 \cap p_1 = \emptyset$, ordered by $(p_0,p_1) \leq (q_0,q_1)$ ("(p_0,p_1) is stronger than (q_0,q_1)") iff $q_i \subseteq p_i$ for $i=0,1$.

The main result of this paper is that under CH there is an atomless BA B such that $\omega = \mathfrak{f}(B) = \mathrm{s_{mm}(B)} < \mathfrak{u}(\mathrm{B}) = \omega_1$. Theorem 2.10 in [Mon08] asserts the existence of an atomless BA with $\mathrm{s_{mm}(B)} < \mathfrak{u}(\mathrm{B})$, but the proof is faulty. The existence of an atomless BA B with $\mathfrak{f}(B) < \mathfrak{u}(B)$ is a problem raised in [Mon11].

From now on, fix a countable, atomless subalgebra A of $\mathscr{P}(\omega)$. Fix some maximal ideal-independent $\mathcal{X} \subseteq A$. Also let $C = \langle c_i : i < \xi\rangle \subseteq A$ be a maximal free sequence such that $c_i \subseteq c_j$ for each $i > j \in \xi$. We will always use u to denote a nonprincipal ultrafilter on A.

We will now define many subsets of $P(A,u)$ and prove their density.

Definition 1.2. 1. For each $a \notin u$ put

$$K_a = \Big\{(p_0,p_1) \in P(A,u) : a \subseteq (p_0 \cup p_1), p_0 \setminus a \neq \emptyset \neq p_1 \setminus a\Big\}.$$

2. For $i \in \omega$, put $F_i = \{(p_0,p_1) \in P(A,u) : i \in p_0 \cup p_1\}$.

   For the next two definitions, we need the following. Fix some $e,f \in A$. For any $p \in P(A,u)$ we define $p^* = (e \cap p_0) \cup (f \cap p_1)$, and $a_p = \omega \backslash (p_0 \cup p_1)$.
3. We define $D_{e,f}$ as follows.
   $p \in D_{e,f}$ iff one of the following conditions holds:
   (a) $p_0 \cup p_1 \supseteq e \triangle f$,
   (b) $\exists n \in \omega\, \exists x_0,\dots,x_n \in \mathcal{X}\, [x_0 \subseteq p^* \cup x_1 \cup \ldots \cup x_n]$,
   (c) $\exists n \in \omega\, \exists x_0,\dots,x_n \in \mathcal{X}\, [p^* \cup a_p \subseteq x_0 \cup \ldots \cup x_n]$.
4. We define $E_{e,f}$ as follows.
   $p \in E_{e,f}$ iff one of the following conditions holds:
   (a) $p_0 \cup p_1 \supseteq e \triangle f$,
   (b) $\exists i < j \in \xi\, [p^* \supseteq c_i \setminus c_j]$,
   (c) $\exists i \in \xi\, [p^* \cup a_p \subseteq \omega \setminus c_i]$,
   (d) $\omega \setminus c_0 \subseteq p^*$.

Lemma 1.1. *The subsets of $P(A,u)$ defined above are dense.*

Proof: 1. (K_a is dense.) If $p = (p_0,p_1) \in P(A,u)$, then we have that $b := p_0 \cup p_1 \cup a \notin u$. Because A is atomless, there are disjoint $x_0,x_1 \subseteq \omega \setminus b$ such that each $x_i \notin u$. Define $q_0 = p_0 \cup x_0$ and $q_1 = p_1 \cup x_1 \cup (a \setminus p_0)$. We have $q_0 \setminus a \neq 0$ since $x_0 \subseteq \omega \setminus a$, hence $x_0 = x_0 \setminus a \subseteq q_0 \setminus a$. Similarly $q_1 \setminus a \neq 0$. So (q_0,q_1) is an extension of p in K_a.

2. (F_i is dense.) Since u is nonprincipal, $\{i\}$ is not a member of u for any $i \in \omega$. Thus if $p = (p_0, p_1) \notin F_i$ then $(p_0 \cup \{i\}, p_1)$ is an extension of p that is a member of F_i.

3. ($D_{e,f}$ is dense.) First note the following observation:

$(\otimes)$ If $p \in P(A, u)$ and $x \notin u$, then there is a $q \leq p$ such that $x \subseteq q_0 \cup q_1$.

In fact, $(\otimes)$ follows from the fact that K_x is dense. Now, to show density, let $p \in P(A, u)$. Recall that for any $p \in P(A, u)$ we define $p^* = (e \cap p_0) \cup (f \cap p_1)$, and $a_p = \omega \setminus (p_0 \cup p_1)$. We also define $e_p = a_p \cap e$, and $f_p = a_p \cap f$. One of the following holds:

(i) $e_p \cap f_p \in u$,
(ii) $\omega \setminus (e_p \cup f_p) \in u$,
(iii) $e_p \setminus f_p \in u$,
(iv) $f_p \setminus e_p \in u$.

Note that $e_p \setminus f_p = a_p \cap (e \setminus f)$, $f_p \setminus e_p = a_p \cap (f \setminus e)$, and $e_p \triangle f_p = a_p \cap (e \triangle f)$. If (i) or (ii) is the case, then $e_p \triangle f_p \notin u$, so also $e \triangle f \notin u$ (as $p_0 \cup p_1 \notin u$). By $(\otimes)$ there is $q \leq p$ such that $q_0 \cup q_1 \supseteq e \triangle f$, so that (a) of the definition of $D_{e,f}$ is satisfied.

Next, suppose that (iii) is the case. Then also $e \setminus f \in u$; by $(\otimes)$ there is $q \leq p$ such that $-(e \setminus f) \subseteq q_0 \cup q_1$, so that $a_q \subseteq e \setminus f$. Now by maximality of $\mathcal{X}$ in A we have that for some $n \in \omega$ and some $x_0, \ldots, x_n \in \mathcal{X}$,

(v) $x_0 \subseteq q^* \cup x_1 \cup \ldots \cup x_n$, or
(vi) $q^* \subseteq x_0 \cup \ldots \cup x_n$.

If (v) is the case, then condition (b) in the definition of $D_{e,f}$ is satisfied. So suppose that (vi) is the case. Again, by maximality of $\mathcal{X}$ in A, there is an $m \in \omega$ and some $y_0, \ldots, y_m \in \mathcal{X}$ such that either:

(vii) $a_q \subseteq y_0 \cup \cdots \cup y_m$, or
(viii) $y_0 \subseteq y_1 \cup \ldots \cup y_m \cup a_q$.

If (vii) holds then $q^* \cup a_q \subseteq x_0 \cup \ldots \cup x_n \cup y_0 \cup \ldots \cup y_m$, so condition (c) of the definition of $D_{e,f}$ is satisfied. Suppose then that (viii) holds.

- Case 1. $a_q \cap y_0 \in u$. Then $a_q \setminus y_0 \notin u$. Let $r_0 = q_0$ and $r_1 = q_1 \cup (a_q \setminus y_0)$. We claim that $r^* \cup a_r \subseteq y_0 \cup x_0 \cup \ldots \cup x_n$, so r satisfies (c) in the definition of $D_{e,f}$. In fact, $a_r = a_q \cap y_0 \subseteq y_0$. Now recall $r^* = (e \cap r_0) \cup (f \cap r_1)$. Note that $r_0 \setminus q_0 = \emptyset$ and $r_1 \setminus q_1 \subseteq a_q$. In particular, since $a_q \subseteq e \setminus f$, $f \cap r_1 = f \cap q_1$. Hence $r^* = q^*$, and by (vi) $q^* \subseteq x_0 \cup \ldots \cup x_n$. So r satisfies condition (c) of $D_{e,f}$.

- Case 2. $a_q \cap y_0 \notin u$. Then let $r_0 = q_0 \cup (a_q \cap y_0)$ and let $r_1 = q_1$. Now using (viii) we have that $y_0 \subseteq y_1 \cup \ldots \cup y_m \cup (a_q \cap y_0)$. Also $a_q \cap y_0 \subseteq a_q \subseteq e$, so $a_q \cap y_0 \subseteq r^*$. Thus we have $y_0 \subseteq y_1 \cup \ldots \cup y_m \cup r^*$. So condition (b) in the definition of $D_{e,f}$ is satisfied.

The case when $f_p \setminus e_p \in u$ is treated similarly. Thus we have proved that the sets $D_{e,f}$ are indeed dense.

4. ($E_{e,f}$ is dense.) We will use the following fact several times:

$$(*) \quad \forall a \in A \Big(\exists i \in \xi \big[a \subseteq (\omega \setminus c_i) \big] \text{ or } \exists i < j \in \xi \big[(c_i \setminus c_j) \subseteq a \big] \text{ or } \omega \setminus c_0 \subseteq a \Big)$$

To see this, suppose that $a \in A$. Clearly the desired conclusion holds if $a = \emptyset$ or $a = \omega$; so suppose that $a \neq \emptyset, \omega$. By maximality of C we have that either

(A) $\exists F \in [\xi]^{<\omega}$ such that $(\bigcap_{i \in F} c_i) \cap a = \emptyset$, or
(B) $\exists F, G \in [\xi]^{<\omega}$, with $\forall i \in F \forall j \in G[i < j]$ such that $(\bigcap_{i \in F} c_i) \cap (\bigcap_{j \in G} \omega \setminus c_j) \cap (\omega \setminus a) = \emptyset$.

If (A) holds then $F \neq \emptyset$ since $a \neq \emptyset$ and then $c_{\max F} \cap a = \emptyset$ so that $a \subseteq (\omega \setminus c_{\max F})$, hence the first part of $(*)$ holds.

If (B) holds then $F \neq \emptyset$ or $G \neq \emptyset$ since $a \neq \omega$. If $F \neq \emptyset \neq G$ then $(c_{\max F} \setminus c_{\min G}) \subseteq a$, giving the second condition of $(*)$. If $F \neq \emptyset = G$ then $c_{\max F} \subseteq a$, giving the second condition of $(*)$ again. Finally if $F = \emptyset \neq G$ then $(\omega \setminus c_{\min G}) \subseteq a$, giving the second or third condition of $(*)$.

Now we will prove that $E_{e,f}$ is dense. Let $p \in P(A, u)$. Recall that for any $p \in P(A, u)$ we define $p^* = (e \cap p_0) \cup (f \cap p_1)$, $a_p = \omega \setminus (p_0 \cup p_1)$, $e_p = a_p \cap e$, and $f_p = a_p \cap f$. One of the following holds:

(i) $e_p \cap f_p \in u$,
(ii) $\omega \setminus (e_p \cup f_p) \in u$,
(iii) $e_p \setminus f_p \in u$,
(iv) $f_p \setminus e_p \in u$.

If (i) or (ii) is the case, then $e_p \bigtriangleup f_p \notin u$, so also $e \bigtriangleup f \notin u$ (as $p_0 \cup p_1 \notin u$). Thus we can extend p to a condition q such that $q_0 \cup q_1 \supseteq e \bigtriangleup f$, so that (a) of the definition of $E_{e,f}$ is satisfied.

Next, suppose that (iii) is the case. Then also $e \setminus f \in u$, so we can first extend p to some condition q so that $a_q \subseteq e \setminus f$. Now $q^* \in A$, so, by $(*)$, either

(v) $\exists i < \xi\, [q^* \subseteq \omega \setminus c_i]$, or
(vi) $\exists i < j \in \xi\, [q^* \supseteq c_i \setminus c_j]$, or
(vii) $\omega \setminus c_0 \subseteq q^*$.

If (vi) holds then q is in $E_{e,f}$ by virtue of condition (b). If (vii), then q is in $E_{e,f}$ by virtue of (d). So we assume now that (v) is the case, and fix $i \in \xi$ as guaranteed by (v). Now also $a_q \in A$, so either

(viii) $\exists j < \xi\, [a_q \subseteq \omega \setminus c_j]$, or
(ix) $\exists j < k \in \xi\, [a_q \supseteq c_j \setminus c_k]$, or
(x) $\omega \setminus c_0 \subseteq a_q$.

First suppose that (viii) holds. Then $a_q \cup q^* \subseteq (\omega \setminus c_i) \cup (\omega \setminus c_j) = \omega \setminus (c_i \cap c_j) = \omega \setminus c_{\max\{i,j\}}$, so $q \in E_{e,f}$ by virtue of condition (c). Next assume that (ix) holds and fix $j < k \in \xi$ as in that case. We consider two cases.

- Case 1. $(c_j \setminus c_k) \in u$. Then extend q to a condition r such that $r_0 = q_0$, and $r_1 = q_1 \cup (-q_0 \cap -(c_j \setminus c_k))$. Then $-(c_j \setminus c_k) \subseteq r_0 \cup r_1$, so $a_r \subseteq c_j \setminus c_k$. Note that $r_1 \setminus q_1 \subseteq a_q \subseteq e \setminus f$, so $(r_1 \setminus q_1) \cap f = 0$. Then $r^* = (r_0 \cap e) \cup (r_1 \cap f) = (q_0 \cap e) \cup (r_1 \cap f)$, and $(r_1 \setminus q_1) \cap f = \emptyset$, so in fact $r^* = q^*$. Recall that $q^* \subseteq (\omega \setminus c_i)$ so $r^* \cup a_r \subseteq (\omega \setminus c_{\max\{i,k\}})$. Thus condition (c) holds for r.

- Case 2. $(c_j \setminus c_k) \notin u$. Then we extend q to a condition r so that $r_0 = q_0 \cup (c_j \setminus c_k)$ and $r_1 = q_1$. Recall that $(c_j \setminus c_k) \subseteq a_q \subseteq e$, so $r^* \supseteq (r_0 \cap e) \supseteq (c_j \setminus c_k) \cap e = c_j \setminus c_k$. Thus r satisfies condition (b) in the definition of $E_{e,f}$.

Finally suppose that (x) is the case. Again, we consider two cases.

- Case 1. $a_q \cap c_0 \notin u$. Then we extend q to a condition r where $r_0 = q_0$ and $r_1 = q_1 \cup (a_q \cap c_0)$. Then $a_r \subseteq (\omega \setminus c_0)$. Also $r^* = q^*$ by the same argument as in Case 1 above. So $a_r \cup r^* \subseteq (\omega \setminus c_i)$, and r satisfies condition (c) of the definition of $E_{e,f}$.

- Case 2. $a_q \cap c_0 \in u$. Then we extend q to a condition r by setting $r_0 = q_0 \cup (a_q \setminus c_0)$ and $r_1 = q_1$. Then $r^* \supseteq r_0 \cap e \supseteq \omega \setminus c_0$, so condition (d) in the definition of $E_{e,f}$ holds.

Thus the sets $E_{e,f}$ are dense. □

We will denote by G a filter in $P(A,u)$ that intersects all the sets mentioned above (for the fixed $\mathcal{X}$ and C, but for all parameters e, f, a, and i). Such a G exists as we have only specified countably many dense sets. Given such a G we define a subset g of ω by

$$g = \bigcup_{(p_0,p_1)\in G} p_0.$$

For brevity in what follows, we may not mention the dense sets or G, but will simply say that a g as above is *generic for $P(A,u)$*. In the following lemmas we prove the crucial facts about extending A by a generic g.

Lemma 1.2. *If g is generic for $P(A,u)$, then $g \notin A$, u does not generate an ultrafilter in $\langle A \cup \{g\}\rangle$, and $\langle A \cup \{g\}\rangle$ is still atomless.*

Proof: First, suppose for a contradiction that $g \in A$. Then either $g \in u$ or $-g \in u$. If $-g \in u$ then $K_g \cap G \neq \emptyset$, so choose $p = (p_0,p_1) \in K_g \cap G$. By definition of g we have $p_0 \subseteq g$. But $p \in K_g$, so also $p_0 \setminus g \neq \emptyset$, a contradiction. We reach a contradiction similarly if $g \in u$. In fact, the same argument works since if $p \in K_{-g} \cap G$ then $p_1 \subseteq -g$. For, if $q \in G$, choose $r \in G$ with $r \leq p, q$. Then $q_0 \cap p_1 \subseteq r_0 \cap r_1 = \emptyset$. So $p_1 \cap q_0 = \emptyset$. Hence $p_1 \cap g = \emptyset$.

Next, suppose that u were to generate an ultrafilter in $\langle A \cup \{g\}\rangle$. So there is an $a \in A \setminus u$ such that either $g \leq a$ or $-g \leq a$. If $g \leq a$ then consider $(p_0,p_1) \in G \cap K_a$. We claim that $g = g \cap a = p_0 \cap a \in A$, a contradiction. In fact, clearly $g \cap a \supseteq p_0 \cap a$. For the other inclusion, consider an arbitrary $q \in G$ and let $r \in G$ be such that $r \leq q, p$. Then since $p \in K_a$, we get $q_0 \cap a \subseteq r_0 \cap (p_0 \cup p_1) \cap a \subseteq p_0$, since $r_0 \cap r_1 = 0$ and $p_1 \subseteq r_1$. Thus $g \cap a \subseteq p_0 \cap a$. To carry out a symmetrical argument in case $-g \leq a$ we just need to see that $-g = \bigcup_{(p_0,p_1)\in G} p_1$. For ($\subseteq$), suppose that $i \in -g$. Let $p \in G \cap F_i$. So $i \in p_0 \cup p_1$. We must have $i \notin p_0$ or else $i \in g$, so $i \in p_1$. For the opposite inclusion, suppose that $p \in G$ and $i \in p_1$. Letting $q \in G$ be arbitrary, it suffices to show that $i \notin q_0$. Find $r \in G$ such that $r \leq p, q$. Then $r_0 \cap r_1 = \emptyset$ implies that $r_0 \cap p_1 = \emptyset$, so $i \notin r_0$. Now, because $r_0 \supseteq q_0$, we see that also $i \notin q_0$.

Next, we will check that $\langle A \cup \{g\}\rangle$ is atomless (since A is). Suppose for a contradiction that $g \cap a$ is an atom for some $a \in A$. If $a \notin u$ then $g \cap a = p_0 \cap a$ for $(p_0, p_1) \in K_a \cap G$ (as proved and used above). As $p_0 \cap a \in A$ this contradicts the fact that A is atomless. So $a \in u$. Now, consider $p := (p_0, p_1) \in K_{-a} \cap G$. We have that $p_0 \setminus (-a) = p_0 \cap a$ is not empty. Also $p_0 \cap a \notin u$. So there is a $q \in K_{a\cap p_0} \cap G$. Then as above we have $q_0 \cap (a \cap p_0) = g \cap (a \cap p_0)$. Note that $g \cap p_0 = p_0$, so the set on the right hand side is equal to $p_0 \cap a$, hence is nonempty, and is in fact equal to the atom $g \cap a$. But the set on the left hand side is in A, a contradiction. If $-g \cap a$ were assumed to be the atom, a symmetric argument yields a contradiction. □

Lemma 1.3. *Assume that $G \subseteq P(A, u)$ is as above. Let $e, f \in A$ and suppose that for some $p \in G$ we have $e \triangle f \subseteq p_0 \cup p_1$. Then the set $b := (g \cap e) \cup (f \setminus g)$ is a member of A.*

Proof: We observe that whenever $p = (p_0, p_1) \in G$ we have $p_0 \subseteq g$ and $p_1 \subseteq \omega\setminus g$. So for $p = (p_0, p_1) \in G$, and $d \in A$ satisfying $d \subseteq p_0 \cup p_1$ we have $d \cap g = d \cap p_0$ and $d \cap (\omega \setminus g) = d \cap p_1$. Applying this observation twice with $d = (e \setminus f)$ and $d = (f \setminus e)$ together with trivial $(g \cap e) \cup ((\omega \setminus g) \cap f) \supseteq e \cap f$ we get that

$$b = (g \cap e) \cup (f \setminus g) = (e \cap f) \cup [p_0 \cap (e \setminus f)] \cup [p_1 \cap (f \setminus e)],$$

so $b \in A$. □

Next, we prove a version of Proposition 3.6 from [Kos99].

Lemma 1.4. *With the above notation, $\mathcal{X}$ is still maximal ideal-independent in the algebra $\langle A \cup \{g\}\rangle$.*

Proof: Suppose that $b \in \langle A \cup \{g\}\rangle$, we will show that $\mathcal{X} \cup \{b\}$ is not ideal-independent. Write $b = (e \cap g) \cup (f \cap (-g))$ for some $e, f \in A$. Now let $p \in D_{e,f}$ be such that $p \in G$. Note that $p_0 \subseteq g$. Also $p_1 \subseteq (-g)$. Suppose that $q \in G$. We want to show that $p_1 \cap q_0 = 0$. Choose $r \in G$ such that $r \leq p, q$. Then $p_1 \cap q_0 \subseteq r_1 \cap r_0 = 0$. So $p^* \subseteq b$. We consider cases according to the definition of $D_{e,f}$.

- Case 1. $p_0 \cup p_1 \supseteq e \triangle f$. Then Lemma 1.3 gives that $b \in A$, so $\mathcal{X} \cup \{b\}$ is not ideal-independent by maximality of $\mathcal{X}$ in A.
- Case 2. $\exists n \in \omega\, \exists x_0, \ldots, x_n \in \mathcal{X}\, [x_0 \subseteq p^* \cup x_1 \cup \ldots \cup x_n]$. Then $x_0 \subseteq b \cup x_1 \cup \ldots \cup x_n$.

- Case 3. $\exists n \in \omega\, \exists x_0, \ldots, x_n \in \mathcal{X}\, [p^* \cup a_p \subseteq x_0 \cup \ldots \cup x_n]$. Clearly $b \cap (p_0 \cup p_1) = p^*$, so $b \subseteq p^* \cup a_p$. So also $b \subseteq x_0 \cup \ldots \cup x_n$. □

Lemma 1.5. *With the above notation, C remains maximal in $\langle A \cup \{g\}\rangle$.*

Proof: Letting $b \in \langle A \cup \{g\}\rangle$ we can write $b = (g \cap e) \cup (f \setminus g)$ for some $e, f \in A$. Let $p \in G \cap E_{e,f}$; we will show that $C^\frown\{b\}$ is no longer free, considering cases according to the definition of $E_{e,f}$.

- Case 1. $p_0 \cup p_1 \supseteq e \triangle f$. By Lemma 1.3, in this case $b \in A$. So b does not extend C by maximality in A.
- Case 2. $\exists i < j \in \xi\, [p^* \supseteq c_i \setminus c_j]$. We have that $p^* \subseteq b$, so also $c_i \setminus c_j \subseteq b$. Then $(c_i) \cap (\omega \setminus c_j) \cap (\omega \setminus b) = \emptyset$, so b does not extend C.
- Case 3. $\exists i \in \xi\, [p^* \cup a_p \subseteq \omega \setminus c_i]$. Clearly $b \cap (p_0 \cup p_1) = p^*$, so $b \subseteq p^* \cup a_p$. So $b \subseteq \omega \setminus c_i$. Thus $c_i \cap b = \emptyset$, and again b does not extend C.
- Case 4. $\omega \setminus c_0 \subseteq p^*$. Since $p^* \subseteq b$, also $\omega \setminus c_0 \subseteq b$ so $(\omega \setminus c_0) \cap (\omega \setminus b) = \emptyset$. □

Theorem 1.6 (CH)**.** *Assuming CH there is an atomless Boolean algebra B such that* $\mathrm{s_{mm}(B)} = \mathfrak{f}(\mathrm{B}) = \omega < \omega_1 = \mathfrak{u}(\mathrm{B})$.

Proof: Let $A_0 = A$, and let $C, \mathcal{X} \subseteq A_0$ be as above. Let $\langle \ell_\alpha : \alpha < \omega_1\rangle$ enumerate the limit ordinals below ω_1. Partition ω_1 into the sets $\{M_i : i \in \omega_1\}$, with each part of size ω_1. For each $i \in \omega_1$ let $\langle k^i_\alpha : \alpha < \omega_1\rangle$ enumerate $M_i \setminus (\ell_i + 1)$. Now we construct a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ of countable atomless subalgebras of $\mathscr{P}(\omega)$ as follows. We have already defined A_0. For any limit ordinal $\alpha = \ell_i$ let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and let $\langle u^i_\beta : \beta < \omega_1\rangle$ enumerate all the nonprincipal ultrafilters on A_α. Now suppose α is the successor ordinal $\gamma + 1$. If $\gamma = k^i_\beta$, we proceed as follows. Note that $\ell_i < k^i_\beta$ and so $u^i_\beta \subseteq A_\gamma$. Let $\overline{u^i_\beta}$ denote the filter on A_γ generated by u^i_β. If $\overline{u^i_\beta}$ is not an ultrafilter or if γ is not in any of the sets $M_i \setminus (\ell_i + 1)$ let $A_\alpha = A_\gamma$. If $\overline{u^i_\beta}$ is an ultrafilter then we let x_γ be generic for $P(A_\gamma, \overline{u^i_\beta})$. Define $A_\alpha = \langle A_\gamma \cup \{x_\gamma\}\rangle$. Note that A_α is atomless and $\overline{u^i_\beta}$ does not generate an ultrafilter on A_α.

Now define $B = \bigcup_{\alpha<\omega_1} A_\alpha$. B is atomless as it is a union of atomless algebras. Suppose that some countable $X \subseteq B$ generates an ultrafilter on B. Then pick a limit ordinal $\alpha = \ell_i < \omega_1$ such that $X \subseteq A_\alpha$. So X generates an ultrafilter of A_α; say it generates u^i_β. Let $\gamma = k^i_\beta$. Then by construction, X does not generate an ultrafilter on $A_{\gamma+1}$, contradiction. Therefore $|B| = \omega_1 = \mathfrak{u}(B)$.

Finally, $\mathrm{s_{mm}(B)} = \omega$ and $\mathfrak{f}(B) = \omega$ by Lemmas 1.4 and 1.5, respectively. □

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